Performance Analysis on LDPC-Coded Systems over Quasi-Static (MIMO) Fading Channels

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***Abstract*—In this paper, we derive closed form upper bounds on the error probability of low-density parity-check (LDPC) coded modulation schemes operating on quasi-static fading channels. The bounds are obtained from the so-called Fano-Gallager's tight bounding techniques, and can be readily calculated when the distance spectrum of the code is available. In deriving the bounds for multiple-input multiple-output (MIMO) systems, we assume the LDPC code is concatenated with the orthogonal space-time block code as an inner code. We obtain an equivalent single-input single-output (SISO) channel model for this concatenated coded-modulation system. The upper bounds derived here indicate good matches with simulation results of a complete transceiver system over Rayleigh and Rician MIMO fading channels in which the iterative detection and decoding algorithm is employed at the receiver.**

***Index Terms*—Tight union bounds, LDPC codes, space-time block code, MIMO systems, quasi-static fading channels.**

I. INTRODUCTION

DURING the past several years, much research effort has been spent on the prediction of the error performance for turbo- and LDPC-coded systems with the maximum likelihood (ML) decoding assumption. This interest has been motivated by the splendid error correction performance of turbo-like codes which comes very close to the theoretical limit for a large block size. In a region close to the capacity limit, it has been known that the usual union bound is loose. Thus, the demand has been very high for finding tight performance bounds that would continuously be useful in this region. Fulfilling this need, there has been a series of substantial progress made recently [1], [2], [3], [4], [5], [6] in the context of single-input single-output channels. They are variations on the so-called Fano-Gallager bounding method, also known as the *limit-before-averaging* bounding technique, which was originally introduced by Fano [7] and then further developed by Gallager for analyses on LDPC codes operating over the additive white Gaussian noise (AWGN) channel [8]. A recent semi-tutorial paper by Shamai and Sason [6] summarizes this family of tight bounding techniques such as the Duman-Salehi bound [9], the Divsalar bound [3], and the Shulman-Feder bound [10], and provides a taxonomy of bounding methods showing how they are related to one another.

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In this paper, we are interested in extending the Fano-Gallager framework to space-time transmission of LDPC-coded multi-level modulation over multiple-input multiple-output channels. The Fano-Gallager bounding technique starts with the following simple decomposition:

$$\begin{aligned}\Pr(\text{error}) &= \Pr(f \in \Re) \Pr(\text{error}|f \in \Re) \\ &\quad + \Pr(f \in \bar{\Re}) \Pr(\text{error}|f \in \bar{\Re}) \\ &\leq \Pr(f \in \Re) \Pr(\text{error}|f \in \Re) + \Pr(f \in \bar{\Re}), \quad (1)\end{aligned}$$

where f is a utility function, called the Fano-Gallager tilting measure, of performance-related random variables such as the additive white Gaussian noise and the multiplicative channel fading gain. $\Re$ is a utility region defined in the received signal space and $\bar{\Re}$ is its complement. The upper bound in the third line is trivially obtained for $\Pr(\text{error}|f \in \bar{\Re}) \leq 1$. A further upper bound can be obtained by applying the conventional union bound on the conditional error probability $\Pr(\text{error}|f \in \Re)$. These utility function and region are the vehicles utilized to obtain tight upper bounds.

Depending on how tight a bound we want, from very simple to very complicated utility function and region can be used, as we examine previous bounding results in the literature. Consider the received signal y over an AWGN channel,

$$y = \alpha x_0 + w, \quad (2)$$

where x_0 is the modulated signal for the all-zero binary codeword c_0, α is the unit channel gain, and w is the AWGN noise. Divsalar [3] defines the utility region $\Re$ to be a hyper-dimensional sphere and takes the approach of optimizing the radius and the location of the sphere. Since the region is a rather simple sphere, the bound is obtained in a closed form. However, we note that this bound is not tight and it is even greater than 1 in the low signal-to-noise ratio (SNR) region. A tight bounding technique can take a rather complicated form. For example, the analyses in [4] focus on the case of fast fading channels, i.e., assume an independent fading gain α in (2) for each channel-symbol. In this case, the utility function and the region depend both on the noise w and the fading gain α. A full blown application of the Fano-Gallager tilting measure technique was taken and it became very complex and cumbersome to evaluate the bound with at least three parameters to be optimized numerically.

There are a small number of previous works on bounds for quasi-static fading channels. In [11], Stefanov and Duman make use of the limit-before-averaging technique,

$\Pr(\text{error}|\alpha) \leq \min\{1, \text{union bound}\}$, for space-time trellis coded MIMO systems. Since the utility region cannot be easily identified for a general MIMO channel, an averaging operation over channel realizations is included in their final expression which again needs to be evaluated numerically. In [12], Vatta, Montosi, and Babich use a further upper bound based on the classical inequality of $\min\{1, \text{union bound}\} \leq \min_{0\leq\rho\leq 1}(\text{ union bound})^{\rho}$, for the analysis on the turbo-coded SISO system. Again, the final upper bound expression is rather complex and should be numerically evaluated and optimized over the parameter ρ. This is mainly because the exponent ρ prevents the exchange of the summation operation within the union bound and the average operation over channel fading.

In this paper, we aim to obtain a simple yet effective approach which does not leave any parameter to be optimized numerically and strike a balance between the tightness and the complexity. This will help us tackle the more complicated situation we have. Namely, our aim is to obtain tight bounds for LDPC-code modulated multi-level space-time transmission over MIMO channels. We select the utility function and region so as to distinguish between "high" and "low" instantaneous SNR events. With a certain threshold value α^*, we define $\{f \in \Re\} := \{\alpha \geq \alpha^*\}$. In the high SNR case, the conventional union bound is used; in the low SNR situation, the trivial bound is used, i.e.,

$$\begin{aligned}\Pr(\text{error}) \quad &\leq \quad \Pr(\alpha \geq \alpha^*) \sum_{c' \neq c_{\underline{0}}} P(c_{\underline{0}} \rightarrow c' \,|\alpha \geq \alpha^*\,) \\ &+ \quad \Pr(\alpha < \alpha^*), \end{aligned} \tag{3}$$

where the summand is the pairwise error probability from c_0 to any other codeword c' conditioned on $f \in \Re$. Namely, we take $f \in \bar{\Re}$ as an "outage" event in which the fading gain is smaller than the threshold. It is worth noting that this approach has been independently taken by Bouzekri and Miller in [13] and Stefanov and Duman in [14] to find tight bounds for the analysis of turbo-coded modulation signals over quasi-static fading channels. However, it should be noted that other than this similarity, our work is independent and more broadly defined and provides a different set of unique contributions.

In [14], the authors consider a general MIMO system. Due to the difficulty caused by this generality, the final expressions of the union bound can only be expressed in multiple integrals over MIMO fading channels. In contrast, we focus on STBC coded MIMO system and obtain a closed form upper bound without any numerical integral for fading (the final expression has a single integral, but it is for the Craig's identity of the Gaussian Q function). In addition, different from the codeword enumerating method in [14], [15], [16]], what's proposed in this paper is a new and simple combinatorial method to identify the cardinality of a set of codewords which lead to the same pairwise error probability. Further discussion on this difference can be found in Section III.

Other contributions of this paper can be summarized as follows. First, instead of the turbo-coded binary modulation over Rayleigh channels in [13], our work (Proposition 2) formulates the Fano-Gallager bounding technique in the more general context such that the bounds can be obtained for multilevel signal constellations and for any fading distributions. We obtain the results for both Rayleigh and Rician channels. Second, utilizing Shannon's classical idea of performance averaging over an ensemble of codes, we develop a statistical property of the ensemble of LDPC codes and identify the set of codewords that lead to an identical pairwise error probability, see section III. This contributes to a concise expression of the upper bound and a systematic approach to evaluating the threshold value α^*, especially for transmission schemes involving multi-level modulation. Our results in this paper are mainly for ensembles of LDPC codes but the general framework can be extendable to other linear block codes that satisfy Proposition 1, including the turbo codes. Third, we make use of the equivalent SISO channel model for orthogonal space-time block coded systems and have the upper bound extended to any orthogonal space-time block coded modulation system. Fourth, we employ the Craig's identity of the Gaussian Q-function instead of its Chernoff bound to pursue the tightness of the overall bound. Roughly, the use of the Craig's identity improves the bound by about 1 dB in SNR.

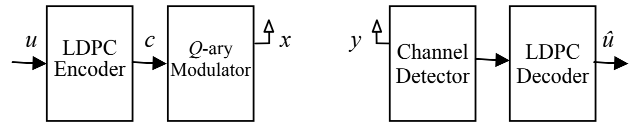

Fig. 1. Coded modulation system over quasi-static fading channel.

The rest of this paper is organized as follows. In Section II, we introduce the system of interest. The statistical property of an ensemble of LDPC codes is developed in Section III. Section IV describes the bounding technique with which upper bounds are derived with closed forms for the SISO system. Section V extends the approach to the MIMO system with inner orthogonal space-time block coding. Section VI discusses the evaluation method of the derived upper bound. Section VII presents simulation results to verify the tightness of the bound. Finally, we make a summary in Section VI.

II. System of Interest

Consider the single-input single-output transmission system illustrated in Fig. 1. A sequence u of K information bits is encoded into an LDPC codeword c of length L. The modulator adopts a constellation of size Q ($Q = 2^{K_b}$, K_b is a positive integer), and its signal points are denoted by $a_0, a_1, \ldots, a_{Q-1}$. The modulated signal vector x of c therefore has $T = L/K_b$ components, each of which is selected from the set $\mathcal{A} = \{a_0, a_1, \ldots, a_{Q-1}\}$. It is transmitted over the channel that can be modeled as follows:

$$y_t = \sqrt{E_s}\alpha x_t + w_t, \qquad (t = 1, 2, \ldots, T), \tag{4}$$

where E_s is the average symbol energy at the transmit antenna; x_t is the t-th component of the transmitted signal vector x and y_t is the corresponding received signal; w_t denotes the independent complex additive white Gaussian noise with zero mean and variance $N_0/2$ per dimension. The fading gain α, sampled according to a certain probability density function, is assumed to be known at the receiver. It is fixed during a block of T channel uses and varies independently from one block to another (quasi-static fading channel). In the case of a Rician channel, the probability density function of the magnitude of fading gain α is given by,

$$p(\alpha) = 2\alpha e^{-(\alpha^2+K_r)} I_0(2\alpha\sqrt{K_r}), \alpha > 0 \tag{5}$$

where $I_0(x) = \frac{1}{\pi}\int_0^{\pi} e^{x\cos\theta} \mathrm{d}\theta$ is the zero-th order modified Bessel function of the first kind, and K_r is the Rician factor which indicates the relative strength of the direct and scattered components of the received signal.

