Union Bounds on Coded Modulation Systems over Fast Fading MIMO Channels

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***Abstract*— Novel closed-form upper bounds on the error performance of coded modulation systems over fast-fading multi-input multi-output (MIMO) channels are obtained for two different space-time schemes: *direct transmission* and *orthogonal space-time block codes*. The concept of the *distance spectrum* for associated space-time codes is developed and used to derive the bounds which can be calculated readily through polynomial expansion. Comparison of the bounds for the two systems indicates the performance of *direct transmission* is superior. Comparison with simulation results shows that the bounds are tight and useful for benchmarking the practical iterative decoding process.**

***Index Terms*— Upper bounds, distance spectrum, LDPC code, space-time code, MIMO systems.**

I. Introduction

LDPC and turbo codes are known to provide near-capacity error performance at moderate to large block sizes. Due to the huge population size of these codes and the lack of information on the code structure, other than the *distance spectrum*, their error-performance measures rarely admit a closed-form expression. A variety of different bounding techniques have been proposed to predict error performance for a number of different classes of channels, such as AWGN channels [1], single-input single-output (SISO) fading channels [2], and quasi-static fading MIMO channels [3], [4]. However, performance bounding for LDPC and turbo-coded systems over fast fading MIMO channels remains an open problem.

While LDPC and turbo codes can provide a significant coding gain, the space-time block codes with orthogonal designs (OSTBC) [5], [6] have the unique advantage of providing the maximum diversity gain for the MIMO channels at a fixed rate. Therefore, it is of interest to investigate the error performance of concatenated transmission schemes in which the LDPC or turbo code is used as an *outer code* and concatenated with an *inner code*: either an OSTBC or a *direct transmission* (i.e., the serial to parallel conversion of modulated symbols without having an explicit space-time coding done). In this paper, we analyze the error performance of these coded modulation systems over fast fading MIMO channels. The distance spectrum of the associated space-time codes are calculated for each scheme. The derived maximum likelihood (ML) upper bounds are in closed forms, requiring only the distance spectrum of outer codes as the input.

The rest of the paper is organized as follows. Section II describes the transmission schemes. Section III derives the upper bounds based on the distance spectrum. In section IV, simulation results are provided and the effectiveness of the bounds is verified. Finally, we make a conclusion in section V.

Manuscript received March 8, 2005. The associate editor coordinating the review of this letter and approving it for publication was Dr. Giorgio Taricco.

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Digital Object Identifier 10.1109/LCOMM.2005.09016.

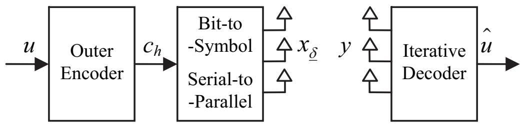

Fig. 1. System model for transmission scheme I.

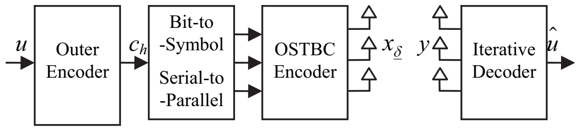

Fig. 2. System model for transmission scheme II.

II. System of Interest

Consider the MIMO system with M-transmit and N-receive antennas. We are interested in the two transmission schemes that are illustrated in Fig. 1 and Fig. 2. In Scheme I, the information message u is encoded into a codeword of length L, say c_h = ($c_{h,1}$, $c_{h,2}$, ..., $c_{h,L}$). We assume L to be a multiple of M, $L/M = T$, for convenience. The modulated codeword is formed by the bit-to-symbol and serial-to-parallel operations. Hence, one $[M \times T]$ space-time block word is obtained as

$$\begin{pmatrix} c'_{h,1} & c'_{h,M+1} & \cdots & c'_{h,L-M+1} \\ c'_{h,2} & c'_{h,M+2} & \cdots & c'_{h,L-M+2} \\ \cdots & \cdots & \ddots & \cdots \\ c'_{h,M} & c'_{h,2M} & \cdots & c'_{h,L} \end{pmatrix}, \qquad (1)$$

where the binary modulation for each entry is obtained by $c'_{h,i} = 1-2c_{h,i}$ (i.e., bit 0 → "+1" and bit 1→ "-1"). When the $[M \times T]$ block of channel-symbols (1) is directly transmitted over the antenna array in T channel uses without any further explicit channel-encoding, we will call this $[M \times T]$ block the associated space-time (AST)-I codeword. Clearly, we note that the outer code and the AST-I code have a one-to-one correspondence, and so do their respective ensembles.

