# Performance Analysis for LDPC-Coded Modulation in MIMO Multiple-Access Systems

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***Abstract*—We consider a low-density parity-check (LDPC)-coded modulation scheme in multi-input multi-output (MIMO) multiple-access systems. The receiver can be regarded as a serially concatenated iterative detection and decoding scheme, where the LDPC decoders perform the role of outer decoder and the multiuser demapper does that of the inner decoder. In this paper, we investigate the performance of the scheme with simulation results and bounds. Union upper bounds are derived, which can be used as additional means to evaluate the performance of the MIMO multiple-access system.**

***Index Terms*—Iterative demapping and decoding, low-density parity-check (LDPC)-coded modulation, multi-input multi-output (MIMO), multiple-access system, union upper bound.**

## I. Introduction

THE channel capacity can be significantly increased by using multiple transmit and multiple receive antennas [1], [2], called the multi-input multi-output (MIMO) system. Interests in turbo-iterative signal processing algorithms have been explosive during the last decade motivated by the debut of the turbo code in 1993 [3] and the return of the low-density parity-check (LDPC) code in 1997 [4], [5]. How to use the turbo-iterative algorithms in multiuser detection and decoding receivers and exploit the capacity potential of the MIMO system in multiple-access channels is of great interest.

A brief survey of literature in the area of iterative detection and decoding relevant to this paper is as follows. In an attempt to approach the capacity limits of single-input single-output (SISO) channels, Narayanan and Stüber [6] propose an iterative detection and decoding scheme with convolutional codes. Hochwald and Brink extend the iterative detection and decoding scheme to the MIMO channel [7]. For multiuser systems, Wang and Poor propose iterative receiver of joint detection and decoding for coded code-division multiple-access (CDMA) systems [8]. In our previous work [9], we address an LDPC-coded modulation scheme for the MIMO multiple-access system and find the optimal choice of constellation mapping rule under different iterative demapping and decoding schemes. As depicted in Fig. 1, the receiver can be regarded as a serially concatenated iterative detection and decoding scheme, where the LDPC decoders for the two senders perform the role of the outer decoders, and the multiuser demapper does that of the inner decoder.

Bounds are useful to assess system performance. Tight union bound techniques based on the Fano–Gallager's tilting measures have been investigated for SISO channels in [10] and [11]. For single-user MIMO systems, combinatorial union bounding techniques have been investigated [12]. In this paper, we extend the combinatorial union bounding techniques to the case of MIMO multiple-access systems. The union upper bound on maximum-likelihood (ML) decoding error probability for turbo-like or LDPC codes provides a performance prediction of the proposed transmission system although the ML decoding is usually prohibitively complex for long block codes. We derive union upper bounds on the ML detection using the distance distribution of the outer LDPC codes. Closed-form expressions are derived, which, with specific SNRs and a constellation mapping rule, can be evaluated efficiently by using a polynomial expansion.

The rest of this paper is organized as follows. In Section II, we introduce an LDPC-coded modulation scheme with the iterative demapping and decoding operation. Section III provides the union upper bounds on error probability for the LDPC-coded modulation scheme. In Section IV, simulated performances of the system are compared with the bounds. Section V provides concluding remarks.

## II. LDPC-Coded Modulation Scheme

In this section, we first introduce the model of MIMO multiple-access systems. Then, the choice of LDPC code and the multiuser iterative soft demapping algorithm are described.

### A. Multiuser Mimo System Model

Consider a MIMO multiple-access system with two senders and one receiver, as shown in Fig. 1. Each sender is equipped with $N_t$ transmit antennas and the receiver with $N_r$ receive antennas. When only one sender is active, the model is reduced to a single-user MIMO system.

At the transmitter, each LDPC encoder is combined with a space–time modulator. For each sender, the binary source is encoded as an LDPC code $\mathbf{c}^u = (c_1^u, \ldots, c_N^u)$, where $u = 1, 2$, is the user index and $N$ is the block length of the code. The mapping device takes a group of $\log_2(M)$ coded bits and maps them into a constellation symbol, where $M$ is the size of the constellation. Then, the mapping device transforms the symbol sequence into a space–time symbol matrix $\mathbf{x}^u$, i.e., the serial to parallel conversion of the symbol sequence without an explicit space–time coding that is the well-known Bell Labs layered space time (BLAST)-type transmission. Two senders simultaneously transmit the space–time symbol vectors in each MIMO channel-use.

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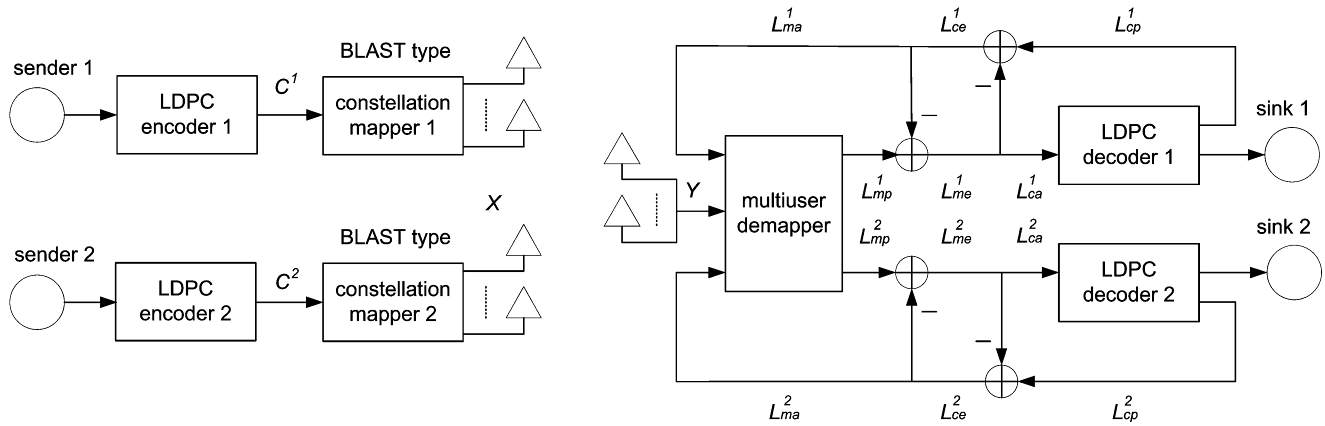


Fig. 1. MIMO multiple access system model with two senders and one receiver.

At the receiver, the received signal at the $t$th channel-use is

$$\mathbf{y}_t = \mathbf{h}^t\mathbf{x}_t + \mathbf{n}_t, \quad \text{for } t = 1, \ldots, T \tag{1}$$

where we define the following vector variables:

$$\mathbf{y}_t := \begin{pmatrix} y_{1t} \\ \vdots \\ y_{N_r t} \end{pmatrix}, \quad \mathbf{n}_t := \begin{pmatrix} n_{1t} \\ \vdots \\ n_{N_r t} \end{pmatrix}, \quad \mathbf{x}_t := \begin{pmatrix} \mathbf{x}_t^1 \\ \mathbf{x}_t^2 \end{pmatrix}$$

$$\mathbf{x}_t^u := \begin{pmatrix} x_{1t}^u \\ \vdots \\ x_{N_t t}^u \end{pmatrix}, \quad \bar{\mathbf{x}}_t^u := \frac{1}{\sqrt{E_{s_u}}}\mathbf{x}_t^u$$

and

$$\mathbf{h}^t := (\mathbf{h}^{1t} \quad \mathbf{h}^{2t}) := \begin{pmatrix} h_{11}^{1t} & \cdots & h_{1N_t}^{1t} & h_{11}^{2t} & \cdots & h_{1N_t}^{2t} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ h_{N_r 1}^{1t} & \cdots & h_{N_r N_t}^{1t} & h_{N_r 1}^{2t} & \cdots & h_{N_r N_t}^{2t} \end{pmatrix}$$

where $T = N/(N_t \log_2(M))$ is the number of channel-uses with an assumption that $N$ is a multiple of $N_t \log_2(M)$ without loss of generality, $E_{s_u}$, for $u = 1, 2$, are the symbol energies of user 1 and user 2, respectively. In this paper, we assume that they are equal, i.e., $E_{s_u} = E_s$. The matrix $\mathbf{h}^t$ is an $[N_r \times 2N_t]$ matrix with independent identically distributed (i.i.d.) complex-valued Gaussian random elements. We assume that each entry is zero mean and unit variance with independent real and imaginary parts. This is intended to model Rayleigh fading channels. The noise $\mathbf{n}_t$ is circularly symmetric complex Gaussian (CSCG) with zero mean and variance $N_0$. Thus, this is a model for equal SNRs for the two senders that will be the assumption throughout the paper. While the model is capable of being generalized to unequal SNRs, the cases with unequal SNRs are not to be considered in this paper. Throughout the paper, the channel state information is assumed to be known at the receiver, but not at the transmitter.