III. Statistical Property of an Ensemble of LDPC Codes

In this paper, we are concerned with the error performance of LDPC-coded modulation systems, averaged over an ensemble of LDPC codes that is specified by three fixed parameters: block length L, variable- and check-node degree distributions [17]. Note that this can be used to define an ensemble for either regular [8] or irregular LDPC codes [17]. Assume each code $\mathcal{C}_{sel}$ in the ensemble $\mathbb{C}$ is selected for use with an equal probability, i.e.,

$$\Pr\left(\mathcal{C}_{sel} \text{ is selected }\right) = |\mathbb{C}|^{-1}, \forall \mathcal{C}_{sel} \in \mathbb{C}, \tag{6}$$

where $|\cdot|$ denotes the cardinality of a set. Denote the average distance spectrum of an ensemble as $\{A_h\}$ in which A_h, the number of codewords of Hamming weight h in one code, is obtained as the average over the ensemble. Then, we have the following statistical property statements for the ensemble.

Proposition 1: If $A_h > 0$ for a certain h, each of the $\binom{L}{h}$ distinct binary sequences of length L and Hamming weight h is a valid codeword in a certain fixed number N_h of codes in the ensemble. Thus, the probability of each of these sequences appearing in the randomly selected code $\mathcal{C}_{sel}$ is equal.

Proof: The proof is given in Appendix A. ■

A sketch of ideas is given here. Instead of the ensemble of codes, we equivalently consider the ensemble of parity-check matrices for the set of codes. The key idea is then to show the closure of the ensemble of parity-check matrices under column permutations. We note that any column permutation of one parity-check matrix generates another one in the same ensemble. Namely, column permutation does not affect the three specified parameters in a particular ensemble. The ensemble of these matrices is thus closed under column permutations. Accordingly, any permutation of a codeword in one code generates a codeword in another code of the same ensemble. Notice that each of the binary sequences of weight h can be regarded as permutation of one another. The statements in Proposition 1 are then readily available.

Indeed, the closure will hold as well for the ensemble of randomly interleaved turbo codes (or any other linear block codes). Thus, the proposition and the analysis scheme in this paper could be applicable to turbo codes as well. In this paper, however, we will stay focused on the analysis of LDPC coded systems due to space limitation.

Proposition 1 will be useful for calculating the number of codewords which lead to the same pairwise error probability in the union bound analysis.

Consider any codeword c of length L. It can be alternatively regarded as a serial concatenation of T binary strings, where each binary string has length K_b, i.e. $K_b = L/T$, and is to be mapped onto a signal point in the Q-ary constellation. Denote the $Q = 2^{K_b}$ possible distinct binary strings as $b_0, b_1, \ldots, b_{Q-1}$, their Hamming weight as $w_0, w_1, \ldots, w_{Q-1}$, and the numbers of their appearance within a single codeword c as $\delta_0, \delta_1, \ldots, \delta_{Q-1}$, respectively. For example, δ_i is the number of appearances of string b_i in a single codeword. Since there are T strings in a codeword, its maximum value is T and minimum is 0; the sum of all these numbers should be equal to T. Namely, we have

$$\delta_i \in \{0, 1, \ldots, T\} \text{ and } \sum_{i=0}^{Q-1} \delta_i = T. \tag{7}$$

Now, the problem is to find the average number $A_{h,\underline{\delta}}$ of the codewords in a code that have the same vector $\underline{\delta} = (\delta_0, \delta_1, \ldots, \delta_{Q-1})$. Let's call this *appearance* vector. Then, we collect all appearance vectors that have the same Hamming weight $h = \Sigma_i \delta_i w_i$ in to a set denoted as Ω_h.

According to Proposition 1 and resorting to simple combinatorial methods, the probability that any codeword c_h of Hamming weight h has an *appearance* vector $\underline{\delta}$ is given by

$$\begin{aligned} &\Pr\left(c_h \text{ has a metric } \underline{\delta}\,|c_h \text{ is of weight } h\right) \\ &= \binom{L}{h}^{-1} \binom{T}{\delta_0, \delta_1, \ldots, \delta_{Q-1}} =: P_{\underline{\delta}|h}, \end{aligned} \tag{8}$$

where $\binom{\sum x_i}{x_0, x_1, \cdots x_{n-1}} = \frac{(\sum x_i)!}{\prod x_i!}$ denotes the multinomial coefficient. Hence, the average number $A_{h,\underline{\delta}}$ is obtained as

$$A_{h,\underline{\delta}} = A_h P_{\underline{\delta}|h} = A_h \binom{L}{h}^{-1} \binom{T}{\delta_0, \delta_1, \ldots, \delta_{Q-1}}. \tag{9}$$

As expected, we can verify that $\sum_{\underline{\delta}\in\Omega_h} P_{\underline{\delta}|h} = 1$ and $\sum_{\underline{\delta}\in\Omega_h} A_{h,\underline{\delta}} = A_h$, where Ω_h denotes the set of all possible $\underline{\delta}$'s leading to the same Hamming weight h,

$$\Omega_h := \left\{ \underline{\delta} \,\middle|\, \delta_i \in \{0, 1, ..., T\}, \sum_{i=0}^{Q-1} \delta_i = T, \sum_{i=0}^{Q-1} \delta_i w_i = h \right\}. \tag{10}$$

It should be noted that all codeword pairs that have the same *appearance* vector $\underline{\delta}$ produce an identical pairwise error probability, and there are $A_{h,\underline{\delta}}$ number of pairs on the average. Thus, we call $A_{h,\underline{\delta}}$ the distance spectral component for weight h and the *appearance* vector $\underline{\delta}$. We use the distance spectrum $\{A_{h,\underline{\delta}}\}$ to obtain the union bound.

Before proceeding with the union bound analysis, it is worth a brief comparison between our codeword enumerating method and the method developed by Duman and Salehi [15], [16]. Their method is also about calculating the set of codewords which result in the same Euclidean distance. The method is applied to the turbo codes while it is applicable to any linear codes whose weight enumerating function (WEF) is given. The turbo code is a concatenation of convolutional codes. Thus, the average distance spectrum of the turbo code can be calculated using the WEFs of constituent convolutional codes via the so-called uniform interleaver technique developed by Benedetto and Montorsi [18]. The number $\bar{f}(\mathbf{n})$ of error sequences of the error type $\mathbf{n}$ is calculated where $\mathbf{n}$ is the vector of numbers $n_{i,j}$, the number of errors with type $(i, j) - i$ message bit errors and j parity bit errors. Then, for each type, a probability mass function (PMF) (see $P[D_n^2 = \Delta_{n,j}^2]$ on page 514 in [15]) is calculated where each "mass point" corresponds to the same PEP. The product of $\bar{f}(\mathbf{n})$ and the PMF takes the similar role of the distance spectrum $A_{h,\underline{\delta}}$ of this paper. Both $\bar{f}(\mathbf{n})$ and PMF should be calculated specifically for different MIMO modulation schemes assuming that all channel symbol errors are independent with each other. One consequence is that they are able to consider any arbitrary codewords as the transmitted codeword. However, this method is quite complex and difficult to evaluate, and thus the union bound is evaluated in a truncation which is the sum of first several pairwise error terms. This might be acceptable for a high SNR case but not good in general, especially for obtaining a tight union bound targeted in this paper.

In our case, being independent of constellations a distance spectral component $A_{h,\underline{\delta}}$ can be obtained relatively easily based on the simple combinatorial method discussed in this section, as long as the size of constellation Q and the distance spectrum $\{A_h\}$ of the LDPC codes are given. Our method, however, relies on a further upper bound (will be discussed in Section IV).

IV. Upper Bounds for SISO Systems

In this section, we make use of Fano-Galager's bounding technique to obtain closed-form upper bounds on the error probability for the LDPC-coded SISO system over quasi-static fading channels. Since the probability of making an error is conditioned on the fading gain α, we may express the word error probability as

$$P_w = \int_0^\infty \Pr(\text{word error}|\alpha) p(\alpha) \mathrm{d}\alpha. \tag{11}$$

For a certain realization of the fading gain α, the conditional probability $\Pr(\text{word error}|\alpha)$ can be upper-bounded by the conventional union bound,

$$\Pr(\text{word error}|\alpha) = E_c\left[P_{w|c}\right] \leq E_c\left[\sum_{c' \neq c} \Pr(c \to c' \,|\alpha)\right], \tag{12}$$

where $E_c[\cdot]$ is the expectation over the equiprobable selection of codeword c for transmission; $P_{w|c}$ is the error probability conditioned on the transmission of c; and $\Pr(c \to c'|\alpha)$ denotes the pairwise error probability between c and any other codeword c' in the code. Suppose c and c' are modulated onto x and x', respectively. We have

$$P(c \to c' \,|\alpha) \leq Q\left(\frac{d\,(x, x')/2}{\sqrt{N_0/2}}\right), \tag{13}$$

where $Q(\cdot)$ is the Gaussian Q-function, $d(x, x')$ is the Euclidean distance between x and x',

$$d^2(x, x') = \alpha^2 E_s \sum_{t=1}^{T} |x_t - x_t'|^2, \tag{14}$$

and x_t and x_t' are the t^{th} components of x and x', respectively.

We note that, for a given constellation and a map, the Euclidean distance profile, $\{d(x, x')|\forall x' \neq x\}$, is generally not the same for different modulated signal vectors x, although the profile of the Hamming distance, $\{d_H(c, c')|\forall c' \neq c\}$, is the same for any LDPC codeword c since the code is linear. Thus, the average operation over c in (12) can not be removed by merely assuming a certain codeword, say the all-zero codeword, is transmitted. This imposes a difficulty on the exact evaluation of the right hand side of (12). We deal with this problem in the following manner. Suppose the error probability $P_{w|\tilde{c}}$ conditioned on a certain transmit codeword $\tilde{c}$ is worse than the average performance, i.e.,

$$E_c\left[P_{w|c}\right] \leq P_{w|\tilde{c}}. \tag{15}$$

Then, the union bound on $P_{w|\tilde{c}}$ can serve as a further upper bound on the average performance.

It is reasonable to postulate that one such codeword would be the one that is close to the mass-center of the hyper-constellation $\mathcal{A}^T$, where $\mathcal{A}$ is the Q-ary alphabet. This makes sense for most practical modulation schemes, such as the phase-shift keying (PSK) and the quadrature amplitude modulation (QAM) constellations. This conjecture is in fact true for all equal energy constellations and for many Q-ary QAM constellations. A theorem is developed in this paper which establishes the validity of the further upper bound approach. The proof is given in Appendix B. We offer a sufficiency condition to the theorem with which one can verify in a systematic manner whether the further upper bound is valid or not for a given constellation with a constellation map. Our results indicate that the further upper bound is indeed valid for many constellations such as Q-ary QAM for Q = 4, 8, 16, 64, 256, 1024, all Q-ary ASK, and all Q-ary PSK. Unfortunately, the test can not verify if the upper bound is valid for Q-ary QAM for Q = 32, 128, 512.