In addition to the operations above, Scheme II further encodes the AST-I codeword by the OSTBC [6]. Taking M=2 as an example, the coding scheme is the Alamouti scheme,

$$\begin{pmatrix} c_a \\ c_b \end{pmatrix} \Rightarrow \begin{pmatrix} c_a & -c_b^* \\ c_b & c_a^* \end{pmatrix}. \qquad (2)$$

Then, the associated space-time block is given by

$$\begin{pmatrix} c'_{h,1} & -c'^*_{h,2} & c'_{h,3} & -c'^*_{h,4} & \cdots & c'_{h,L-1} & -c'^*_{h,L} \\ c'_{h,2} & c'^*_{h,1} & c'_{h,4} & c'^*_{h,3} & \cdots & c'_{h,L} & c'^*_{h,L-1} \end{pmatrix}, \qquad (3)$$

which is transmitted over $T' = L$ channel uses. Let us call this the AST-II codeword. Note that from the repetitive characteristics of OSTBC each leading column of sub-blocks in the AST-II codeword is identical to each column of the AST-I codeword, and the rest $(M-1)$ columns in each sub-block are repetitions of the leading column with operations such as permutation, negation, and conjugation.

Assume that an arbitrary AST codeword x of the form (1) or (3) is transmitted over the antennas. The receive signal y_t^n at the n^{th} antenna at the time instant t can be expressed as

$$y_t^n = \sqrt{\rho_s} \sum_{m=1}^{M} \alpha_{n,m}(t) x_{m,t} + z_t^n, \quad t = 1, 2, \ldots, T, \quad (4)$$

where $\rho_s = \rho_b R/M$ and $\rho_b = E_b/N_o$; R is the transmission rate in information bits per channel use; E_b is the information bit energy; $N_o/2$ is the two-sided power spectral density of the white Gaussian noise present at the receiver; $x_{m,t}$ denotes the m^{th} row and t^{th} column element of x; z_t^n denotes the independent complex additive white Gaussian noises with zero mean and variance 0.5 per dimension; and $\alpha_{n,m}(t)$ denotes the independent Rayleigh fading gain from the m^{th} transmit to the n^{th} receive antenna during the t^{th} channel use.

III. Error Performance Analysis

In this section, we consider a class of outer codes and develop a set of statistical properties which renders the error performance analysis.

A. Outer Codes of Interest

First, we consider an ensemble of LDPC codes whose parity-check matrices are defined by a set of three fixed parameters: block length, variable- and check-node degree distributions [7] (i.e., the column- and row-weights for regular code [8]). It is clear under column permutation the set is closed: any column permutation of a particular parity-check matrix drawn randomly from the ensemble produces another matrix in the same ensemble. Accordingly, the permutation of a codeword in one codebook exists as a codeword in another codebook. Define $\mathcal{C}_h$ as the set of all the codewords with the same Hamming weight h. Note that each codeword in $\mathcal{C}_h$ can be regarded as the permutation of another. If each codebook (or the parity-check matrix) is selected equi-probably from the ensemble, the following statistical properties shall hold for each codeword in $\mathcal{C}_h$:

Property I: The probability of one codeword in $\mathcal{C}_h$ belonging to the selected codebook is identical for every codeword in $\mathcal{C}_h$.

Property II: The probability of the i^{th} bit, b_i, in any codeword in $\mathcal{C}_h$ taking either 0 or 1 is identical for $i = 1, 2, \ldots, L$ and is given by $P(b_i = 1) = \frac{h}{L}$ and $P(b_i = 0) = \frac{L-h}{L}$.

Property II actually holds for a class of outer codes, including turbo codes and fully random block codes. For simplicity, we consider the LDPC code as an outer code in this paper, although the same analysis can be applied to any code satisfying Property II.

B. Pairwise Error Probability

For a fast Rayleigh fading channel, the maximum likelihood decoding pairwise error probability for any two space-time codewords x and x' is given by Tarokh in [9],

$$P(x \to x') \leq \prod_{t=1}^{T} \left(1 + |x_t - x_t'|^2 \frac{\rho_s}{4}\right)^{-N}, \quad (5)$$

where x_t and x_t' are the t^{th} columns of x and x', respectively.

Although MIMO channels are generally not *symmetrical*, it is interesting to note that the right hand side of (5) is *symmetrical* for any BPSK-modulated AST codeword. Thus, we can use the assumption that the all-zero LDPC codeword is transmitted for the derivation of upper bounds.