Referring to the receiver part of Fig. 1, the received signal is iteratively demapped and decoded by mutually exchanging soft information between the inner multiuser demapper and the outer LDPC decoders. The demapper computes the posterior log-likelihood ratios (LLRs) $L_{mp}^u$ for each coded bit by using the channel observation $\mathbf{y}$ and the prior information $L_{ma}^u$, which is zero initially. This posterior information $L_{mp}^u$, after subtracting the prior part $L_{ma}^u$, becomes the so-called "extrinsic information" $L_{me}^u$, which is passed to the corresponding LDPC decoder for the initialization of the LDPC message-passing algorithm. The decoder then calculates the posterior LLRs $L_{cp}^u$ as the output of the decoder. Subtracting the prior part $L_{ca}^u$ forwarded from the demapper, we obtain the extrinsic information $L_{ce}^u$ of the decoder that is fed back to the multiuser demapper.

In this decoding scheme, there are two kinds of iterations: one is *super* iteration, the iteration between the demapping and the decoding block; the other is *internal* iteration, the iteration within the LDPC decoder itself. It should be noted that in this system, the two independent code words from the two senders are simultaneously demapped and decoded.

### B. Brief Note on the Design of LDPC Code

An $(N, J, K)$ LDPC code can be represented by a bipartite graph, whose rate is $R_c = 1 - J/K$. Each edge in the graph is related to the nonzero entry of the parity-check matrix $H.H$ has $J$ 1's in each column and $K$ 1's in each row. The low-density matrix $H$ of size $N(1 - R_c) \times N$ is randomly generated. By performing the Gaussian elimination, matrix $H$ can be represented in a systematic way as $[\mathbf{I}; \mathbf{P}]$, where $\mathbf{I}$ is the identity matrix of size $N(1 - R_c) \times N(1 - R_c)$ and $\mathbf{P}$ is the parity-check part of size $N(1 - R_c) \times NR_c$. The generator matrix is constructed as $[\mathbf{P}'; \mathbf{I}]$, where $'$ denotes the transpose. Code words are generated by using the generator matrix, which is the same for the two senders in our system.

### C. Multiuser Iterative Soft Demapping and Decoding

In this section, we focus on the multiuser iterative soft demapping and decoding operations. For simplicity, the script $t$ is omitted in this section since the demapping algorithm is the same for any time $t$.

The multiuser demapper calculates the posterior probability on each unmapped bit in the received signal vector from both senders. We arrange all the bits in an ascending order from 0 to $A = 2N_t \log_2(M) - 1$, where the first group of $N_t \log_2(M)$ bits are from sender 1 and the second group of $N_t \log_2(M)$ bits from sender 2. We keep all calculations in log-domain in this section.

Let $L_{mp}(c_k)$, $L_{ma}(c_k)$, and $L_{me}(c_k)$ denote the posterior probability, the prior probability, and the extrinsic information on the $k$th bit of the multiuser demapper, respectively. We name the collection of the corresponding probability of the first group of $N_t \log_2(M)$ bits as $L^1_{mp}, L^1_{ma}$, and $L^1_{me}$, and the second group of $N_t \log_2(M)$ bits as $L^2_{mp}$, $L^2_{ma}$, and $L^2_{me}$ that are shown in Fig. 1. Let $\tau(\mathbf{c}_{0\dots A,k})$ denote all the possible combinations of bit sequence $c_0 \cdots c_A$ excluding $c_k$. By using the total probability theorem, the log ratio of the posterior probability on each bit of the demapper can be written as

$$L_{mp}(c_k) = \log \frac{\sum p(c_k = 1, \tau(\mathbf{c}_{0\dots A,k})|\mathbf{y})}{\sum p(c_k = 0, \tau(\mathbf{c}_{0\dots A,k})|\mathbf{y})}. \tag{2}$$

Since the parity-check matrix $H$ is randomly generated, it assures near independence until a convergence is reached. Thus, we write the joint probabilities as the product of individual terms. Using Bayes' rule, (2) can be rewritten as

$$\begin{aligned} &L_{mp}(c_k) \\ &= \log \frac{\sum p(\mathbf{y}|c_k = 1, \tau(\mathbf{c}_{0\dots A,k}))p(c_k = 1, \tau(\mathbf{c}_{0\dots A,k}))}{\sum p(\mathbf{y}|c_k = 0, \tau(\mathbf{c}_{0\dots A,k}))p(c_k = 0, \tau(\mathbf{c}_{0\dots A,k}))} \\ &= \log \frac{p(c_k = 1)}{p(c_k = 0)} \\ &\quad + \log \frac{\sum p(\mathbf{y}|c_k = 1, \tau(\mathbf{c}_{0\dots A,k}))p(\tau(\mathbf{c}_{0\dots A,k}))}{\sum p(\mathbf{y}|c_k = 0, \tau(\mathbf{c}_{0\dots A,k}))p(\tau(\mathbf{c}_{0\dots A,k}))} \\ &= L_{ma}(c_k) + L_{me}(c_k). \end{aligned} \tag{3}$$

Let $\mathbf{c}_{bin(j,k)} := c_0 \cdots c_{k-1}c_{k+1} \cdots c_A$ be the binary decomposition of $j$ such that $j = \sum_{i=0,i\neq k}^{A} c_i\, 2^{(i-u(i-k))}$, where $u(t) = 1$ if $t \geq 0$ and $u(t) = 0$ if $t < 0$. Let $B = 2^A - 1$. Then, the extrinsic information on the $k$th bit equals to

$$\begin{aligned} &L_{me}(c_k) \\ &= \log \frac{\displaystyle\sum_{j=0}^{B} p\big(\mathbf{y}|c_k = 1, \mathbf{c}_{\text{bin}(j,k)}\big) \exp\left(\sum_{\substack{i=0,i\neq k\\ c_i=1}}^{A} L_{ma}\, c_i\right)}{\displaystyle\sum_{j=0}^{B} p\Big(\mathbf{y}|c_k = 0, \mathbf{c}_{\text{bin}(j,k)}\big) \exp\left(\sum_{\substack{i=0,i\neq k\\ c_i=1}}^{A} L_{ma}\, c_i\right)}. \end{aligned} \tag{4}$$

To calculate $L_{me}(c_k)$, we need the channel output $\mathbf{y}$. The transition probability can be expressed as

$$p(\mathbf{y}|\mathbf{x}, \mathbf{h}) = \frac{1}{(\pi N_o)^{n/2}} \exp\left(-\frac{1}{N_o}\|\mathbf{y} - \mathbf{h}\mathbf{x}\|^2\right) \tag{5}$$

where $n = 1$ if the signal is real, otherwise $n = 2$ if the signal is complex.

Let $\mathbf{x}_{k,1,j} := \text{map}(c_k = 1, \mathbf{c}_{\text{bin}(j,k)})$; $\mathbf{x}_{k,0,j} := \text{map}(c_k = 0, \mathbf{c}_{\text{bin}(j,k)})$. The function map transforms the included bits into a constellation symbol. Substituting (5) into (4), we obtain the extrinsic information as

$$\begin{aligned} &L_{me}(c_k) \\ &= \log \frac{\displaystyle\sum_{j=0}^{B} \exp\left(-\frac{1}{N_o}\|\mathbf{y} - \mathbf{h}\mathbf{x}_{k,1,j}\|^2 + \sum_{\substack{i=0,i\neq k\\ c_i=1}}^{A} L_{ma}\, c_i\right)}{\displaystyle\sum_{j=0}^{B} \exp\left(-\frac{1}{N_o}\|\mathbf{y} - \mathbf{h}\mathbf{x}_{k,0,j}\|^2 + \sum_{\substack{i=0,i\neq k\\ c_i=1}}^{A} L_{ma}\, c_i\right)}. \end{aligned} \tag{6}$$

This extrinsic information is used as the prior information at the LDPC graph decoder. On the code graph, we perform the message passing algorithm [13] to decode the LDPC code.

## III. Performance Analysis

In this section, we derive union upper bounds for the LDPC-coded modulation scheme in the MIMO multiple access system. We first discuss the properties of an ensemble of LDPC codes. Next, the pairwise error probability is calculated by averaging over the channel state. We further compute the pairwise error probability averaged over the column distance distribution, which will be discussed in detail in Section III-C. The union upper bound is obtained by summing up the averaged pairwise error probabilities for the binary modulation scheme. Then, the derivation of the union upper bound is extended to the case of an $M$-ary modulation scheme. Finally, we provide an example to illustrate the calculation of the union upper bound.

### *A. Discussion on the Ensemble of LDPC Codes*

Let $\mathcal{H}$ be the ensemble of the LDPC matrices $H$, each of which defines an $(N, J, K)$ LDPC code. Let $\mathcal{C}$ be the ensemble of $(N, J, K)$ LDPC codes defined by $\mathcal{H}$. The ensemble $\mathcal{H}$ is closed under column permutation. That is, a column permutation of a particular $H \in \mathcal{H}$ produces another LDPC matrix belonging to the same ensemble. Accordingly, a permutation of a codeword in a codebook is a codeword in another codebook in the ensemble $\mathcal{C}$. Let $\mathcal{C}_d$ be the set of all codewords with hamming distance $d$ from any codebook in the ensemble $\mathcal{C}$, and denote the corresponding set of space–time symbol matrices as $\mathcal{X}_d$.