Thus, we proceed with two assumptions that (i) the transmitted signal x is mapped from the all-zero codeword c_0, i.e., $x = (a_0, a_0, \ldots, a_0)$, and (ii) the channel symbol, $a_0 \in \mathcal{A}$, is selected to be the one closest to the origin (or more precisely, it can be any channel symbol that satisfies the sufficiency test in Appendix B).

Now proceeding with the derivation of bound, suppose c' has *appearance* vector $\underline{\delta}$ = $(\delta_0, \delta_1, \ldots, \delta_{Q-1})$. A codeword has T binary strings. Each element δ_i indicates the number of binary string b_i in T strings. Namely, the appearance vector can tell how many times each channel symbol a_i appears in x'. Therefore we have

$$\begin{aligned} &\sum_{c' \neq c_0} P(c_0 \to c' \,|\alpha) \\ &\quad \leq \sum_{c' \neq c_0} Q\left(\alpha \sqrt{\frac{E_s}{2N_0} \sum_{t=1}^{T} |x_t - x_t'|^2}\right) \\ &\quad = \sum_{h=1}^{L} \sum_{\underline{\delta} \in \Omega_h} A_{h,\underline{\delta}} Q\left(\alpha \sqrt{\frac{E_s}{2N_0} \sum_{i=0}^{Q-1} \delta_i \,|a_0 - a_i|^2}\right) \\ &\quad =: \phi_w(\alpha), \end{aligned} \tag{16}$$

where in the second step we re-enumerate all codewords $c' \neq c_0$ according to their Hamming weight h and *appearance* vector $\underline{\delta}$, and the argument of the Gaussian Q function is accordingly rewritten based on the associated *appearance* vector $\underline{\delta}$. As we recall, $A_{h,\delta}$ denotes the number of codewords that have the same *appearance* vector $\underline{\delta}$ and it can be easily calculated according to (9) once the distance spectrum $\{A_h\}$ of the binary code is available.

On the other hand, note that $\phi_w(\alpha)$ $(\alpha \geq 0)$ is a strictly monotone function decreasing from $\phi_w(0) = \frac{1}{2}\Sigma_h A_h$ to $\phi_w(\infty) = 0$. In practice, we have $\phi_w(0) \geq 1 \geq \delta_w$ since it equals one half of the codebook size (any practical codebook has at least two codewords).

The conventional union bound is useful only when the instantaneous SNR is high. It becomes meaningless, resulting in a bound greater than 1, at the low instantaneous SNR region. In this case, a straightforward bound is

$$\Pr(\text{word error}|\alpha) \leq 1. \tag{17}$$

Since both (16) and the unit 1 are upper bounds on the conditional error probability, i.e., $\Pr(\text{word error}|\alpha) \leq \min\{1, \phi_w(\alpha)\}$, a tight bound can be obtained as

$$\begin{aligned} P_{w|0} &\leq \int_0^\infty \min\{1, \phi_w(\alpha)\}\, p(\alpha)\mathrm{d}\alpha \\ &= \int_0^{\alpha^*} p(\alpha)\mathrm{d}\alpha + \int_{\alpha^*}^\infty \phi_w(\alpha)p(\alpha)\mathrm{d}\alpha, \end{aligned} \tag{18}$$

where α^* is the unique solution to the equation $\phi_w(\alpha) = 1$, i.e.,

$$\begin{aligned} &\phi_w(\alpha) \\ &= \sum_{h=1}^{L}\sum_{\underline{\delta}\in\Omega_h} A_{h,\underline{\delta}} Q\left(\alpha\sqrt{\frac{E_s}{2N_0}\sum_{i=0}^{Q-1}\delta_i\left|a_0 - a_i\right|^2}\right) \\ &= 1, \end{aligned} \tag{19}$$

for $\alpha \geq 0$.

The second line of (18) follows from the fact that $\phi_w(\alpha)$ for $\alpha \geq 0$ is a monotone decreasing function with $\phi_w(0) \geq 1$. A systematic method for the evaluation of (19) will be discussed later in Section VI.

To summarize the result compactly, we have the following proposition.

Proposition 2: For an LDPC-coded modulation system over a single-input single-output quasi-static fading channel with multiplicative gain α and additive white Gaussian noise, an upper bound on the word error probability is given by

$$P_{w|0} \leq \int_0^{\alpha^*} p(\alpha)\mathrm{d}\alpha + \int_{\alpha^*}^\infty \phi_w(\alpha)p(\alpha)\mathrm{d}\alpha =: P_w^1 + P_w^2, \tag{20}$$

where α^* is the unique solution of (19). Furthermore, for all constellations that satisfy the sufficiency test in Appendix B, the right hand side of (20) also serves as an upper bound for the word error probability, i.e.,

$$P_w \leq P_w^1 + P_w^2. \tag{21}$$

Proof: The first part has been proved throughout our discussion in (17) and (18). The proof of the second part is given in Appendix B. ■

The bound in (20) is essentially a formal statement of the bound in (3) we consider in the introduction. It may not be further tightened by applying the 1965 Gallager bounding technique or other variations in [6] which are developed from an original formula

$$\Pr(\text{word error } |\alpha) \leq (\phi_w(\alpha))^\rho, \text{for } 0 \leq \rho \leq 1. \tag{22}$$

This is clear considering that $\min\{1,\ \phi_w(\alpha)\} \leq \min_{0\leq\rho\leq 1}(\phi_w(\alpha))^\rho$. An expression similar to (20) was obtained in [13] by applying the limit-before-averaging technique to binary modulation systems over Rayleigh fading channels. However, a systematic approach to find the optimal α^* (i.e., the h_0 in [13]) was not discussed. The value of the optimal fading level α^* is universal for all orders and classes of fading channels, and thus should be determined only once for a given constellation (recall the argument inside the Gaussian Q-function in (19)). Once we find the optimal solution α^* per constellation, we only need to re-scale it for different SNR. See Section VI for detailed discussions.

As a special case of Proposition 2, we can derive the bound for Rician channels. Again, we provide the results in the following proposition. For this, we define $F(x|2m, s^2) = 1 - Q_m(s, \sqrt{x})$ is the cumulative density function of the non-central chi-square distribution with $2m$ degrees of freedom and a non-centrality parameter $s^2, Q_m(a, b)$ is the m-th order Marcum Q function

$$Q_m(a, b) = \frac{1}{a^{m-1}}\int_b^\infty x^m e^{-(x^2+a^2)/2} I_{m-1}(ax)\mathrm{d}x, \tag{23}$$

where $I_k(x)$ denotes the k-th order modified Bessel function of the first kind, i.e., $I_k(x) = \frac{1}{\pi}\int_0^\pi e^{x\cos\theta}\cos k\theta \mathrm{d}\theta$, and for compact notation, we use a utility variable $g_{\underline{\delta},\theta}$ defined as

$$g_{\underline{\delta},\theta} \triangleq \frac{E_s}{4N_0\sin^2\theta}\sum_{i=0}^{Q-1}\delta_i\left|a_0 - a_i\right|^2 + 1. \tag{24}$$

Proposition 3: For an LDPC-coded modulation system over a Rician distributed quasi-static fading channel with multiplicative gain α and additive Gaussian noise, the first of the two probability terms on the right hand side of (20) is given by

$$P_w^1 = F\left(2\alpha^{*2}|2, 2K_r\right) = 1 - Q_1\left(\sqrt{2K_r}, \sqrt{2}\alpha^*\right), \tag{25}$$

and the second one given at the top of next page:

Proof: See Appendix C for proof. ■

Eq. (25) is the outage probability for Rician channels. Eq. (26) is the union bound conditioned upon fading gain. Substituting (25) and (26) into (20), we obtain an upper bound on the word error probability

$$\begin{aligned} P_w \leq 1 - Q_1\left(\sqrt{2K_r}, \sqrt{2}\alpha^*\right) + \sum_{h=1}^{L}\sum_{\underline{\delta}\in\Omega_h}\frac{A_{h,\underline{\delta}}}{\pi} \\ \cdot \int_0^{\pi/2} g_{\underline{\delta},\theta} e^{-K_r\left(1-g_{\underline{\delta},\theta}\right)} Q_1\left(\sqrt{2g_{\underline{\delta},\theta}K_r}, \sqrt{\frac{2}{g_{\underline{\delta},\theta}}}\alpha^*\right)\mathrm{d}\theta. \end{aligned} \tag{27}$$

In the case of Rayleigh channels, the upper bound reduces to

$$P_w \leq 1 - e^{-\alpha^{*2}} + \sum_{h=1}^{L}\sum_{\underline{\delta}\in\Omega_h}\frac{A_{h,\underline{\delta}}}{\pi}\int_0^{\pi/2} g_{\underline{\delta},\theta}^{-1} e^{-g_{\underline{\delta},\theta}\alpha^{*2}}\mathrm{d}\theta, \tag{28}$$

$$\begin{aligned} P_w^2 &= \sum_{h=1}^{L} \sum_{\underline{\delta}\in\Omega_h} \frac{A_{h,\underline{\delta}}}{\pi} \int_0^{\pi/2} \frac{e^{-K_r\left(1-1/g_{\underline{\delta},\theta}\right)}}{g_{\underline{\delta},\theta}} \left[1 - F\left(2g_{\underline{\delta},\theta}\alpha^{*2} \,\middle|\, 2, \frac{2}{g_{\underline{\delta},\theta}}K_r\right)\right] \mathrm{d}\theta \\ &= \sum_{h=1}^{L} \sum_{\underline{\delta}\in\Omega_h} \frac{A_{h,\underline{\delta}}}{\pi} \int_0^{\pi/2} \frac{e^{-K_r\left(1-1/g_{\underline{\delta},\theta}\right)}}{g_{\underline{\delta},\theta}} Q_1\left(\sqrt{\frac{2}{g_{\underline{\delta},\theta}}K_r}, \sqrt{2g_{\underline{\delta},\theta}}\alpha^*\right) \mathrm{d}\theta. \end{aligned} \tag{26}$$

by setting K_r = 0 in (27) and resorting to the property of the Marcum Q function

$$Q_m\left(0, b\right) = \frac{\Gamma\left(m, b^2/2\right)}{\Gamma\left(m\right)} = e^{-b^2/2} \sum_{k=0}^{m-1} \frac{\left(b^2/2\right)^k}{k!}, \tag{29}$$

where $\Gamma(m)$ and $\Gamma(m,x)$ are the complete and incomplete Gamma Functions, respectively.