C. Distance Spectrum of AST

The product terms in (5) have at most $M+1$ different values since $|x_t - x_t'|^2 = 4m$ with $m = 0, 1, ..., M$. Denote $x_{\underline{0}}$ as the AST codeword mapped from the all-zero LDPC codeword. The entries in $x_{\underline{0}}$ are thus all "+1" if AST-I is concerned. For any AST codeword, let δ_m denote the number of its columns each of which has exactly *m* differences with the corresponding column of $x_{\underline{0}}$. Let us name the collection of these column weights the *column weight distribution* (CWD), $\underline{\delta} := (\delta_0, \delta_1, ..., \delta_M)$. It should be noted that the column weight distribution determines the pairwise error probability. Denote $\chi_{\underline{\delta}}$ as the set of all AST codewords with a particular column weight distribution $\underline{\delta}$. Then, we can express the pairwise error probability between $x_{\underline{0}}$ and any codeword $x_{\underline{\delta}}$ in $\chi_{\underline{\delta}}$ as

$$P(x_{\underline{0}} \to x_{\underline{\delta}}) \leq \prod_{m=0}^{M} (1 + m\rho_s)^{-\delta_m N}, \quad (6)$$

by arranging groups of like product-terms in (5) into a power term with respect to m. That is, there are δ_m product terms with the same value $\left|x_{\underline{0},t} - x_{\underline{\delta},t}\right|^2 = 4m$.

Consider the distance spectrum of AST-I first. Denote χ_h as the image of $\mathcal{C}_h$ in the AST-I code. It can be decomposed into disjoint subsets $\chi_{\underline{\delta}}$, each of which is composed of all the AST-I codewords with the same CWD $\underline{\delta}$. Accordingly, the pre-image $\mathcal{C}_{\underline{\delta}}$ of $\chi_{\underline{\delta}}$ forms a decomposition of $\mathcal{C}_h$. Namely, we have $\mathcal{C}_{\underline{\delta}} \bigcap \mathcal{C}_{\underline{\delta}'} = \emptyset$, for $\underline{\delta} \neq \underline{\delta}'$ and $\mathcal{C}_h = \bigcup_{\underline{\delta} \in \Omega_h} \mathcal{C}_{\underline{\delta}}$ where

$$\Omega_h := \left\{ \underline{\delta} \middle| \delta_m \in \{0, 1, \ldots, T\}, \sum_{m=0}^{M} \delta_m = T, \sum_{m=0}^{M} m\delta_m = h \right\}. (7)$$

The cardinalities of $\chi_{\underline{\delta}}$ and $\mathcal{C}_{\underline{\delta}}$ are the same; i.e., $\left|\chi_{\underline{\delta}}\right| = \left|\mathcal{C}_{\underline{\delta}}\right|$. Therefore, the distance spectrum of AST-I can be obtained by focusing on $\mathcal{C}_{\underline{\delta}}$ and computing the probability $p_h(\underline{\delta})$ with which a randomly selected LDPC codeword in $\mathcal{C}_h$ is contained in its subset $\mathcal{C}_{\underline{\delta}}$. Let us define $A_\delta := \left|\mathcal{C}_{\underline{\delta}}\right| = \left|\chi_{\underline{\delta}}\right|$ and $A_h := |\mathcal{C}_h|$. Applying Property II to the set $\mathcal{C}_h$ and resorting to a repeated application of combinatorial techniques, we obtain

$$A_{\underline{\delta}} = |\mathcal{C}_h| \, p_h(\underline{\delta}) = A_h \binom{L}{h}^{-1} \binom{T}{\delta_0, \delta_1, \cdots, \delta_M} \prod_{m=0}^{M} \binom{M}{m}^{\delta_m}, (8)$$

where $\binom{\sum x_i}{x_1, x_2, \cdots, x_n}$ denotes the multinomial coefficients.

D. Error Performance of the Transmission Scheme I

The union bound to word error probability can be obtained by summing up the pairwise error probabilities. Based on (6) and (8), we have

$$P_w \leq \sum_{h=1}^{L} \sum_{\underline{\delta} \in \Omega_h} A_{\underline{\delta}} P\left(x_{\underline{0}} \to x_{\underline{\delta}}\right) = \sum_{h=1}^{L} \binom{L}{h}^{-1} A_h \phi_h, \quad (9)$$

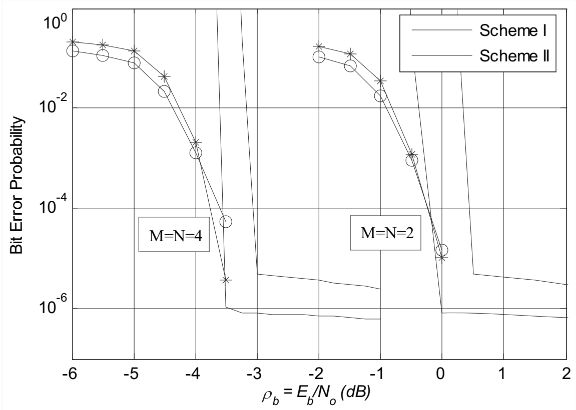

Fig. 3. ML upper bounds vs. simulation results. LDPC (3000, 3, 4) and LDPC (1500, 3, 6) are used for scheme I and scheme II, respectively.

where $\phi_h := \sum_{\underline{\delta}\in\Omega_h} \binom{T}{\delta_0,\delta_1,\cdots,\delta_M} \prod_{m=0}^{M} \beta_m^{\delta_m}$ and $\beta_m := \binom{M}{m}(1+m\rho_s)^{-N}$.