We assume that every code $C \in \mathcal{C}$ is equiprobably selectable; so does every code word $\mathbf{c} \in C$. Thus, each bit within a randomly selected codeword in $\mathcal{C}_d$ can be modeled as a Bernouli distributed random variable with the parameter $d/N$. We will refer to this as the equiprobable property of the ensemble of LDPC codes.

Next, we will introduce ensemble-averaged distance distribution of LDPC codes. Litsyn and Shevelev propose a number of ways to calculate the distance distributions in [14]. Our ensemble is the same as *ensemble B* in their paper. For a particular code $C \in \mathcal{C}$, the distance distribution is defined as $\mathbf{S}(C) := \big(S_0(C) = 1, S_1(C), \dots, S_N(C)\big)$, where $S_d(C) = |\{\mathbf{c} \in C : \theta(\mathbf{c}) = d\}|$, for $d = 0, \dots, N$, where $\theta(\cdot)$ denotes the Hamming weight and $|\cdot|$ denotes the cardinality of the set.

Then, the ensemble-averaged distance distribution can be defined as $\mathbf{S}(\mathcal{C}) := \big(S_0(\mathcal{C}), S_1(\mathcal{C}), \ldots, S_N(\mathcal{C})\big)$, where $S_d(\mathcal{C}) = \frac{1}{|\mathcal{C}|}\sum_{C\in\mathcal{C}} S_d(C) = \frac{|\mathcal{C}_d|}{|\mathcal{C}|}$.

Both senders use the codes from the ensemble $\mathcal{C}$; thus, they share the same equiprobable property of the ensemble of LDPC codes and the same ensemble-averaged distance distribution. These properties will be utilized in the derivation of the union upper bound.

### B. Pairwise Error Probability Averaged Over Channel State

Let $\mathbf{x}_{(d_1,d_2)} := \begin{pmatrix}\mathbf{x}^1_{(d_1)}\\ \mathbf{x}^2_{(d_2)}\end{pmatrix}$ denote the space-time symbol matrix, where $\mathbf{x}^1_{(d_1)}$ is mapped from the codeword with Hamming weight $d_1$ for user 1 and $\mathbf{x}^2_{(d_2)}$ mapped from the codeword with Hamming weight $d_2$ for user 2. Assume the all-zero codewords are transmitted for both senders. We consider the probability of transmitting $\mathbf{x}_{(0,0)}$ in favor of deciding $\mathbf{x}_{(d_1,d_2)}$, conditioned on the channel state. This pairwise error probability based on ML detection can be upper bounded by using the Chernoff bound

$$p\big(\mathbf{x}_{(0,0)} \to \mathbf{x}_{(d_1,d_2)}|\mathbf{h}\big) \le \exp\left(-\frac{d^2\big(\mathbf{x}_{(0,0)}\,\mathbf{x}_{(d_1,d_2)}\big)}{4N_0}\right) \tag{7}$$

where

$$d^2\big(\mathbf{x}_{(0,0)},\,\mathbf{x}_{(d_1,d_2)}\big) = \sum_{t=1}^{T}\|\mathbf{h}^t\mathbf{x}_{t,(0,0)} - \mathbf{h}^t\mathbf{x}_{t,(d_1,d_2)}\|_F^2$$

where $\mathbf{x}_{t,(0,0)}$ and $\mathbf{x}_{t,(d_1,d_2)}$ are the $t$th column of $\mathbf{x}_{(0,0)}$ and $\mathbf{x}_{(d_1,d_2)}$, respectively, $F$ denotes the Frobenius norm.

To calculate an upper bound on the pairwise error probability averaged over the channel state, we take the average of R.H.S. of the (7) with respect to the channel fading matrix $\mathbf{h}$ [15]. Then, the pairwise error probability can be bounded by

$$\begin{aligned} p\big(&\mathbf{x}_{(0,0)} \to \mathbf{x}_{(d_1,d_2)}\big)\\ &\le \prod_{t=1}^{T}\left(1+\frac{|\mathbf{x}_{t,(0,0)} - \mathbf{x}_{t,(d_1,d_2)}|^2}{4N_0}\right)^{-N_r}\\ &= \prod_{t=1}^{T}\left(1+|\bar{\mathbf{x}}_{t,(0,0)} - \bar{\mathbf{x}}_{t,(d_1,d_2)}|^2\rho\right)^{-N_r} \end{aligned} \tag{8}$$

where $\bar{\mathbf{x}}_{t,(0,0)} = \frac{1}{\sqrt{E_s}}\mathbf{x}_{t,(0,0)}$, $\bar{\mathbf{x}}_{t,(d_1,d_2)} = \frac{1}{\sqrt{E_s}}\mathbf{x}_{t,(d_1,d_2)}$, and $\rho = E_s/4N_0$.

To illustrate the reasoning clearly, we first consider the BPSK mapping case. Let $w_t^u := |\bar{\mathbf{x}}^u_{t,(0)} - \bar{\mathbf{x}}^u_{t,(d_u)}|^2/4$ denote the *weight* of the $t$th column error of user $u$, for $t = 1,\ldots,T$. The weight $w_t^u$ takes a value from the set of $\{0,\ldots,N_t\}$, and thus, it has at most $N_t+1$ different values. Then, (8) can be rewritten as

$$p\big(\mathbf{x}_{(0,0)} \to \mathbf{x}_{(d_1,d_2)}\big) \le \prod_{t=1}^{T}\left(1+4(w_t^1+w_t^2)\rho\right)^{-N_r}. \tag{9}$$

We notice that the pairwise error probability is determined by $T$ numbers of *column weight pairs* $(w_t^1, w_t^2)$.

### C. Pairwise Error Probability Averaged Over Column Distance Distribution

We observe that the columns with the identical column weight pair $(w_t^1, w_t^2)$ result in the same term in the product of (9). Thus, we let $l_{i,j}$ denote the number of columns of the difference matrix $(\bar{\mathbf{x}}_{(0,0)} - \bar{\mathbf{x}}_{(d_1,d_2)})$, each of which has weight $i$ from user 1 and weight $j$ from user 2. That is

$$l_{i,j} := \sum_{t=1}^{T}\mathbf{1}_t \tag{10}$$

where $\mathbf{1}_t$ is the indicator function of $t$, which is either 1 if $(w_t^1, w_t^2) = (i,j)$ or 0 otherwise. By grouping the identical terms, (9) can be further simplified as

$$p\big(\mathbf{x}_{(0,0)} \to \mathbf{x}_{(d_1,d_2)}\big) \le \prod_{i=0}^{N_t}\prod_{j=0}^{N_t}(1+4(i+j)\rho)^{-N_r l_{i,j}}. \tag{11}$$

Now, we aim to compute the average pairwise error probability for any $\mathbf{x}_{(d_1,d_2)} \in \mathcal{X}_{d_1,d_2}$, where

$$\mathcal{X}_{d_1,d_2} := \begin{pmatrix}\mathcal{X}_{d_1}\\ \mathcal{X}_{d_2}\end{pmatrix}.$$

Collect $l_{i,j}$'s into an $(N_t+1)\times(N_t+1)$ matrix, which is denoted as $\mathbf{L}$. Let us name $\mathbf{L}$ as the column distance distribution (CDD) matrix. Also, define $\mathcal{L}_{d_1,d_2}$ as the collection of all CDD matrices, i.e.,

$$\mathcal{L}_{d_1,d_2} := \left\{\mathbf{L}\,|\,l_{i,j}\in\{0,1,\ldots,T\}, \sum_{i=0}^{N_t}\sum_{j=0}^{N_t} l_{i,j} = T,\right.$$
$$\left.\sum_{i=0}^{N_t} i l_{i,j} = d_1, \sum_{j=0}^{N_t} j l_{i,j} = d_2\right\}$$

That is, each CDD matrix in the set $\mathcal{L}_{d_1,d_2}$ must satisfy the following constraints.

1) $\sum_{i=0}^{N_t}\sum_{j=0}^{N_t} l_{i,j} = T$: The total number of columns should add up to $T$.
2) $\sum_{i=0}^{N_t} i l_{i,j} = d_1$ and $\sum_{j=0}^{N_t} j l_{i,j} = d_2$: the weight of the first user's codeword and that of the second user's should add up to $d_1$ and $d_2$, respectively.