The bound obtained above is based on the Craig's identity of Gaussian Q function and thus involves a finite range integral in the final expression. We can remove the integral operation in (27) and (28) by using the Chernoff bound, $Q(x) \le \frac{1}{2}\exp(-x^2/2)$, on the Gaussian Q function. That is, similar to (28), we have

$$P_w \le 1 - e^{-\alpha^{*2}} + \sum_{h=1}^{L} \sum_{\underline{\delta}\in\Omega_h} A_{h,\underline{\delta}} g_{\underline{\delta}}^{-1} e^{-g_{\underline{\delta}}\alpha^{*2}}, \tag{30}$$

where $g_{\underline{\delta}} = \frac{E_s}{4N_0}\sum_{i=0}^{Q-1}\delta_i \left|a_0 - a_i\right|^2 + 1$. Note that this leads to a bound looser than (28).

We next move on to the bit error performance of the system. An upper bound similar to (17) can be considered for the bit error probability

$$\Pr(\text{bit error } |\alpha) \le \frac{1}{2}. \tag{31}$$

On the other hand, as shown in [3], a union bound on the bit error probability can be obtained by replacing A_h in (16) and (9) with A'_h,

$$A'_h = \sum_{\omega=1}^{K} \frac{\omega}{K} A_{\omega,h}, \tag{32}$$

where $A_{\omega,h}$ is the number of the codewords with input weight ω and output weight h. For the ensemble of codes that satisfies Proposition 1, (32) can be simplified as (See Appendix D)

$$A'_h = \frac{h}{L} A_h. \tag{33}$$

Thus, we make the following proposition on the bit error performance.

Proposition 4: The upper bounds presented in Propositions 2 and 3 can be applied to the bit error case by respectively replacing A_h with A'_h in (33) and calculating the corresponding α^* according to (19).

Proof: See proof of (33) in Appendix D and this proposition is obvious. ■

V. Upper Bounds for MIMO Systems with OSTBC

In this section, we are interested in applying the techniques developed so far to MIMO channels. Namely, the transmission of LDPC code concatenated with the orthogonal space-time block code (OSTBC) [19][20] over quasi-static MIMO fading channel is considered. The feature of OSTBC we utilize here, other than its capability of achieving full transmit diversity, is its capability of transforming the quasi-static MIMO channels into an equivalent SISO channel. The structure of the OSTBC allows a simple linear processing on the received signal which enables us to decouple the effect of MIMO coupling done on the constituent channel-symbols of each OSTBC codeword. As a result, the joint maximum likelihood detection of channel symbols in each OSTBC codeword can be transformed into the separate detection of each component channel symbol without loss of optimality. This point was originally introduced by Alamouti [21], and later noted and used in [19], [20], [22], [23].

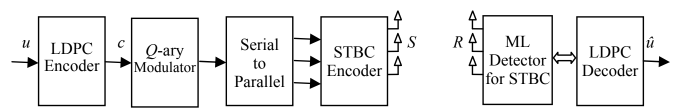

Fig. 2. Concatenated coding modulation system and iterative decoder.

Previous treatments on this subject were not clearly done, however, perhaps because it was not the main focus of these papers. For example, equivalent SISO channels were obtained explicitly for equal energy constellations such as PSK signals but not for constellations with unequal energy signal points such as the multilevel QAM constellations. Thus, we will spend some time discussing how to modify the previous procedures and obtain equivalent SISO channels even for unequal energy constellations. Hence, the bounding technique developed for SISO systems in the previous section can be applied to the MIMO systems without any restriction to signal constellations.

Let us consider an M-transmit, N-receive MIMO system. The concatenation of OSTBC with the LDPC-coded modulation system is illustrated in Fig. 2. The space-time block codeword is expressed in an $M \times T_s$ transmission matrix S, each entry of which is a linear combination of a group of K_s input symbols x_k and their conjugates x_k^* (k = 1, 2, … K_s). In order to achieve full transmit diversity, the signal matrix S is constructed based on the orthogonality design criterion [19], [20]. The signal matrix S is transmitted across the M transmit antennas T_s channel uses. The channel can be modeled as[1]

$$R = \sqrt{E_s} H S + W, \tag{34}$$

where E_s is the average symbol energy at each transmit antenna, $R = \{r_{n,t}\}$ is the received $N \times T_s$ signal matrix, $W = \{w_{n,t}\}$ is the $N \times T_s$ complex white Gaussian noise matrix,

[1]The channel model in [19] can be expressed in a form of $R = SH + W$. Thus, the transmission matrix S in this paper can be regarded as the transpose of that in [19] ignoring the coefficient $\sqrt{E_s}$.

each entry of which has zero mean and variance N_0, H is the $N \times M$ channel matrix known at the receiver and its (n,m)-th entry, $\alpha_{n,m}$, represents the independent fading gain from the $m-$th transmit antenna to the $n-$th receive antenna. The channel matrix is assumed to be fixed during a block of one LDPC codeword transmission, and varies independently from one block to another.

The ML detection of this system can be efficiently conducted by a series of linear processing operations on the received signal. For this, one may consider the Alamouti code [21] and the OSTBC with real orthogonal designs in [19] as specific examples.

We now state the proposition first and discuss the rationale after.

Proposition 5: A single-input single-output channel model which is equivalent to the system (34) under the maximum likelihood detection criterion is given by

$$y_k = \sqrt{E_s}\alpha_{st}x_k + w_k, \tag{35}$$

where w_k is the independent, complex white Gaussian noise with zero-mean and variance N_0, and α_{st} is the channel fading which is given by

$$\alpha_{st} := \left(\sum_{n=1}^{N}\sum_{m=1}^{M}|\alpha_{n,m}|^2\right)^{1/2}. \tag{36}$$

Proof: Without loss of generality, we will consider the case of Alamouti code over 2 × 2 MIMO channels [21] for brevity of discussion. The other OSTBC cases can be treated in a similar manner.

We will start out by briefly introducing the key arguments made in [19]-[21] which show the equivalent SISO channel model good only for equal energy constellations. We will then identify our approach that brings them into the general SISO channel model good for any constellation.

Assuming the perfect channel state information is available, the receiver calculates the maximum likelihood decision metric, $\sum_t\sum_n |r_{n,t} - \sum_m \alpha_{n,m}\hat{s}_{m,t}|^2$ over all hypothetical codewords $\hat{S} = \{\hat{s}_{m,t}\}$ and decides in favor of the codeword that minimizes the metric. As shown in [19]-[21], the minimization of the ML metric can be done equivalently (without loss of optimality) by minimizing K_s individual metrics

$$(\alpha_{st}^2 - 1)\left|\sqrt{E_s}\hat{x}_k\right|^2 + \left|y_k' - \sqrt{E_s}\hat{x}_k\right|^2, \tag{37}$$

for $k = 1, 2, \cdots, K_s$, where $\hat{x}_k$ is the hypothetical symbol coded into $\hat{S}$, y_k' is a linear combination of the receive signals $r_{n,t}$, and α_{st} defined in (36) can be regarded as a constant for the channel is known at the receiver.

As a special case of (37), the ML decision metric for Alamouti scheme over 2 × 2 MIMO channels is given by

$$(\alpha_{st}^2 - 1)\left|\sqrt{E_s}\hat{x}_k\right|^2 + \left|y_k' - \sqrt{E_s}\hat{x}_k\right|^2. \tag{38}$$

This is what was obtained in [21] (Eq. 17 there), by replacing their notations, $\tilde{s}_k$, s_k, and $\alpha_0^2 + \cdots + \alpha_3^2$, with our y_k', $\sqrt{E_s}\hat{x}_k$, and α_{st}^2 respectively. Note that the receive signal y_k' is expressed as

$$y_k' = \alpha_{st}^2\sqrt{E_s}\hat{x}_k + w_k', \tag{39}$$

where w_k' represents the noise term in [21] (Eq. 16), which is independent, additive white Gaussian distributed with zero mean and variance $\alpha_{st}^2 N_0$. As noted in [21], the first term in (38) can be ignored in the ML decision for equal-energy modulations, say PSK. This leads to an equivalent SISO channel whose ML metric is the second term in (38), i.e., the squared Euclidean distance between the hypothetical transmit signal $\sqrt{E_s}\hat{x}_k$ and the receive signal y_k'. Thus, following the discussion of these previous contributions, the equivalent channel looks valid only for equal energy signals.

At this point, we introduce our simple manipulation to the problem which leads to the equivalent SISO channel good for any arbitrary signal constellation. Let us add a term $\left(\alpha_{st}^{-2} - 1\right)|y_k'|^2$ to (38) and obtain

$$\left|\alpha_{st}^{-1}y_k' - \sqrt{E_s}\alpha_{st}\hat{x}_k\right|^2, \tag{40}$$

which is equivalent to (38) under ML hypothesis testing since the added term is free of the hypothetical candidate $\hat{x}_k$. Note that this operation is applicable to any OSTBCs in general by utilizing their orthogonality structure. Denote $y_k = \alpha_{st}^{-1}y_k'$ and $w_k = \alpha_{st}^{-1}w_k'$. Then, we can rewrite (39) with the variables of the SISO channel model given in (35) with ML metric expressed as (40),

$$\left|y_k - \sqrt{E_s}\alpha_{st}\hat{x}_k\right|^2. \tag{41}$$

This completes the proof. ■

Note that the result obtained till now is applicable to the Alamouti code or the OSTBC with real orthogonal designs. This can be easily extended to the OSTBC with complex-valued orthogonal designs, such as those formulated by (37)–(40) in [19]. Take (37) and (38) in [19] as examples. The linear processing of ML detection is to minimize each of the K_s individual decision metric [20]

$(2\alpha_{st}^2 - 1)\left|\sqrt{E_s}\hat{x}_k\right|^2 + \left|y_k - \sqrt{E_s}\hat{x}_k\right|^2, \quad k = 1, 2, \ldots, K_s$.

Recall the expression of (37) for real-valued orthogonal design. By simply setting $\alpha_{st} = \sqrt{2}\alpha_{st}$ and we can make the equivalent SISO channel (35) work for complex valued cases.

Based on the equivalent SISO channel, the bounding framework developed in the previous section can be applied to the MIMO fading channels. The key and only difference is that the fading gain in (35) is a higher-order non-central chi-square distribution. We have the upper bound on the error performance as follows:

Proposition 6: For M-transmit N-receive MIMO Rician fading channels, an upper bound on the word error probability using the OSTBC is given by (20) with P_w^1

$$\begin{aligned} P_w^1 &= F\left(2\alpha^{*2}\,|2MN, 2MNK_r\right) \\ &= 1 - Q_{MN}\left(\sqrt{2MNK_r}, \sqrt{2}\alpha^*\right), \end{aligned} \tag{42}$$

and P_w^2 given as the equation given at the top of next page.

A similar upper bound can be obtained for the bit error probability by respectively replacing A_h with A_h' and calculating the corresponding α^* according to (19).