At first glance, ϕ_h seems complex and not easily computable since the cardinality of Ω_h is large. Resorting to the polynomial expansion $(\sum_{m=0}^{M} \beta_m x^m)^T$, however, it can be readily obtained. Recalling the definition of Ω_h in (7), we note

$$\left(\sum_{m=0}^{M} \beta_m x^m\right)^T = \sum_{h=0}^{L} \phi_h x^h. \tag{10}$$

The bit error probability of the coded bits is obtained by

$$P_b \le \sum_{h=1}^{L} \frac{h}{L}\binom{L}{h}^{-1} A_h \phi_h. \tag{11}$$

In practice, it serves as a good approximation, if not exact, of the bit error probability of the information bits.

E. Error Performance of the Transmission Scheme II

As mentioned in the last section, the columns of an AST-II codeword can be regarded as repetitions of the columns of the corresponding AST-I codeword; thus operations such as permutation, negation, and conjugation can be ignored for the purpose of computing the column weight distribution. Define $\chi'_{\underline{\delta}}$ as the collection of all the AST-II codewords $x_{\underline{\delta}}$ with the same column weight distribution $\underline{\delta}$. The distance spectrum of AST-II is then given by

$$A'_{\underline{\delta}} := \left|\chi'_{\underline{\delta}}\right| = A_{\frac{\underline{\delta}}{M}} = A_h \binom{L}{h}^{-1} \binom{\frac{T'}{M}}{\frac{\delta_0}{M}, \frac{\delta_1}{M}, \ldots, \frac{\delta_M}{M}} \prod_{m=0}^{M} \binom{M}{m}^{\frac{\delta_m}{M}}, \tag{12}$$

for $\underline{\delta} \in \Omega'_h$ $(h = 1, 2, \ldots, L)$ which is defined analogously to (7) as

$$\Omega'_h =: \left\{ \underline{\delta} \,\middle|\, \frac{\delta_m}{M} \in \left\{0, 1, \ldots, \frac{T'}{M}\right\}, \sum_{m=0}^{M} \frac{\delta_m}{M} = \frac{T'}{M}, \sum_{m=0}^{M} m\frac{\delta_m}{M} = h \right\}. \tag{13}$$

Similar to (11), the bit error probability of Scheme II is

$$P'_b \le \sum_{h=1}^{L} \frac{h}{L}\binom{L}{h}^{-1} A_h \phi'_h, \tag{14}$$

where $\phi'_h := \sum_{\underline{\delta}\in\Omega'_h} \binom{T'/M}{\delta_0/M, \delta_1/M, \cdots, \delta_M/M} \prod_{m=0}^{M} {\beta'_m}^{\frac{\delta_m}{M}}$ and $\beta'_m := \binom{M}{m}(1+m\rho_s)^{-MN}$. By the same reasoning, ϕ'_h can be obtained by

$$\left(\sum_{m=0}^{M} \beta'_m x^m\right)^{T'/M} = \sum_{h=0}^{L} \phi'_h x^h. \tag{15}$$

IV. RESULTS

The comparison of derived upper bounds and simulation results is illustrated in Fig. 3. We use the sum-product iterative decoding algorithm with fifty iterations. For meaningful comparison, the Gallager's (3000, 3, 4) and (1500, 3, 6) codes [8] are used for Scheme I and II respectively. Alamouti code and the one in [6, Eq. (3.46)] are adopted as OSTBC in Scheme II for $M = 2$ and $M = 4$, respectively. The transmission rate of Scheme II is therefore the same as that of Scheme I when $M = 2$ or a half of that when $M = 4$. The bounds for Scheme I indicate good matches with the simulation results. They are a little loose for Scheme II, but with differences always less than 0.5 dB. As expected, OSTBC does not improve performance much in the fast fading case where the diversity order is already sufficient. In fact, it performs no better than the direct transmission, even when $M = 4$ in which case its transmission rate is only a half of the direct transmission. The higher error floor for Scheme II seems due to the larger population of low-weight codewords in the (1500, 3, 6) code compared with the (3000, 3, 4) code.

V. CONCLUSION

In this paper, we proposed upper bounds on the error performance of coded modulation systems over fast fading MIMO channels. These upper bounds are used to compare two space-time transmission schemes: direct transmission and orthogonal space-time block codes. Surprisingly, the direct transmission scheme has been shown to be better in both simulations and bounds than space-time block coding for fast fading MIMO channels. This implies that the coding rate should be better spent at the outer code instead of at the inner space-time block code for robust performance.

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