Applying the equiprobable property of the ensemble of the codes, the probability of a given $\mathbf{L}$ is the number of selections satisfying $\theta(\mathbf{c}^u) = d_u$ for a particular $\mathbf{L}$ divided by the total number of such selections for any $\mathbf{L}$. Using the usual combinatorial techniques, we obtain the probability distribution of $\mathbf{L}$ as

$$p(\mathbf{L}) = \begin{cases} \binom{T}{\mathbf{L}}\prod_{i=0}^{N_t}\prod_{j=0}^{N_t}\binom{N_t}{i}^{l_{i,j}}\binom{N_t}{j}^{l_{i,j}} \\ \quad\times\prod_{u=1}^{2}\binom{N}{d_u}^{-1}, & \text{if } \mathbf{L}\in\mathcal{L}_{d_1,d_2}\\ 0, & \text{otherwise}\end{cases}$$

where $\binom{T}{\mathbf{L}}$ is the multinomial coefficient, i.e.,

$$\binom{T}{\mathbf{L}} = \frac{T!}{\prod_{i=0}^{N_t}\prod_{j=0}^{N_t} l_{i,j}!}.$$

The multinomial coefficient $\binom{T}{\mathbf{L}}$ denotes the number of every possible way that the $T$ distinct columns can be partitioned into $(N_t+1)^2$ unordered subsets. The $(i,j)$th subset has $l_{i,j}$ columns, where $(i,j)$ with $i=0,\ldots,N_t; j=0,\ldots,N_t$ is the index of the subset.

Thus, we obtain the upper bound of the pairwise error probability averaged over the column distance distribution as follows:

$$\begin{aligned}\overline{p\big(\mathbf{x}_{(0,0)} \to \mathbf{x}_{(d_1,d_2)}\big)} &= \sum_{\mathbf{L}\in\mathcal{L}_{d_1,d_2}} p\big(\mathbf{x}_{(0,0)} \to \mathbf{x}_{(d_1,d_2)}\big) p(\mathbf{L}) \\ &\le \sum_{\mathbf{L}\in\mathcal{L}_{d_1,d_2}} \binom{T}{\mathbf{L}} \prod_{i=0}^{N_t}\prod_{j=0}^{N_t} \binom{N_t}{i}^{l_{i,j}} \binom{N_t}{j}^{l_{i,j}} \\ &\times \Big(1+4(i+j)\rho\Big)^{-N_r l_{i,j}} \prod_{u=1}^{2}\binom{N}{d_u}^{-1}. \qquad (12)\end{aligned}$$

### *D. Union Upper Bound for LDPC-Coded Modulation in MIMO Multiple-Access Systems: BPSK Case*

The union bound on block error probability for ML detection is to sum up the average pairwise error probabilities of (12), each of which is weighted by the ensemble-averaged distance distribution. In a MIMO multiple-access system, a pairwise error happens if the decoder is in favor of $\mathbf{x}_{(d_1,d_2)}$, which is not equal to $\mathbf{x}_{(0,0)}$. Applying the ensemble-averaged distance distribution property, the average word error probability in a MIMO multiple-access system can be upper bounded by

$$\begin{aligned}\overline{P_e} \le& \sum_{d_1=0}^{N}\sum_{d_2=0}^{N} S_{d_1}S_{d_2}\, \overline{p\big(\mathbf{x}_{(0,0)} \to \mathbf{x}_{(d_1,d_2)}\big)} \\ &- S_0^2\, \overline{p\big(\mathbf{x}_{(0,0)} \to \mathbf{x}_{(0,0)}\big)} \qquad (13)\end{aligned}$$

where $S_{d_u} = S_{d_u}(\mathcal{C})$ for $u=1,2$.

By applying (12) and defining $\alpha_{i,j}$ and $\Phi_{d_1,d_2}$, (13) can be written as

$$\begin{aligned}\overline{P_e} \le& \sum_{d_1=0}^{N}\sum_{d_2=0}^{N} \binom{N}{d_1}^{-1}\binom{N}{d_2}^{-1} S_{d_1}S_{d_2} \\ &\times \sum_{\mathbf{L}\in\mathcal{L}_{d_1,d_2}} \binom{T}{\mathbf{L}} \prod_{i=0}^{N_t}\prod_{j=0}^{N_t} (\alpha_{i,j})^{l_{i,j}} - S_0^2 \\ =& \sum_{d_1=0}^{N}\sum_{d_2=0}^{N} \binom{N}{d_1}^{-1}\binom{N}{d_2}^{-1} S_{d_1}S_{d_2}\Phi_{d_1,d_2} - S_0^2 \quad (14)\end{aligned}$$

where

$$\alpha_{i,j} := \binom{N_t}{i}\binom{N_t}{j}\Big(1+4(i+j)\rho\Big)^{-N_r} \qquad (15)$$

and

$$\Phi_{d_1,d_2} := \sum_{\mathbf{L}\in\mathcal{L}_{d_1,d_2}} \binom{T}{\mathbf{L}} \prod_{i=0}^{N_t}\prod_{j=0}^{N_t} (\alpha_{i,j})^{l_{i,j}}. \qquad (16)$$

To evaluate the RHS of (14) efficiently, we resort to the method of a polynomial expansion. Let $\mathbf{L}$ denote a square matrix with $(N_t+1)\times(N_t+1)$ elements, then the following equation holds:

$$\left(\sum_{i=0}^{N_t}\sum_{j=0}^{N_t} x_{i,j}\right)^T = \sum_{\mathbf{L}\in\mathcal{L}} \binom{T}{\mathbf{L}} \prod_{i=0}^{N_t}\prod_{j=0}^{N_t} (x_{i,j})^{l_{i,j}} \qquad (17)$$

where

$$\mathcal{L} := \Big\{\mathbf{L}\,|\, l_{i,j}\in\{0,\ldots,T\}, \sum_{i=0}^{N_t}\sum_{j=0}^{N_t} l_{i,j} = T\Big\}.$$

By applying (17), we get

$$\begin{aligned}\left(\sum_{i=0}^{N_t}\sum_{j=0}^{N_t} \alpha_{i,j}\, y^i z^j\right)^T &= \sum_{\mathbf{L}\in\mathcal{L}} \binom{T}{\mathbf{L}} \prod_{i=0}^{N_t}\prod_{j=0}^{N_t} \left(\alpha_{i,j}\, y^i z^j\right)^{l_{i,j}} \\ &= \sum_{\mathbf{L}\in\mathcal{L}} \binom{T}{\mathbf{L}} y^{\sum_{i=0}^{N_t} i\, l_{i,j}} z^{\sum_{j=0}^{N_t} j\, l_{i,j}} \prod_{i=0}^{N_t}\prod_{j=0}^{N_t} (\alpha_{i,j})^{l_{i,j}} \\ &= \sum_{d_1=0}^{N}\sum_{d_2=0}^{N}\sum_{\mathbf{L}\in\mathcal{L}_{d_1,d_2}} \binom{T}{\mathbf{L}} \prod_{i=0}^{N_t}\prod_{j=0}^{N_t} (\alpha_{i,j})^{l_{i,j}} y^{d_1} z^{d_2} \\ &= \sum_{d_1=0}^{N}\sum_{d_2=0}^{N} \Phi_{d_1,d_2} y^{d_1} z^{d_2} \qquad (18)\end{aligned}$$

where $N = N_t T$.

Then, substituting (18) into (14), $\Phi_{d_1,d_2}$ can be calculated by collecting the coefficients $\alpha_{i,j}$'s in $\left(\sum_{i=0}^{N_t}\sum_{j=0}^{N_t} \alpha_{i,j}\, y^i z^j\right)^T$.

Using the equiprobable property of the ensemble of the codes again, we can show that the union upper bound for bit error probability is

$$\overline{P_b} \le \sum_{d_1=0}^{N}\sum_{d_2=0}^{N} \frac{d_1}{N}\frac{d_2}{N}\binom{N}{d_1}^{-1}\binom{N}{d_2}^{-1} S_{d_1}S_{d_2}\Phi_{d_1,d_2}. \quad (19)$$

### *E. Union Upper Bound for LDPC-Coded Modulation in MIMO Multiple-Access Systems: M-Ary Case*

The previous analysis is based on LDPC space–time code with the binary phase-shift keying (BPSK) modulation. For an $M$-ary modulation, we need to recalculate the average pairwise error probability by reconsidering the distribution of $\mathbf{L}$. We take the following approach.

Each entry of $\bar{\mathbf{x}}_{(d_1,d_2)}$ is selected from the $M$-ary symbols $\{s_i\}_{i=0}^{M-1}$. We have $s_0$ mapped from the string of all-zero bits of length $\log_2(M)$. Let $\delta_i$ denote the Hamming weight of the bits mapped to $s_i$ and $r_i^u$ the number of $s_i$'s contained in a particular column of $\bar{\mathbf{x}}^u_{(d_u)}$ for $u=1,2$. Collect $\delta_i$'s and $r_i^u$'s into $M$-tuples, i.e., $\delta = (\delta_\mathbf{0},\ldots,\delta_\mathbf{M-1})$ and $\mathbf{r}^u = (r_0^u,\ldots,r_{M-1}^u)$. Then, the column weight can be obtained by the inner product of the two, i.e., $w_t^u = \mathbf{r}^u\cdot\delta$, for $t=1,\ldots,T$. Each column weight $w_t^u$ takes a value from the set $\{0,\ldots,N_t\,\log_2(M)\}$. Note the cardinality of this set is $N_t\,\log_2(M)+1$. We also note that the column weight $w_t^u$ is completely determined by $\mathbf{r}^u$ for a fixed constellation mapping rule (i.e., for a fixed $\delta$).