Proof: See Appendix C. ■

$$\begin{aligned} P_w^2 &= \sum_{h=1}^{L} \sum_{\underline{\delta}\in\Omega_h} \frac{A_{h,\underline{\delta}}}{\pi} \int_0^{\pi/2} \frac{e^{-MNK_r\left(1-1/g_{\underline{\delta},\theta}\right)}}{g_{\underline{\delta},\theta}^{MN}} \left(1 - F\left(2g_{\underline{\delta},\theta}\alpha^{*2} \,\middle|\, 2MN, \frac{2}{g_{\underline{\delta},\theta}}MNK_r\right)\right) \mathrm{d}\theta \\ &= \sum_{h=1}^{L} \sum_{\underline{\delta}\in\Omega_h} \frac{A_{h,\underline{\delta}}}{\pi} \int_0^{\pi/2} \frac{e^{-MNK_r\left(1-1/g_{\underline{\delta},\theta}\right)}}{g_{\underline{\delta},\theta}^{MN}} Q_{MN}\left(\sqrt{\frac{2}{g_{\underline{\delta},\theta}}MNK_r}, \sqrt{2g_{\underline{\delta},\theta}}\alpha^*\right) \mathrm{d}\theta. \end{aligned} \tag{43}$$

Similar to the SISO case in (30), the expression of the upper bound can be simplified for Rayleigh MIMO channels as follows:

$$P_w \le 1 - e^{-\alpha^{*2}} \sum_{k=0}^{MN-1} \frac{\alpha^{*2k}}{k!} + \sum_{h=1}^{L} \sum_{\underline{\delta}\in\Omega_h} \frac{A_{h,\underline{\delta}}}{\pi} \cdot \int_0^{\pi/2} g_{\underline{\delta},\theta}^{-MN} e^{-g_{\underline{\delta},\theta}\alpha^{*2}} \sum_{k=0}^{MN-1} \frac{1}{k!} \left(g_{\underline{\delta},\theta}\alpha^{*2}\right)^k \mathrm{d}\theta. \tag{44}$$

If the Chernoff bound on the Gaussian Q function is applied, (44) can be further upper bounded by

$$P_w \le 1 - e^{-\alpha^{*2}} \sum_{k=0}^{MN-1} \frac{\alpha^{*2k}}{k!} + \sum_{h=1}^{L} \sum_{\underline{\delta}\in\Omega_h} A_{h,\underline{\delta}} g_{\underline{\delta}}^{-MN} e^{-g_{\underline{\delta}}\alpha^{*2}} \sum_{k=0}^{MN-1} \frac{1}{k!} \left(g_{\underline{\delta}}\alpha^{*2}\right)^k. \tag{45}$$

It is worth noting that the evaluation of these upper bounds is not as difficult as imaginable at the first look. The optimal threshold value α^*, the cardinality $A_{h,\delta}$ and the utility variable $g_{\underline{\delta}}$ are calculated only once without regard to the order of fading and SNR. In addition, the integral due to the use of Craig's identity is taken for a smooth function over a finite range interval, and thus can be easily evaluated via numerical methods.

VI. Discussions

In the previous sections, we have derived tight upper bounds for transmission of LDPC-code concatenated with OSTBCs operating over quasi-static MIMO fading channels. The upper bound can be evaluated by solving (19) for the optimal threshold α^* and then calculating the two error terms in (42) and (43). In this section, we briefly discuss the complexity involved in the evaluation of the upper bound.

The first is the step involved in finding the optimal α^*. As mentioned earlier, the optimal α^* can be obtained regardless of channel fading distributions. Thus, the good news is that it needs to be calculated only once for a given signal constellation. From observation of the argument of the Gaussian Q-function in (19), we can find the optimal value at $E_s/N_0 = 1$, say $\alpha^*_{0\mathrm{dB}}$, and then re-scale it, i.e., $\alpha^* = \alpha^*_{0\mathrm{dB}}/\sqrt{E_s/N_0}$, for different E_s/N_0 values. To solve (19), we can make use of the monotonic property of the function $\phi_w(\alpha)$. The primary difficulty in finding the optimal threshold lies in the evaluation of $\phi_w(\alpha)$ at each α. The summation operation over $\underline{\delta} \in \Omega_h$ is rather cumbersome as the cardinality of the set Ω_h tends to be very large. For this, we make use of the polynomial expansion idea proposed in [24]. That is, by making use of the Craig's identity [25] of the Gaussian Q function (we can do this with the Chernoff bound as well),

$$Q(x) = \frac{1}{\pi} \int_0^{\pi/2} \exp\left(\frac{-x^2}{2\sin^2\theta}\right) \mathrm{d}\theta, \tag{46}$$

and by substituting the expression of $A_{h,\underline{\delta}}$ in (9), we can rewrite (19) as

$$\frac{1}{\pi} \int_0^{\pi/2} \sum_{h=1}^{L} \begin{pmatrix} L \\ h \end{pmatrix}^{-1} A_h \varphi_h \mathrm{d}\theta = 1, \tag{47}$$

where

$$\varphi_h \triangleq \sum_{\underline{\delta}\in\Omega_h} \begin{pmatrix} T \\ \delta_0, \delta_1, \ldots, \delta_{Q-1} \end{pmatrix} \prod_{i=0}^{Q-1} \beta_i^{\delta_i}, \tag{48}$$

and

$$\beta_i \triangleq \exp\left(-\frac{E_s \alpha^2 \left|a_0 - a_i\right|^2}{4N_0 \sin^2\theta}\right).$$

It can be proved [24] (see Appendix A) that φ_h's are the coefficients of a polynomial expansion

$$\left(\sum_{i=0}^{J-1} \beta_i z^{w_i}\right)^T = \sum_{h=0}^{L} \varphi_h z^h, \tag{49}$$

where w_j, as defined in section III, is the Hamming weight of the bit string b_i that is mapped to the constellation point a_i. It is worth noting that the polynomial expansion method is originally proposed in [24] for performance analyses on fast Rician fading MIMO channels. Its applicability in the context of this paper lies in the fact that the left hand side of (19) or (47) is indeed a union bound for an AWGN channel (i.e., a quasi-static fading channel with fixed channel gain α), which in turn can be regarded as a special case of the fast Rician fading channel with factor $K_r = \infty$.

Second, the other difficulty may lie in the numerical evaluation of the Marcum Q function in (23) because the routine is numerically sensitive. We take the approach of using the series representation of the Marcum Q function such as

$$Q_m(a, b) = e^{-(a^2+b^2)/2} \sum_{k=1-m}^{+\infty} \left(\frac{a}{b}\right)^k I_k(ab), \tag{50}$$

which can be truncated at $k = 50$ without losing much precision [26]. Interested readers are referred to [27] (see Ch. 4.2) for other evaluation methods.

VII. Results

We compare the derived ML upper bounds with the simulation results of a complete, practical transceiver system over quasi-static Rician fading channel. The bounds are evaluated for the ensemble of Galager's (3000, 3, 6) LDPC codes [8]. The distance spectrum of the ensemble is calculated according to [28].

In simulations, the receiver is assumed to know the channel exactly and employ an iterative detection and decoding algorithm as illustrated in Fig. 3. While interested readers are referred to [29] for detailed explanations on a similar example, we sketch the iterative algorithm as follows. The detector

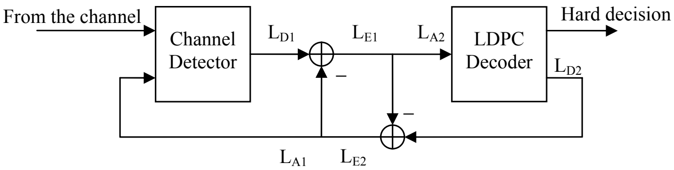

Fig. 3. Iterative detection and decoding algorithm. L_{D1} and L_{D2} are the posteriori LLRs (log-likelihood ratios) from the detector and the decoder, respectively. L_{E1} and L_{E2} are the corresponding extrinsic information, and are treated as the prior information, L_{A2} and L_{A1}, at the detector and the decoder, respectively.

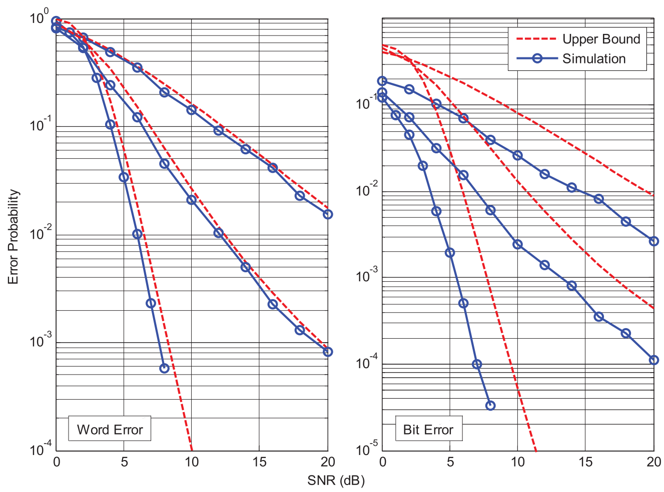

Fig. 4. Error performance of the (3000, 3, 6) LDPC coded 4PSK modulation over quasi-static SISO Rician fading channels. In order of error performance at high SNR, the three pairs of bound-simulation curves in each plot correspond to the cases in which the Rician factor K_r equals 0, 5, and 20, respectively.

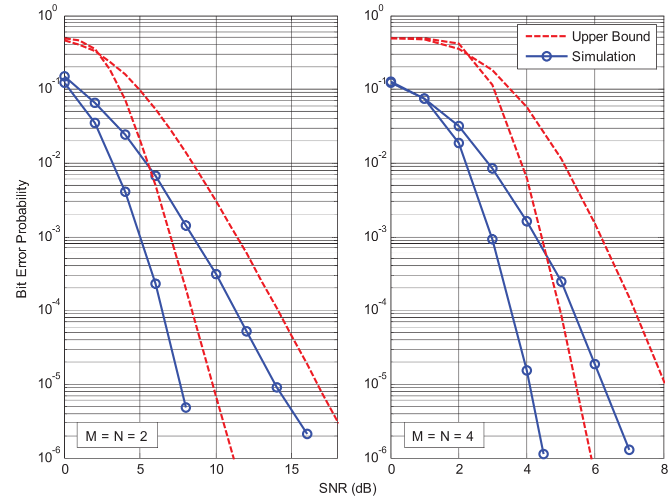

Fig. 5. Error performance of the LDPC (3000, 3, 6) coded 4PSK modulation over quasi-static MIMO Rician fading channels. In order of error performance at high SNR, the two pairs of bound-simulation curves in each plot correspond to the cases in which the Rician factor K_r equals 0 and 5, respectively.

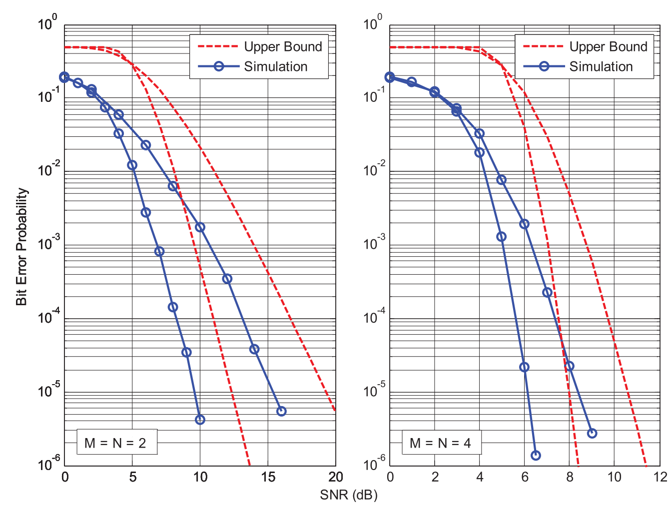

Fig. 6. Error performance of the LDPC (3000, 3, 6) coded 8QAM modulation over quasi-static MIMO Rician fading channels. In order of error performance at high SNR, the two pairs of bound-simulation curves in each plot correspond to the cases in which the Rician factor K_r equals 0 and 5, respectively.

takes channel observations and the *a priori* information L_{A1} to compute the new *a posteriori* information L_{D1} on each coded bit. The difference, $L_{E1} = L_{D1} - L_{A1}$, is referred to as "extrinsic" message and is forward to the decoder as the *a priori* input, L_{A2}. Then, the decoder generates the *a posteriori* information L_{D2}, and feedbacks the corresponding extrinsic information $L_{E2} = L_{D2} - L_{E1}$ as *a priori* knowledge to the detector. This complete a single iteration of messages between the detector and the decoder. We call this *super*-iteration as compared to the decoder's own iteration which we call *internal*-iteration of the decoder. In our simulation, we use three *super* and ten *internal*-iterations.

To average the performance of the code ensemble, we randomly generate 5,000 LDPC codes and use each of them for ten codeword transmissions; the error probability is averaged over 50,000 randomly selected transmit codewords. For fair comparison, the error performance is plotted with respect to the normalized SNR,

$$\frac{E_b}{N_0} := \frac{(1+K_r)E_s M N}{N_0 R_t}, \tag{51}$$

where $(1 + K_r)$ is the average value of squared magnitude of the fading gain and R_t is the transmission rate of the system in information bits/channel use.

We first verify the effectiveness of the upper bounds for the coded system over SISO fading channel. As shown in Fig. 4, the bound on the word error probability indicates a good match with the simulation result for different Rician channels. The SNR difference between the bit error probabilities and the corresponding upper bounds is about 2 - 4 dB, a relatively large deviation compared to that of the word error case. It is interesting to observe that the error probability decreases faster as the Rician factor increases from zero, five to twenty. This is of no surprise since a Rician channel converges to an AWGN channel as the Rician factor goes to infinity. The final SNR gap as Rician factor goes up is thus expected to be about 1.5 dB, as reported for AWGN channels in the literature [6].

The performance of the concatenated MIMO system is illustrated in Fig. 5 and Fig. 6 for 4PSK and 8QAM modulation, respectively. In the case of 2-by-2 MIMO system, we adopt the Alamouti scheme as the inner space-time block code. The derived bounds are about 2.5 dB away from the simulation results. The difference decreases to 1.5 dB for the system with four transmit and four receive antennas, where the orthogonal space-time block code in [19], (see Eq. 38) is adopted. Also note that, in all investigated scenarios, the bound becomes tighter when the channel has a larger Rician factor and therefore stiffer an error curve is. Hence, the derived upper bounds will be useful to benchmark the performance of the turbo-iterative algorithm, especially when the system has more transmit and receive antennas and operates over channels with large Rician factors.

VIII. Conclusion

We have presented an error performance bounding approach for quasi-static fading channels. Under the proposed approach, the Fano-Gallager bounding technique is formulated in a manner that divides the range of the fading gain into two disjoint regions. The critical fading level which is optimally selected at each average SNR divides the two regions. The proposed approach seems to overcome the excessive codeword-multiplicity problem in conventional union bounds and works well for different channels and coded modulation scenarios. For applications in MIMO systems, we show that a linear processing technique can be applied which effectively transforms the MIMO system into an equivalent SISO system regardless of constellations. We note that this technique enables us, by leveraging on the bounding technique developed for the SISO systems, to obtain tight closed-form bounds for the concatenated MIMO systems.

Appendix A
Proof of Proposition 1

Instead of an ensemble of LDPC code, it is easy for us to equivalently consider the ensemble H of the corresponding parity-check matrices.

$A_h > 0$ means that at least one codeword, say c_h, of Hamming weight h exists in certain codes in the ensemble. Assume c'_h to be any arbitrary permutation of c_h. That is, $c'_h = \pi(c_h)$, where $\pi(\cdot)$ is the pattern of the associated column permutation. Denote H_1 and H_2 as the sets of all parity-check matrices in H that c_h and c'_h satisfy, respectively; i.e., $\mathrm{H}_1 := \left\{H \,\middle|\, H \in \mathrm{H},\ Hc_h^T = \underline{0}\right\}$ and $\mathrm{H}_2 := \left\{H \,\middle|\, H \in \mathrm{H},\ Hc_h'^T = \underline{0}\right\}$. (53)

The cardinality, $N_h := \Delta|H_1|$, of H_1 is nonzero. Note that there is a one-to-one correspondence between H_1 and H_2; i.e.,

$$H_2 = \{\pi(H)\,|H \in \mathrm{H}_1\}, \tag{52}$$

considering that $H \in \mathrm{H}$ implies $\pi(H) \in \mathrm{H}$ and vice versa (any arbitrary column permutation of a parity-check matrix does not change the variable- and the check-node degree distributions). Therefore, $|\mathrm{H}_2| = |\mathrm{H}_1| = N_h$. Since each of the $\begin{pmatrix} L \\ h \end{pmatrix}$ binary sequences of Hamming weight h can be regarded as a permutation of c_h, the first statement of the theorem is proved. With the assumption of equiprobable selection of codes in (6), the probability of each of these sequences appearing in the randomly selected code X_{sel} is equal; i.e.,

$$\Pr\left(c_h \in \mathcal{C}_{sel}\right) = N_h/|\mathbb{C}|. \tag{53}$$

Appendix B
Proof of Propositions 2 (the Second Statement)

In this section, we prove that, given a sufficiency condition satisfied, the general union bound averaged over the transmission of all possible codewords can be further upper bounded by a simpler bound which is based merely on the transmission of the all zero codeword.

The union bound for a given fading channel coefficient α can be written as

$$\begin{aligned}\Pr(\text{word error }|\alpha) &= E_c\left[P_{w|c}\right]\\ &\leq \quad E_c\left[\sum_{c'\neq c}\Pr(x(c)\rightarrow x'(c')\,|\alpha)\right],\end{aligned} \tag{54}$$

where $x(c)$ and $x'(c')$ denote the respective modulated codewords of c and c' transmitted over antennas. For simplicity, we set α = 1 since it does not affect the derivations below.

We want to find a codeword $\tilde{c}$ which has the worse-than-average pairwise error performance, i.e.,

$$E_c\left[P_{w|c}\right] \leq P_{w|\tilde{c}}. \tag{55}$$

Our aim is to prove that, given a sufficiency condition satisfied, a codeword $\tilde{c}$ whose modulated sequence is of the form $\tilde{x} = (a_0\ a_0\ \cdots\ a_0)$ satisfies the inequality of (55).

To state the result first, a theorem with a sufficiency condition is developed which establishes the validity of the further upper bound (55). The sufficiency condition provides a simple, systematic way to verify if a constellation contains such a channel symbol a_0 which makes the further upper bound valid. Our results indicate that such a further upper bound can be established for a number of constellations such as Q-ary QAM for Q = 4, 8, 16, 64, 256, 1024, all Q-ary ASK, and all Q-ary PSK. Unfortunately, we have not been able to show that the method holds for Q-ary QAM for Q = 32, 128, 512.

Development of the Theorem

While the precise condition will be given later on in (63), in a loose sense, a_0 could be selected from among the so-called mass center points within a constellation $\mathcal{A} = \{a_0, a_1, \cdots, a_{Q-1}\}$ that satisfy the following inequality

$$\frac{1}{Q}\sum_{i=0}^{Q-1}\sum_{j=0}^{Q-1}|a_i - a_j|^2 \geq \sum_{j=0}^{Q-1}|a_0 - a_j|^2. \tag{56}$$

In general, there are multiple such symbols, $a_0 \in \mathcal{A}$, which satisfy (56). Among them, we note that, the symbol that maximizes the right hand side of (56) can be selected for a tight further upper bound.

The condition (56) can be rewritten as

$$\sum_{i=0}^{Q-1}\sum_{j=0}^{Q-1}|a_i - a_j|^2 \geq Q\sum_{j=0}^{Q-1}|a_0 - a_j|^2. \tag{57}$$

There are Q^2 magnitude square terms on each side of (57). We can make a couple of observations:

1) On the left hand side (LHS) of (57), the Q^2 summands constitutes every possible basic building block in the Euclidean distance $d(\tilde{x}, x)$ for any pair of codewords.

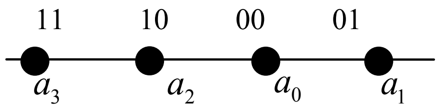

Fig. 7. A Gray labeled 4-ASK constellation

They cover every possible case for every possible pair of codewords.

2) On the right hand side (RHS) of (57), there are only Q distinct summands, but each is summed Q times; and thus, RHS also has Q^2 summands. These summands are the basic building blocks of the Euclidean distance $d(\tilde{x}, x)$ between $\tilde{x} = (a_0\ a_0\ \cdots a_0)$ and any other codeword x.

We now define a number of useful structures for the development of the theorem. To make our definitions more clearly understandable, we provide examples for a simple 4-ASK constellation along the way. We assume the minimum distance of 4-ASK constellation is 1, i.e., $|a_0 - a_2| = 1$. As shown in Fig. 7, 4-ASK constellation is labeled with the channel symbol index i and its binary string (which is the natural map on the index i). We note that an index pair comes with its unique Euclidean distance (ED) as well as Hamming distance (HD). For example, a symbol pair (a_1, a_3), or simply an index pair $(1, 3)$, comes with ED of 3 and HD of 1, for the 4-ASK constellation.

Each summand in (57) can be identified by its index pair. A summand is an index pair which again comes with its Euclidean and Hamming distance. In fact, a quadruplet is formed – a summand, an index pair, ED and HD. In either side of (57), we can put these Q^2 quadruplets into an ordered sequence of quadruplets, first based on HD and then on ED.