We note that any two columns with an identical column weight pair $(w_t^1, w_t^2)$ produce the same term in the product of the pairwise error probability. Thus, we denote $l_{\mathbf{r}^1,\mathbf{r}^2}$ as the number of the columns in the difference matrix $(\bar{\mathbf{x}}_{(0,0)} - \bar{\mathbf{x}}_{(d_1,d_2)})$, each of which has a weight tuple $\mathbf{r}^1$ from user 1 and a weight tuple $\mathbf{r}^2$ from user 2, i.e., it can be written as

$$l_{\mathbf{r}^1,\mathbf{r}^2} := \sum_{t=1}^{T} \mathbf{1}_t \tag{20}$$

where $\mathbf{1}_t = 1$ if $(w_t^1, w_t^2) = (\mathbf{r}^1 \cdot \delta, \mathbf{r}^2 \cdot \delta)$ and 0 otherwise. By grouping the identical terms together, the expression of the upper bound on the pairwise error probability can be written as

$$p\left(\mathbf{x}_{(0,0)} \to \mathbf{x}_{(d_1,d_2)}\right) \leq \prod_{\mathbf{r}^1\in\mathcal{R}_1} \prod_{\mathbf{r}^2\in\mathcal{R}_2} \left(1 + \sum_{u=1}^{2} \sum_{i=0}^{M-1} r_i^u |s_i - s_0|^2 \rho\right)^{-N_r l_{\mathbf{r}^1,\mathbf{r}^2}}. \tag{21}$$

We aim to compute the average pairwise error probability for any $\mathbf{x}_{(d_1,d_2)} \in \mathcal{X}_{d_1,d_2}$. Let us collect $l_{\mathbf{r}^1,\mathbf{r}^2}$'s into an $(N_t \log_2(M)+1) \times (N_t \log_2(M)+1)$ matrix $\mathbf{L}$. Define $\mathcal{L}_{d_1,d_2}$ as the collection of all such matrices that satisfy a set of constraints, i.e.,

$$\mathcal{L}_{d_1,d_2} := \Big\{\mathbf{L} \,|\, l_{\mathbf{r}^1,\mathbf{r}^2} \in \{0,1,\ldots,T\}, \quad \sum_{\mathbf{r}^1\in\mathcal{R}_1}\sum_{\mathbf{r}^2\in\mathcal{R}_2} l_{\mathbf{r}^1,\mathbf{r}^2} = T$$
$$\sum_{\mathbf{r}^1\in\mathcal{R}_1} (\mathbf{r}^1\cdot\delta)\, l_{\mathbf{r}^1,\mathbf{r}^2} = d_1, \qquad \sum_{\mathbf{r}^2\in\mathcal{R}_2} (\mathbf{r}^2\cdot\delta)\, l_{\mathbf{r}^1,\mathbf{r}^2} = d_2\Big\}$$

where $\mathcal{R}_u = \left\{\mathbf{r}^u \,|\, r_i^u \in \{0,1,\ldots,N_t\}, \sum_{i=0}^{M-1} r_i^u = N_t\right\}$ for $u = 1, 2$.

Then, the distribution of $\mathbf{L}$ is as follows:

$$p(\mathbf{L}) = \begin{cases} \binom{T}{\mathbf{L}} \prod_{u=1}^{2} \binom{N}{d_u}^{-1} \prod_{\mathbf{r}^u\in\mathcal{R}_u} \binom{N_t}{\mathbf{r}^u}^{l_{\mathbf{r}^1,\mathbf{r}^2}}, & \text{if } \mathbf{L} \in \mathcal{L}_{d_1,d_2} \\ 0, & \text{otherwise} \end{cases}$$

where

$$\binom{T}{\mathbf{L}} = \frac{T\,!}{\prod_{\mathbf{r}^1\in\mathcal{R}_1}\prod_{\mathbf{r}^2\in\mathcal{R}_2} l_{\mathbf{r}^1,\mathbf{r}^2}\,!}$$

and

$$\binom{N_t}{\mathbf{r}^u} = \frac{N_t\,!}{\prod_{i=0}^{M-1} r_i^u\,!}.$$

Thus, the upper bound of the average pairwise error probability can be obtained as

$$\overline{p\big(\mathbf{x}_{(0,0)} \to \mathbf{x}_{(d_1,d_2)}\big)} = \sum_{\mathbf{L}\in\mathcal{L}_{d_1,d_2}} p\big(\mathbf{x}_{0,0} \to \mathbf{x}_{d_1,d_2}\big) p(\mathbf{L})$$
$$\leq \sum_{\mathbf{L}\in\mathcal{L}_{d_1,d_2}} \binom{T}{\mathbf{L}} \prod_{\mathbf{r}^1\in\mathcal{R}_1}\prod_{\mathbf{r}^2\in\mathcal{R}_2} \left(\binom{N_t}{\mathbf{r}^1}\binom{N_t}{\mathbf{r}^2}\right)^{l_{\mathbf{r}^1,\mathbf{r}^2}}$$
$$\times \left(1 + \sum_{u=1}^{2}\sum_{i=0}^{M-1} r_i^u |s_i - s_0|^2\rho\right)^{-N_r l_{\mathbf{r}^1,\mathbf{r}^2}} \prod_{u=1}^{2}\binom{N}{d_u}^{-1}. \tag{22}$$

By summing up the average pairwise error probabilities, we obtain the upper bound of the word error probability for multiuser space–time coded $M$-ary modulation of LDPC codes as

$$\begin{aligned}\overline{P_e} &\leq \sum_{d_1=0}^{N}\sum_{d_2=0}^{N} S_{d_1} S_{d_2}\, \overline{p\big(\mathbf{x}_{(0,0)} \to \mathbf{x}_{(d_1,d_2)}\big)} \\ &\quad - S_0^2\, \overline{p\big(\mathbf{x}_{(0,0)} \to \mathbf{x}_{(0,0)}\big)} \\ &\leq \sum_{d_1=0}^{N}\sum_{d_2=0}^{N} \binom{N}{d_1}^{-1}\binom{N}{d_2}^{-1} S_{d_1}S_{d_2} \\ &\quad \times \sum_{\mathbf{L}\in\mathcal{L}_{d_1,d_2}} \binom{T}{\mathbf{L}} \prod_{\mathbf{r}^1\in\mathcal{R}_1}\prod_{\mathbf{r}^2\in\mathcal{R}_2} (\beta_{\mathbf{r}^1,\mathbf{r}^2})^{l_{\mathbf{r}^1,\mathbf{r}^2}} - S_0^2 \\ &= \sum_{d_1=0}^{N}\sum_{d_2=0}^{N} \binom{N}{d_1}^{-1}\binom{N}{d_2}^{-1} S_{d_1}S_{d_2}\Psi_{d_1,d_2} - S_0^2\end{aligned} \tag{23}$$

where

$$\beta_{\mathbf{r}^1,\mathbf{r}^2} := \binom{N_t}{\mathbf{r}^1}\binom{N_t}{\mathbf{r}^2}\left(1 + \sum_{u=1}^{2}\sum_{i=0}^{M-1} r_i^u|s_i - s_0|^2\rho\right)^{-N_r} \tag{24}$$

and

$$\Psi_{d_1,d_2} := \sum_{\mathbf{L}\in\mathcal{L}_{d_1,d_2}} \binom{T}{\mathbf{L}} \prod_{\mathbf{r}^1\in\mathcal{R}_1}\prod_{\mathbf{r}^2\in\mathcal{R}_2} (\beta_{\mathbf{r}^1,\mathbf{r}^2})^{l_{\mathbf{r}^1,\mathbf{r}^2}}. \tag{25}$$

By applying (17), we get

$$\begin{aligned}&\left(\sum_{\mathbf{r}^1\in\mathcal{R}_1}\sum_{\mathbf{r}^2\in\mathcal{R}_2} \beta_{\mathbf{r}^1,\mathbf{r}^2}\, y^{(\mathbf{r}^1\cdot\delta)} z^{(\mathbf{r}^2\cdot\delta)}\right)^T \\ &= \sum_{\mathbf{L}\in\mathcal{L}} \binom{T}{\mathbf{L}} \prod_{\mathbf{r}^1\in\mathcal{R}_1}\prod_{\mathbf{r}^2\in\mathcal{R}_2} \left(\beta_{\mathbf{r}^1,\mathbf{r}^2}\, y^{(\mathbf{r}^1\cdot\delta)} z^{(\mathbf{r}^2\cdot\delta)}\right)^{l_{\mathbf{r}^1,\mathbf{r}^2}} \\ &= \sum_{\mathbf{L}\in\mathcal{L}} \binom{T}{\mathbf{L}} \left(y^{\sum_{\mathbf{r}^1\in\mathcal{R}_1}(\mathbf{r}^1\cdot\delta)\, l_{\mathbf{r}^1,\mathbf{r}^2}}\right)\left(z^{\sum_{\mathbf{r}^2\in\mathcal{R}_2}(\mathbf{r}^2\cdot\delta)\, l_{\mathbf{r}^1,\mathbf{r}^2}}\right) \\ &\quad \times \prod_{\mathbf{r}^1\in\mathcal{R}_1}\prod_{\mathbf{r}^2\in\mathcal{R}_2} \left(\beta_{\mathbf{r}^1,\mathbf{r}^2}\right)^{l_{\mathbf{r}^1,\mathbf{r}^2}} \\ &= \sum_{d_1=0}^{N}\sum_{d_2=0}^{N}\sum_{\mathbf{L}\in\mathcal{L}_{d_1,d_2}} \binom{T}{\mathbf{L}} \prod_{\mathbf{r}^1\in\mathcal{R}_1}\prod_{\mathbf{r}^2\in\mathcal{R}_2} \left(\beta_{\mathbf{r}^1,\mathbf{r}^2}\right)^{l_{\mathbf{r}^1,\mathbf{r}^2}} y^{d_1} z^{d_2} \\ &= \sum_{d_1=0}^{N}\sum_{d_2=0}^{N} \Psi_{d_1,d_2} y^{d_1} z^{d_2}\end{aligned} \tag{26}$$

where $N = N_t T \log_2 M$ and

$$\mathcal{L} := \left\{\mathbf{L} \,|\, l_{\mathbf{r}^1,\mathbf{r}^2} \in \{0,1,\ldots,T\}, \sum_{\mathbf{r}^1\in\mathcal{R}_1}\sum_{\mathbf{r}^2\in\mathcal{R}_2} l_{\mathbf{r}^1,\mathbf{r}^2} = T\right\}.$$

Applying (26) to (23), $\Psi_{d_1,d_2}$ can be evaluated by collecting the coefficients $\beta_{\mathbf{r}^1,\mathbf{r}^2}$'s in $\left(\sum_{\mathbf{r}^1\in\mathcal{R}_1}\sum_{\mathbf{r}^2\in\mathcal{R}_2}\beta_{\mathbf{r}^1,\mathbf{r}^2}\, y^{(\mathbf{r}^1\cdot\delta)} z^{(\mathbf{r}^2\cdot\delta)}\right)^T$.

Then, the union upper bound of bit error probability for the multiuser space–time coded $M$-ary modulation of LDPC codes is

$$\overline{P_b} \le \sum_{d_1=0}^{N}\sum_{d_2=0}^{N}\frac{d_1}{N}\frac{d_2}{N}\binom{N}{d_1}^{-1}\binom{N}{d_2}^{-1}S_{d_1}S_{d_2}\Psi_{d_1,d_2}. \quad (27)$$

### F. An Illustrative Example

We provide an example to illustrate how to calculate the union upper bound. Consider a MIMO multiple-access system with two senders under BPSK modulation, each of which is equipped with two transmit antennas, and the receiver with two receive antennas. By applying (15), we first form a $[3\times 3]$ matrix with elements $\alpha_{i,j}$, which are obtained as

$$\alpha := \begin{pmatrix}\alpha_{0,0} & \alpha_{0,1} & \alpha_{0,2}\\ \alpha_{1,0} & \alpha_{1,1} & \alpha_{1,2}\\ \alpha_{2,0} & \alpha_{2,1} & \alpha_{2,2}\end{pmatrix} = \begin{pmatrix}1 & 2(1+4\rho)^{-2} & (1+8\rho)^{-2}\\ 2(1+4\rho)^{-2} & 4(1+8\rho)^{-2} & 2(1+12\rho)^{-2}\\ (1+8\rho)^{-2} & 2(1+12\rho)^{-2} & (1+16\rho)^{-2}\end{pmatrix}.$$

We take $\log_2(T)$-fold two-dimensional convolution to obtain the $[(N+1)\times(N+1)]$ coefficient matrix $[\Phi_{d_1,d_2}]$. For an illustration, suppose $N=4$ and, thus, $T=2$.

$$\begin{pmatrix}\alpha_{0,0} & \alpha_{0,1}z & \alpha_{0,2}z^2\\ \alpha_{1,0}y & \alpha_{1,1}yz & \alpha_{1,2}yz^2\\ \alpha_{2,0}y^2 & \alpha_{2,1}y^2z & \alpha_{2,2}y^2z^2\end{pmatrix} * \begin{pmatrix}\alpha_{0,0} & \alpha_{0,1}z & \alpha_{0,2}z^2\\ \alpha_{1,0}y & \alpha_{1,1}yz & \alpha_{1,2}yz^2\\ \alpha_{2,0}y^2 & \alpha_{2,1}y^2z & \alpha_{2,2}y^2z^2\end{pmatrix}$$
$$= \begin{pmatrix}\gamma_{0,0} & \gamma_{0,1}z & \gamma_{0,2}z^2 & \gamma_{0,3}z^3 & \gamma_{0,4}z^4\\ \gamma_{1,0}y & \gamma_{1,1}yz & \gamma_{1,2}yz^2 & \gamma_{1,3}yz^3 & \gamma_{1,4}yz^4\\ \gamma_{2,0}y^2 & \gamma_{2,1}y^2z & \gamma_{2,2}y^2z^2 & \gamma_{2,3}y^2z^3 & \gamma_{2,4}y^2z^4\\ \gamma_{3,0}y^3 & \gamma_{3,1}y^3z & \gamma_{3,2}y^3z^2 & \gamma_{3,3}y^3z^3 & \gamma_{3,4}y^3z^4\\ \gamma_{4,0}y^4 & \gamma_{4,1}y^4z & \gamma_{4,2}y^4z^2 & \gamma_{4,3}y^4z^3 & \gamma_{4,4}y^4z^4\end{pmatrix} = \gamma\cdot V$$

where $*$ and $\cdot$ denote the two-dimensional convolution and the dot product for matrices, respectively; in addition, we define

$$\gamma := \begin{pmatrix}\gamma_{0,0} & \gamma_{0,1} & \gamma_{0,2} & \gamma_{0,3} & \gamma_{0,4}\\ \gamma_{1,0} & \gamma_{1,1} & \gamma_{1,2} & \gamma_{1,3} & \gamma_{1,4}\\ \gamma_{2,0} & \gamma_{2,1} & \gamma_{2,2} & \gamma_{2,3} & \gamma_{2,4}\\ \gamma_{3,0} & \gamma_{3,1} & \gamma_{3,2} & \gamma_{3,3} & \gamma_{3,4}\\ \gamma_{4,0} & \gamma_{4,1} & \gamma_{4,2} & \gamma_{4,3} & \gamma_{4,4}\end{pmatrix}$$

and

$$V := \begin{pmatrix}1 & z & z^2 & z^3 & z^4\\ y & yz & yz^2 & yz^3 & yz^4\\ y^2 & y^2z & y^2z^2 & y^2z^3 & y^2z^4\\ y^3 & y^3z & y^3z^2 & y^3z^3 & y^3z^4\\ y^4 & y^4z & y^4z^2 & y^4z^3 & y^4z^4\end{pmatrix}.$$

We note that $\gamma = \alpha * \alpha$, which are exactly the coefficients of expanded $\left(\sum_{i=0}^{2}\sum_{j=0}^{2}\alpha_{i,j}\, y^i z^j\right)^2$. For $T>2$, we repeat the convolution operation for $\log_2(T)$ times.

## IV. Numerical Evaluation and System Simulation Results

In this section, we provide the simulation results of the LDPC-coded modulation scheme for the MIMO multiple access system depicted in Fig. 1, and illustrate the performance of the turbo-iterative multiuser detection and decoding processing receiver. The system simulation results will be compared with the union upper bounds, as well as with well-established performance prediction measures such as the constrained channel capacities and the threshold values obtained from the extrinsic information transfer (EXIT) chart analysis [16]–[18]. For all simulations, the regular LDPC codes with the bit node degree 3 and the check node degree 6 are used, i.e., $J=3, K=6$. The rate $R_c$ of this binary code is, thus, 1/2. We simulate various block lengths, i.e., 256, 512, and 1024, to see how the bounds and the simulation results scale with increase in block length. The signal-to-noise ratio (SNR) is defined as the ratio of the received signal energy per user to the energy of the noise whose one sided power spectral density is $N_0$. The received signal energy per user is the symbol energy times the number of transmit antennas. Thus, the SNR is defined as SNR $= N_tE_s/N_0$. In addition, each sender is equipped with two transmit antennas and the receiver with two receive antennas. Two different modulation cases, BPSK and 4-QAM with the Gray constellation mapping, are considered.