Definition (*Hamming Distance Profile, HDP***)** We order the Q^2 quadruplets in the ascending order of Hamming distances from which we can obtain two sequences of Hamming distances, one for the RHS (called $\mathbf{w}_R$) and the other for LHS (called $\mathbf{w}_L$), and refer to these two vectors as the right and the left *Hamming distance profile* respectively. In fact, we notice that $\mathbf{w}_R = \mathbf{w}_L$. But we will keep both notations for clarity.

In what follow, we use $\mathbf{w}_0 := \mathbf{w}_R$ so as to explicitly denote the relationship of the right Hamming distance profile with the reference codeword $\tilde{x} = (a_0\ a_0\ \cdots a_0)$. This will not cause any confusion.

For the example of 4-ASK constellation. There are $Q^2 = 4^2$ summands. Each Hamming distance profile is a vector of size Q^2,

$$\mathbf{w}_L = \mathbf{w}_0 = (0\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 2\ 2\ 2\ 2)\,. \tag{58}$$

We now sort the obtained quadruplet sequence one more time. Notice that the two sequences of quadruplets are already in the ascending order of Hamming distances. This time we want to arrange them in the ascending order of Euclidean distance, but this re-ordering is done only amongst those indexes whose Hamming distances are the same. Thus, this does not affect the order in terms of Hamming distance. Thus, $\mathbf{w}_L = \mathbf{w}_0$ both remain to be the same as (58). We refer to the resulting sequences of Euclidean distance squares as the *Euclidean distance profiles* (EDP).

Definition (*Right Euclidean Distance Profile*) For the quadruplet sequence of the RHS of (57), we sort it first in the ascending order of Hamming distance, and second re-order those with the same Hamming distance in the ascending order of Euclidean distance. We call the resulting EDP the right *Euclidean Distance Profile*, i.e.,

$$\mathbf{D}_R = \left[|a_0 - a_{0_0}|^2, |a_0 - a_{0_1}|^2, \cdots, \left|a_0 - a_{0_{Q^2-1}}\right|^2 \right]. \tag{59}$$

where the index 0_p, $p = 0, 1, \cdots, Q^2 - 1$, can be found.

Definition (*Left Euclidean Distance Profile*) For the quadruplet sequence of the LHS of (57), we sort it first in the ascending order of Hamming distance, and second re-order those with the same Hamming distance in the ascending order of Euclidean distance. We call the resulting EDP the left *Euclidean Distance Profile*, i.e.,

$$\mathbf{D}_L := \left[|a_{i_0} - a_{j_0}|^2, |a_{i_1} - a_{j_1}|^2, \cdots, \left|a_{i_{Q^2-1}} - a_{j_{Q^2-1}}\right|^2 \right], \tag{60}$$

where the index pairs (i_p, j_p), $p = 0, 1, \cdots, Q^2 - 1$, can be found.

Similar to notation for $\mathbf{w}_0$, we use $\mathbf{D}_0 := \mathbf{D}_R$ to explicitly denote the relationship of the right Euclidean distance profile with the reference codeword $\tilde{x} = (a_0\ a_0\ \cdots a_0)$.

Now again consider the 4-ASK constellation. The right and left EDPs are

$$\mathbf{D}_0 = (0\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 2^2\ 2^2\ 2^2\ 2^2), \tag{61}$$

and

$$\mathbf{D}_L = (0\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 3^2\ 3^2\ 2^2\ 2^2\ 2^2\ 2^2) \tag{62}$$

respectively. Note that $\mathbf{D}_L$ is not in the ascending order of Euclidean distance. The 10^{th} and 11^{th} elements are for the index pairs $(i_p, j_p) = (2, 3)$ or $(3, 2)$, whose Hamming distance is 1 and whose Euclidean distance square is $|a_3 - a_2|^2 = 3^2$. This is the consequence of our ordering procedure that the Hamming distance takes the priority in ordering and the Euclidean distance the second. The ordering based on Euclidean distance was done only among those indexes which have the same Hamming distance.

We note that $\mathbf{D}_L$in (62) is greater than or equal to $\mathbf{D}_0$ in the element-by-element manner, i.e., $\mathbf{D}_L \succ \mathbf{D}_0$. We will show that $\mathbf{D}_L \succ \mathbf{D}_0$ together with $\mathbf{w}_L = \mathbf{w}_0$ constitutes a sufficiency condition to the theorem which guarantees the validity of the further upper bound in (55). Namely, the upper bound conditioned on the transmission of $\tilde{x} = (a_0\ a_0\ \cdots a_0)$ is a valid further upper bound to the union bound averaged over the transmission of all possible codewords. Thus, for a particular alphabet $\mathcal{A}$ and a selection $a_0 \in \mathcal{A}$, a testing if $\mathbf{D}_L \succ \mathbf{D}_0$ with $\mathbf{w}_L = \mathbf{w}_0$ is true or not, can be done to validate the upper bound. If the sufficiency condition is not met, the further upper bound cannot be corroborated.

A Sufficiency Test: Given a constellation $\mathcal{A}$ and a constellation mapping which labels the constellation points, we say that the constellation together with the mapping satisfy the sufficiency test to the Theorem if the constellation contains a channel symbol $a_0 \in \mathcal{A}$ that satisfies $\mathbf{D}_L \succ \mathbf{D}_0$, i.e.,

$$\left[|a_{i_0} - a_{j_0}|^2, |a_{i_1} - a_{j_1}|^2, \cdots, \left|a_{i_{Q^2-1}} - a_{j_{Q^2-1}}\right|^2\right] \succeq \left[|a_0 - a_{0_0}|^2, |a_0 - a_{0_1}|^2, \cdots, \left|a_0 - a_{0_{Q^2-1}}\right|^2\right], \quad (63)$$

which comes with the identical right and left HDPs, $\mathbf{w}_L = \mathbf{w}_0$. Such a channel symbol $a_0 \in \mathcal{A}$ is usually found among those satisfying the following

$$\frac{1}{Q}\sum_{i=0}^{Q-1}\sum_{j=0}^{Q-1}|a_i - a_j|^2 \geq \sum_{j=0}^{Q-1}|a_0 - a_j|^2. \quad (64)$$

In fact, when (63) is met, (64) is always met.

Theorem 7: For a constellation $\mathcal{A}$ and a constellation map which satisfy the sufficiency test, the union bound based on the transmission of the all zero codeword whose all-zero binary string of length $\log_2(Q)$ is mapped to the channel symbol a_0 is greater than or equal to the union bound averaged over all codeword transmission. Equality is achieved when the sufficiency test is met with equality. Namely, the following inequality holds

$$\sum_{h=1}^{L}\sum_{\underline{\delta}_L \in \Omega_h} A_{h,\underline{\delta}_L} Q\left(\sqrt{\frac{E_s}{2N_0}\sum_{p=0}^{Q^2-1}\delta_{i_p,j_p}\left|a_{i_p} - a_{j_p}\right|^2}\right) \leq \sum_{h=1}^{L}\sum_{\underline{\delta}_R \in \Omega'_h} A_{h,\underline{\delta}_R} Q\left(\sqrt{\frac{E_s}{2N_0}\sum_{p=0}^{Q^2-1}\delta_{0,0_p}\left|a_0 - a_{0_p}\right|^2}\right), \quad (65)$$

where $\delta_{i_p,j_p} \in \{0, 1, 2, \cdots, T\}$ represents the number of Euclidean distance squares $\left|a_{i_p} - a_{j_p}\right|^2$ in the Euclidean distance square $d^2(\tilde{x}, x)$ of any two codewords; $\underline{\delta}_L := (\delta_{i_0,j_0}\,\delta_{i_1,j_1}\cdots\delta_{i_{Q^2-1},j_{Q^2-1}})$ and $\underline{\delta}_R := (\delta_{0,0_0}\,\delta_{0,0_1}\cdots\delta_{0,0_{Q^2-1}})$ are the respective collections of δ_{i_p,j_p} and $\delta_{0,0_p}$; Ω_h and Ω'_h are the respective sets of $\underline{\delta}_L$ and $\underline{\delta}_R$ that correspond to a Hamming distance h:

$$\Omega_h := \left\{\underline{\delta}_L \,\middle|\, \delta_{i_p,i_p} \in \{0, 1, \cdots, T\}, \sum_{p=0}^{Q^2-1}\delta_{i_p,i_p} = T, \mathbf{w}_L\underline{\delta}_L = h\right\}, \quad (66)$$

$$\Omega'_h := \left\{\underline{\delta}_R \,\middle|\, \delta_{0,0_p} \in \{0, 1, \cdots, T\}, \sum_{p=0}^{Q^2-1}\delta_{0,0_p} = T, \mathbf{w}_0\underline{\delta}_R = h\right\}. \quad (67)$$

and $A_{h,\underline{\delta}_L} = A_h\binom{L}{h}^{-1}\binom{T}{\underline{\delta}_L}$ and $A_{h,\underline{\delta}_R} = A_h\binom{L}{h}^{-1}\binom{T}{\underline{\delta}_R}$ are the corresponding distance spectra.

Proof: The LHS of (65) is the general union bound averaged over the transmission of all possible codewords. This union bound is obtained by summing all pairwise error probabilities between the transmitted codeword c and any other codeword c'. Similar to analysis in Section III, the pairwise error probability between c and c' is solely determined by the *appearance* vector $\underline{\delta}_L := (\delta_{i_0,j_0}\,\delta_{i_1,j_1}\cdots\delta_{i_{Q^2-1},j_{Q^2-1}})$ between them. Thus, we can partition the codebook according to the *appearance* vector $\underline{\delta}_L$ between a codeword and c. This is indeed a further decomposition of the codebook in addition to the partition based on the Hamming distance between any codeword and c. The summation on the LHS of (65) is thus taken with respect to the Hamming distance h and the *appearance* vectors $\underline{\delta}_L \in \Omega_h$ which correspond to the same h.