As mentioned in Section II-A, there are two kinds of iterations for the receiver. We investigate the optimal ratio of the number of *super* iterations (NSI) to the number of *internal* iterations (NII), given the total number of iterations (TNI). For compact description of the simulation results, Table I tabulates the required SNRs to achieve bit-error rate (BER) of $10^{-4}$ at each option for the (1024,3,6) LDPC code with BPSK modulation in a MIMO multiple-access system. The TNIs considered are 30, 60, and 120. For each TNI, we vary the ratio between the NSI and the NII, and make a notice on the best option. We note that the performance benefit is about 0.3 dB when the TNI varies from 30 to 60; the benefit becomes about 0.1 dB when the TNI varies from 60 to 120. At the TNI of 60, the best combination is found to be 6 *super* iterations and 10 *internal* iterations.

In addition, we find from extensive simulations that increasing the NSI is beneficial when the number of bits per channel-use is increased. For example, using 4-QAM modulation and fixing the TNI at 60, it is better off to have the NSI increased to 12 while the NII decreased to 5. In what follows, we fix the TNI at 60 and use the best ratio obtained for each constellation option.

TABLE I
COMPARISON OF SNRS TO ACHIEVE BER OF $10^{-4}$

| TNI | 30 | 30 | 60 | 60 | 60 | 120 |
|---|---|---|---|---|---|---|
| NSI | 3 | 6 | 3 | 6 | 12 | 6 |
| NII | 10 | 5 | 20 | 10 | 5 | 20 |
| SNR (dB) | 1.5 | 2.1 | 1.4 | 1.2 | 2.1 | 1.1 |

For EXIT chart analysis, we note that, there are at least two methods available in the literature. In [17], the MIMO demapper is combined with the bit nodes of the LDPC decoder, which is treated as a single entity for the EXIT chart analysis. The check node in the decoder is the other entity. In another approach, e.g., in [18], the MIMO demapper is treated as one entity for the EXIT chart analysis, and the entire LDPC decoder as the other. We provide our EXIT results based on the latter method. That is, the multiuser demapper is regarded as one entity and the LDPC decoder as the other. The latter method gives us a clear view of the separate effects of the demapper and the decoder. For the decoder transfer curve, a randomly selected (1024,3,6) LDPC code is used, and the number of the LDPC internal iterations is fixed at 10. In particular, the mutual information at the output of the LDPC decoder is calculated from the extrinsic information obtained at the end of ten internal iterations; the mutual information at the input of the LDPC decoder is obtained from the prior information, which is, as usual, assumed to be Gaussian distributed. The relationship of mutual information between the input and the output of the decoder is used to generate the transfer characteristic curve for the LDPC decoder.

Figs. 2 and 3 show the comparison of performance curves with the upper bounds, the constrained capacities, and the thresholds from the EXIT chart analysis for single-user and multiuser MIMO systems, respectively. The constrained capacity is calculated from the Monte Carlo evaluation of the mutual information between the vector input and the vector output of the channel. Each entry of the vector input is assumed to be equally likely selected from a constellation such as BPSK and 4-QAM, and the channel is assumed to be ergodic [7].

There are four performance curves in total in the two figures. The first two curves in Fig. 2 are for the single-user MIMO system with BPSK and 4-QAM modulations. The next two curves in Fig. 3 are for the multiuser MIMO system with BPSK and 4-QAM modulations. For convenience, we may refer to them as the first, the second, the third, and the fourth scenarios, respectively. Given the TNI at 60, the NSIs and NIIs are divided into four cases such as (3, 20), (6, 10), (6, 10), and (12, 5) for the four scenarios, respectively. Since both senders are equipped with two transmit antennas, the number of transmitted coded bits are 2, 4, 4, and 8 in one channel use. The second and the third scenarios are similar in that they are both sending four coded bits per channel-use, and 4-QAM can be treated as two orthogonal BPSKs. Also, note that in Fig. 3, the $x$-axis is SNR per user. If the two users are treated as one super-user, there will be a 3-dB shift in SNR to the right. With this adjustment, we note that the second and the third scenarios, indeed, show a very similar performance, as expected. For the fourth scenario, we note that the simulation result goes above the union upper bound in high SNR region, which indicates that more iterations are needed, especially the internal iterations. We can see that the iterative processing in MIMO multipleaccess systems still has much potential to be improved.

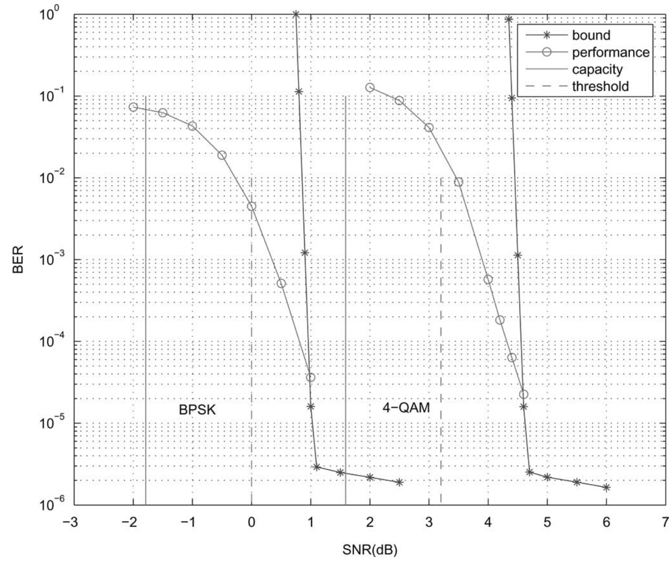

Fig. 2. Comparison of performance, bounds, capacities, and thresholds in single-user MIMO systems.

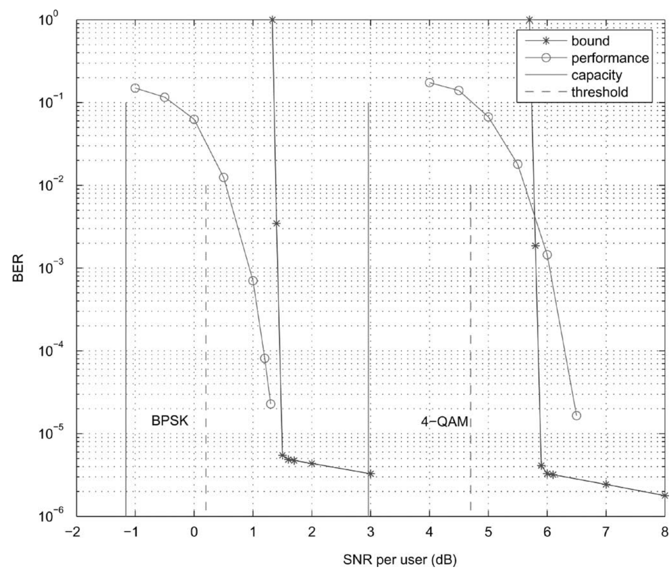

Fig. 3. Comparison of performance, bounds, capacities, and thresholds in multiuser MIMO systems.

There are two observations from system simulations: (1) more number of *super* iterations is helpful when the number of bits/channel-use is increased and (2) more *internal* iterations are always helpful and bring the waterfall region closer to the capacity limit.

We note that different perspective on the performance can be obtained from the two evaluation methods. The union upper bound is able not only to predict the waterfall region for the iterative detection and decoding receiver, but also to provide

TABLE II
COMPARISON OF BOUNDS, PERFORMANCE CURVES, AND COMPLEXITY OF ENUMERATION FOR BOUNDS

| Block length (bits) | Number of Tx and Rx antennas | Error floor of bound (log) | Bound gap from capacity (dB) | Performance gap from capacity (dB) | Number of mults for $\Phi_{d_1,d_2}$ $(O(N^{2U}))$ |
|---|---|---|---|---|---|
| 256 | $2 \times 2$ | $-4.45$ | 2.8 | 3.6 | $2.8 \times 10^8$ |
| 512 | $2 \times 2$ | $-5.13$ | 2.7 | 2.8 | $4.4 \times 10^9$ |
| 1024 | $2 \times 2$ | $-5.48$ | 2.6 | 2.4 | $6.9 \times 10^{10}$ |
| 1024 | $4 \times 4$ | $-5.91$ | 2.3 | 2.0 | $6.9 \times 10^{10}$ |

information on the error floor behavior of the ensemble of the codes. On the other hand, the threshold values from the EXIT chart analysis seem to indicate the starting point of the waterfall region for short block to moderate block codes, and the ultimate convergent point of any practical turbo receiver can achieve at any large block length. Table II shows the results of upper bounds and simulated performances for different block lengths up to a thousand as well as for different number of antennas. From the table, we can see that both the bounds and the performance curves move toward the capacity limits as the block length increases. In fact, we note that the performance curves move toward the capacity limits faster than the union bounds do.

In addition, we investigate the number of multiplication required to evaluate the union upper bounds. The results for BPSK modulation for two users are tabulated in Table II. The required multiplications are for the convolution operation that is on the order of $O(N^{2U})$, where $U$ is the number of users.

It is worthwhile to note that if the block length is further increased to a level beyond a thousand, both the threshold value from the EXIT chart and the waterfall SNR from system simulation would tend to move toward the capacity limit; while the union bound will converge only to a cutoff SNR. This calls for tight union bound techniques for MIMO channels that continue to work beyond the cutoff SNR point. There are significant recent developments in this direction for SISO channels [10]. Finding tight union bounds for single-user and multiuser MIMO systems is an open research area.

### A. *Error Floors, Why?*

The error floors are eminent in the union bounds for the LDPC codes; but when you, in fact, select a code and simulate for the code performance, there is no error floors. Why? This is due to the fact that the distance distribution, and thus, the union bound as well, is obtained as an average for an ensemble of LDPC codes. In an ensemble, there are many codes. Some codes in this ensemble are very bad such that they contain codewords with small Hamming weights, say weight 2. The proportion of such codes in an ensemble is small in general, and thus, the probability of selecting such a bad code in practice is small, but not to the level that it can be completely ignored.

For example, $S_2$ is about 0.0089 at the block length of 1024 for the regular $(3, 6)$ LDPC code. First, the spectral component is fractional. What does it mean to have a fractional distance spectral component? One way to interpret it is that if ten thousand (1024, 3, 6) LDPC codes are selected randomly, there are about 89 bad codes that contain one weight-2 codeword. This small but bad codes in the ensemble is the culprit of the eminent error floor in the ensemble averaged union bound as more explanation will be given shortly later. Second, this number decreases proportionally to the block length of the code. Thus, this number vanishes at the infinite block length. The *minimum distance* of an ensemble of LDPC codes is defined as the minimum weight that do not vanish at the infinite length—the minimum weight that gives nonfractional component.

The distance spectral component stays factional for other weights such as 4, 6, 8, 10, ..., until the weight finally reaches up to the *minimum distance* of the code ensemble. All distance spectral components $S_d$, for even $d < d_{\min}$, are fractional numbers, rather than whole numbers. (All odd weights are not valid in the regular (1024, 3, 6) LDPC codes.) The *minimum distance* for (1024,3,6) LDPC code ensemble is 28. The distance spectral component $S_{28}$, the first whole number, is the average number of codewords of a code in the ensemble. This implies that with probability close to 1, any randomly selected LDPC code contains roughly $S_{28}$ number of weight 28 codewords. The probability of a random selection of a code from the ensemble that has at least one weight 2 codeword (or any codeword with weight smaller than the minimum distance) can be made arbitrarily small by increasing the block length. More formal discussion in this direction can be found in [19, Th. 2.4].

Returning to our answer to the question, the eminent error floor in the union bound, therefore, is due to the pairwise error probability associated with the weight 2 codewords. Recall that the union bound is the addition of all pairwise error probabilities, each multiplied with the codeword multiplicity (a component in the distance distribution). For example, in the case of AWGN channel with BPSK modulation, $S_2 P_2(2)$ is the most dominant term in the union bound for high SNR region, where $P_2(2)$ denotes the pairwise error probability from the all-zero transmitted codeword to a neighboring codeword with Hamming weight 2. For MIMO MAC channels, the same trends are observed.

### B. *The Gaps to the Constraint Channel Capacity*

The constrained capacity, as mentioned earlier, is calculated from the Monte Carlo evaluation of the mutual information between the vector input and the vector output of the MIMO channel. Each entry of the vector input is assumed to be equally likely selected from a constellation such as BPSK and 4-QAM, and the channel is assumed to be *ergodic*. For a fair comparison, the simulation results of the BLAST-type transmission are compared with the constrained capacities calculated from the

BLASTtype transmission scheme. Looking at the results, there are significant gaps of about 3 dB between the constrained capacity and the simulation results of the turbo-iterative receiver as indicated in the figures. This gap could have resulted from such factors as the use of less-than-infinite block length and the imperfection of the code itself. From looking at the results given in Figs. 2 and 3, it is reasonable to conjecture that the turbo-receiver would converge to the threshold values at the infinite complexity (infinite number of iterations), and the threshold would move further toward the capacity as the length of the code increases. The gap from the threshold to the capacity may indicate the imperfectness of the code, especially at the block length of 1024. Gallager's thesis [19, Th. 3.3] states that the regular LDPC code cannot achieve the capacity. Although this result is obtained for binary symmetric channels, it certainly indicates the imperfectness of the code. Further elucidation on precise causes of these gaps is interesting, but have been left as future work.

## V. Conclusion

The channel capacity can be significantly increased by using multiple transmit and multiple receive antennas. The LDPC-coded modulation scheme with iterative demapping and decoding can be utilized to exploit the capacity potential available in the MIMO multiple access system. We proposed novel union upper bound techniques for the MIMO multipleaccess system. We provided some system simulation results that can be contrasted with the upper bounds. It was shown that the union bounds can be used in combination with the EXIT chart analysis as a performance evaluation tool. It is worthy to note that while the EXIT chart analysis provides the threshold value that are only a single SNR point on the BER graph, the union bounds provide information on waterfall region as well as the error floor behavior.

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## References

[1] I. E. Telatar, "Capacity of multi-antenna Gaussian channels," Bell Labs. Paris, France Tech. Memorandum, 1995.
[2] G. J. Foschini and M. J. Gans, *On Limits of Wireless Communications in a Fading Environment When Using Multiple Antennas*. Norwell, MA: Kluwer, 1998.
[3] C. Berrou, A. Glavieux, and P. Thitimajshima, "Near Shannon limit error-correcting coding and decoding: Turbo-codes," in *Proc. IEEE Int. Conf. Commun. (ICC 1993)*, vol, 2, May, pp. 1064–1070.
[4] D. J. C. MacKay and R. M. Neal, "Near Shannon limit performance of low density parity check codes," *Electron. Lett.*, vol. 33, no. 6, pp. 457–458, Mar. 1997.
[5] D. J. C. Mackay, "Good error-correcting codes based on very sparse matrices," *IEEE Trans. Inf. Theory*, vol. 45, no. 2, pp. 399–431, Mar. 1999.
[6] K. R. Narayanan and G. L. Stüber, "A serial concatenation approach to iterative demodulation and decoding," *IEEE Trans. Commun.*, vol. 47, no. 7, pp. 956–961, Jul. 1999.
[7] B. M. Hochwald and S. ten Brink, "Achieving near-capacity on a multiple-antenna channel," *IEEE Trans. Commun.*, vol. 51, no. 3, pp. 389–399, Mar. 2003.
[8] X. Wang and H. V. Poor, "Iterative (turbo) soft interference cancellation and decoding for coded CDMA," *IEEE Trans. Commun.*, vol. 47, no. 7, pp. 1046–1061, Jul. 1999.
[9] J. Wu and H.-N. Lee, "Best mapping for LDPC coded modulation on SISO, MIMO and MAC channels," *IEEE Wireless Commun. Networking Conf. (WCNC 2004)*, vol. 4, Mar. pp. 2428–2431.
[10] I. Sason, S. Shamai, and D. Divsalar, "Tight exponential upper bounds on the ML decoding error probability of block codes over fully interleaved fading channels," *IEEE Trans. Commun.*, vol. 51, no. 8, pp. 1296–1305, Aug. 2003.
[11] D. Divsalar, "Simple tight bound on error probability of block codes with application to turbo code," presented at the IEEE Commun. Theory Workshop. Aptos, CA, 1999.
[12] J. Zhang and H.-N. Lee, "Closed form union bounds on coded modulation over fast fading MIMO channels," *IEEE Commun. Lett.*, vol. 9, no. 9, pp. 796–798, Sep. 2005.
[13] R. M. Tanner, "A recursive approach to low complexity codes," *IEEE Trans. Inf. Theory*, vol. IT-27, no. 5, pp. 533–547, Sep. 1981.
[14] S. Litsyn and V. Shevelev, "On ensembles of low-density parity-check codes: Asymptotic distance distributions," *IEEE Trans. Inf. Theory*, vol. 48, no. 4, pp. 887–908, Apr. 2002.
[15] V. Tarokh, N. Seshadri, and A. R. Calderbank, "Space–time codes for high data rate wireless communication: Performance criterion and code construction," *IEEE Trans. Inf. Theory*, vol. 44, no. 2, pp. 744–765, Mar. 1998.
[16] S. ten Brink, "Designing iterative decoding schemes with extrinsic information transfer chart," *AEU Int. J. Electron. Commun.*, vol. 54, no. 6, pp. 389–398, Nov. 2000.
[17] S. ten Brink, G. Kramer, and A. Ashikhmin, "Design of low-density parity-check codes for modulation and detection," *IEEE Trans. Commun.*, vol. 52, no. 4, pp. 670–678, Apr. 2004.
[18] J. Hou, P. H. Siegel, and L. B. Milstein, "Design of multi-input multi-output systems based on low-density parity-check codes," *IEEE Trans. Commun.*, vol. 53, no. 4, pp. 601–611, Apr. 2005.
[19] R. G. Gallager, "Low-density parity-check codes," Ph.D. dissertation, Massachusetts Inst. Technol., Cambridge, 1963.

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