The RHS of (65) is the union bound of error performance conditioned on the transmission of $\tilde{x} = (a_0\ a_0\ \cdots\ a_0)$. This is a special case of the LHS and thus the analysis on the LHS applies. We note that the $\underline{\delta}_L$ is indeed a dummy variable in the definition of Ω_h in (66) . In addition, we have $\mathbf{w}_L = \mathbf{w}_0$; and thus, the set Ω_h on the LHS of (65) is the same as the set Ω'_h on the RHS. In other words, we can consider $\underline{\delta}_L = \underline{\delta}_R$ in the calculation on both sides. As the result, the sufficiency condition $\mathbf{D}_L \succ \mathbf{D}_0$ leads to $\mathbf{D}_L\underline{\delta}_L \geq \mathbf{D}_R\underline{\delta}_R$, i.e.,

$$\sum_{p=0}^{Q^2-1}\delta_{i_p,j_p}\left|a_{i_p} - a_{j_p}\right|^2 \geq \sum_{p=0}^{Q^2-1}\delta_{0,0_p}\left|a_0 - a_{0_p}\right|^2. \quad (68)$$

From (68), we can write, after multiplying both sides by $\frac{E_s}{2N_0}$,

$$Q\left(\sqrt{\frac{E_s}{2N_0}\sum_{p=0}^{Q^2-1}\delta_{i_p,j_p}\left|a_{i_p} - a_{j_p}\right|^2}\right) \leq Q\left(\sqrt{\frac{E_s}{2N_0}\sum_{p=0}^{Q^2-1}\delta_{0,0_p}\left|a_0 - a_{0_p}\right|^2}\right). \quad (69)$$

In addition, given $\underline{\delta}_L = \underline{\delta}_R$, the distance spectrum is the same on both sides of (65), i.e., $A_{h,\underline{\delta}_L} = A_{h,\underline{\delta}_R}$. By multiplying $A_{h,\underline{\delta}_L} = A_{h,\underline{\delta}_R}$ to both sides of (69) and summing over all h and δ, we have

$$\sum_{h=1}^{L}\sum_{\underline{\delta}_L \in \Omega_h} A_{h,\underline{\delta}_L} Q\left(\sqrt{\frac{E_s}{2N_0}\sum_{p=0}^{Q^2-1}\delta_{i_p,j_p}\left|a_{i_p} - a_{j_p}\right|^2}\right) \leq \sum_{h=1}^{L}\sum_{\underline{\delta}_R \in \Omega'_h} A_{h,\underline{\delta}_R} Q\left(\sqrt{\frac{E_s}{2N_0}\sum_{j=0}^{Q^2-1}\delta_{0,0_j}\left|a_0 - a_{0_j}\right|^2}\right). \quad (70)$$

The L.H.S. is the general union bound averaged over the transmission of every codeword while the R.H.S. is the union bound conditioned on the transmission of $\tilde{x} = (a_0\ a_0\ \cdots\ a_0)$. ■

We note that the RHS of (70) is equal to (16) (with $\alpha = 1$ of course). Note that there are only Q distinct terms among the Q^2 summands under the square root on the RHS of (70). We treated them to be Q^2 distinct summands in order to have the set partition structure comparable to the LHS of (70) which have Q^2 distinct summands. This was needed for the proof of the Theorem. Now, it is trivial to show that the partitioned sets can be grouped together to form an expression of (16).

Discussion

Several examples would clarify the Theorem. First, we note that every equi-energy constellation satisfies the sufficiency test with equality. Thus, for equi-energy constellations such as 4-QAM and Q-ary PSK, the theorem holds with equality. In addition, the theorem holds for the 4-ASK we have been using as an example since the sufficiency test is met. For a systematic search, we wrote a MATLAB program and performed a search on constellations which satisfy the sufficiency

TABLE I
MODULATION SCHEME AND SATISFACTION PERCENTAGE

Modulation Scheme	Percentage of elements that satisfy $\mathbf{D}_L \succeq \mathbf{D}_R$
32 QAM	1016/1024 (99.22%)
128 QAM	16288/16384 (99.41 %)
512 QAM	261136/262144 (99.62%)

test, especially for Q-ary QAM. The constellations that satisfy the test are found to be Q = 4, 8, 16, 64, 256, 1024. The constellation that do not satisfy the sufficiency test are Q = 32, 128, and 512. The percentage of elements in the EDP vector pairs which do not satisfy $\mathbf{D}_L \succ \mathbf{D}_0$ with $\mathbf{w}_0 = \mathbf{w}_L$ is usually very small as Table I indicates.

Discussion of the reason why some of the constellations, such as 32-QAM and 128 QAM, do not pass the sufficiency test is beyond the scope of this paper. One observation we could report in this paper, however, is that these constellations with the Gray constellation map, have a few points at the edge of constellation whose Hamming distance is greater than 1 while being only a minimum Euclidean distance away from each other. These are the points that violate the sufficiency test.

APPENDIX C
PROOF OF PROPOSITIONS 3 & 6

First, we note that Proposition 3 can be regarded as a special case of Proposition 6 by setting $M = N = 1$. We next prove the latter by making use of the equivalent SISO channel model of (35).

For simplicity, we introduce a new random variable $\beta = 2\alpha_{st}^2$ which follows the non-central chi-squared distribution with $2MN$ degrees of freedom and non-centrality parameter $2MNK_r$. According to [30], Eq. 2-1-118], the probability distribution function of β is given by

$$p_\beta(\beta) = \frac{1}{2}\left(\frac{\beta}{2MNK_r}\right)^{(MN-1)/2} \cdot e^{-\frac{\beta+2MNK_r}{2}} I_{MN-1}(\sqrt{2\beta MNK_r}). \quad (71)$$

Therefore, from (20) we have

$$P_w^1 = \int_0^{2\hat{\alpha}^2} p_\beta(\beta)\mathrm{d}\beta = F\left(2\hat{\alpha}^2 \,|2MN, 2MNK_r\right) = 1 - Q_{MN}\left(\sqrt{2MNK_r}, \sqrt{2}\hat{\alpha}\right). \quad (72)$$

The probability term P_w^2 is obtained by substituting (71) into its definition in (20) and then resorting to the Craig's identity of the Gaussian Q function in (46). Detailed derivation is presented as follows at the top of next page, where the third equation is obtained by resorting to the expression of Gaussian Q function in (46) and setting $g := g_{\underline{\delta},\theta} = \frac{E_s}{4N_0 \sin^2\theta}\sum_{i=0}^{Q-1}\delta_i \left|a_0 - a_i\right|^2 + 1$, and the fourth step follows from the integration by substitution, $x = g\beta$.

APPENDIX D
PROOF OF EQUATION (33)

Consider the ensemble of LDPC codes in Section III. Each LDPC code maps K information bits into a codeword of length L. Denote the generating matrix of the code as G; we can find its equivalent, systematic form $G_s = (P : I_K)$ by Gauss-Jordan elimination, where P is the $K \times (L - K)$ resultant matrix, and I_K is the $K \times K$ identity matrix. Therefore, the last K bits of each codeword are exactly the repetition of the information bits. For any codeword with input weight ω and output weight h, the weights of its first $L-K$ bits and last K bits are $h - \omega$ and ω, respectively. For simplicity, we denote this weight pair $h - \omega,\ \omega)$ as a metric of the codeword.

Similar to the approach in Section III, we resort to Proposition 1 and obtain the probability that any codeword $c$of weight h has a metric $(h - \omega,\ \omega)$ as follows,

$$\Pr\left(c \text{ has a metric } (h-\omega,\ \omega)\,|c \text{ is of weight } h\right) = \binom{L}{h}^{-1}\binom{L-K}{h-\omega}\binom{K}{\omega} =: P_{(h-\omega,\omega)|h}. \quad (74)$$

The average number $A_{\omega,h}$ of the codewords of metric $(h-\omega,\ \omega)$ in one code is therefore given by

$$A_{\omega,h} = A_h P_{(h-\omega,\omega)|h} = A_h \binom{L}{h}^{-1}\binom{L-K}{h-\omega}\binom{K}{\omega}. \quad (75)$$

By simple manipulation, we have from (32) that

$$A'_h = \sum_{\omega=1}^{K}\frac{\omega}{K}A_{\omega,h} = A_h\binom{L}{h}^{-1}\sum_{\omega=1}^{K}\binom{L-K}{h-\omega}\binom{K}{\omega} = A_h\binom{L-1}{h-1}^{-1}\sum_{\omega=1}^{K}\binom{L-K}{h-\omega}\binom{K-1}{\omega-1} = \frac{h}{L}A_h. \quad (76)$$

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$$
\begin{aligned}
P_w^2 = & \sum_{h=1}^{L} \sum_{\underline{\delta}\in\Omega_h} A_{h,\underline{\delta}} \int_{\hat{\alpha}}^{\infty} Q\left(\alpha\sqrt{\frac{E_s}{2N_0}\sum_{i=0}^{Q-1}\delta_i\left|a_0-a_i\right|^2}\right) p\left(\alpha\right)\mathrm{d}\alpha \\
= & \sum_{h=1}^{L} \sum_{\underline{\delta}\in\Omega_h} A_{h,\underline{\delta}} \int_{2\hat{\alpha}^2}^{\infty} Q\left(\sqrt{\beta\frac{E_s}{4N_0}\sum_{i=0}^{Q-1}\delta_i\left|a_0-a_i\right|^2}\right) p_\beta\left(\beta\right)\mathrm{d}\beta \\
= & \sum_{h=1}^{L} \sum_{\underline{\delta}\in\Omega_h} A_{h,\underline{\delta}} \int_{2\hat{\alpha}^2}^{\infty} \frac{1}{\pi}\int_0^{\pi/2} \frac{1}{2}\left(\frac{\beta}{2MNK_r}\right)^{(MN-1)/2} e^{-(\beta g+2MNK_r)/2} I_{MN-1}(\sqrt{2\beta MNK_r})\mathrm{d}\theta\mathrm{d}\beta \\
= & \sum_{h=1}^{L} \sum_{\underline{\delta}\in\Omega_h} \frac{A_{h,\underline{\delta}}}{\pi} \int_0^{\pi/2}\int_{2\hat{\alpha}^2}^{\infty} \frac{\beta^{(MN-1)/2}e^{-(\beta g+2MNK_r)/2}}{2(2MNK_r)^{(MN-1)/2}} I_{MN-1}(\sqrt{2\beta MNK_r})\mathrm{d}\beta\mathrm{d}\theta \\
= & \sum_{h=1}^{L} \sum_{\underline{\delta}\in\Omega_h} \frac{A_{h,\underline{\delta}}}{\pi} \int_0^{\pi/2} \frac{e^{-MNK_r(1-1/g)}}{2g^{MN}} \int_{2g\hat{\alpha}^2}^{\infty} \frac{x^{(MN-1)/2}e^{-(x+2MNK_r)/2}}{(2MNK_r/g)^{(MN-1)/2}} I_{MN-1}\left(\sqrt{\frac{2}{g}xMNK_r}\right)\mathrm{d}x\mathrm{d}\theta \\
= & \sum_{h=1}^{L} \sum_{\underline{\delta}\in\Omega_h} \frac{A_{h,\underline{\delta}}}{\pi} \int_0^{\pi/2} \frac{e^{-MNK_r(1-1/g)}}{g^{MN}} \left(1-F\left(2g\hat{\alpha}^2\left|2MN,\frac{2}{g}MNK_r\right.\right)\right)\mathrm{d}\theta \\
= & \sum_{h=1}^{L} \sum_{\underline{\delta}\in\Omega_h} \frac{A_{h,\underline{\delta}}}{\pi} \int_0^{\pi/2} \frac{e^{-MNK_r(1-1/g)}}{g^{MN}} Q_{MN}\left(\sqrt{\frac{2}{g}MNK_r},\sqrt{2g}\hat{\alpha}\right)\mathrm{d}\theta,
\end{aligned}
\tag{73}
$$

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