

Computation of an Equilibrium in Spectrum Markets for Cognitive Radio Networks

Sang-Seon Byun, *Member, IEEE*, Ilanko Balasingham, *Senior Member, IEEE*,
Athanasios V. Vasilakos, *Senior Member, IEEE* and Heung-No Lee, *Member, IEEE*

Abstract—In this paper, we investigate a market equilibrium in multi-channel sharing cognitive radio networks (CRNs): it is assumed that every subchannel is orthogonally licensed to a single primary user (PU), and can be shared with multiple secondary users (SUs). We model this sharing as a spectrum market where PUs offer SUs their subchannels with limiting the interference from SUs; the SUs purchase the right to transmit over the subchannels while observing the interference limits set by the PUs and their budget constraints. Moreover, we consider each SU limits the total interference that can be invoked from all other SUs, and assume that every transmitting SU marks the interference charges to other transmitting SUs. The utility function of SU is defined as least achievable transmission rate, and that of PU is given by the net profit. We define a market equilibrium in the context of extended Fisher model, and show that the equilibrium is yielded by solving an optimization problem, Eisenberg-Gale convex program. In order to make the solutions of the convex program meet the market equilibrium, we apply monotone-transformation to the utility function of each SU. Furthermore, we develop a distributed algorithm that yields the stationary solutions asymptotically equivalent to the solutions given by the convex program.

Index Terms—Cognitive Radio Networks, Fisher Model, Market Equilibrium, Eisenberg-Gale Convex Program, Distributed Algorithm.

1 INTRODUCTION

IN general markets, when the demand and supply depend upon current price only, there exists an equilibrium under certain conditions, called market equilibrium, where (i) all the traders (i.e., suppliers and consumers) can achieve maximum utilities at least in the Pareto sense; (ii) the total demand for each commodity is equal to the total supply of that commodity; (iii) all the budgets possessed by the consumers are spent completely [1].

Recently, market-based approaches have started being deployed to various cognitive radio network (CRN) scenarios since the behaviors of the wireless users in CRN can be cast easily into those of the traders in the general market. Furthermore the market equilibria comply with the key requirement of the CRN, i.e., spectral efficiency; therefore lately the US Federal Communications Commission (FCC) has employed policies and procedures to bring spectrum trading into CRN, and analogous regulatory efforts are commenced by EU [2], [3], [4].

In the market-based approaches for CRNs, spectra

and interference are regarded as marketable products. In words, primary users (PUs) offer the interference on their licensed spectra to transmitting secondary users (SUs)¹ with collecting certain monetary rewards from SUs; SUs purchase the offered interference on each spectrum by adjusting their transmission powers. Furthermore, every PU has a limitation in her production (i.e., spectra and interference); meanwhile, every SU has limited budget.

In this work, we consider a multi-channel sharing CRN where the frequency range is divided into multiple subchannels, and each subchannel is orthogonally licensed to a single PU. Each PU offers the interference on her subchannels to SUs with limiting the interference from the SUs; the SUs purchase the offered interference observing their budget constraints and the interference limits set by the PUs. Moreover, we reflect the interference among SUs to the spectrum market: every SU limits the total interference from other SUs, and every transmitting SU is required to pay charges for interfering with other SUs. The utility function of SU is defined as the least achievable transmission rate, and the utility of PU is given by the net profit the PU makes.

During the history of market theory, there have been lots of market models developed in accordance with various market scenarios: for instance, *Cournot model* for oligopoly, *Stackelberg model* considering leadership in the production, *Bertrand model* for describing the fierce competition among the sellers, and *Edgeworth model* considering a capacity-limited market². In our work, we apply the market model developed in [6] where the au-

- S. S. Byun is with the School of Information and Communications, Gwangju Institute of Science and Technology, Gwangju, South Korea. E-mail: ssbyun@gist.ac.kr
- I. Balasingham is with the Intervention Center, Rikshospitalet, Oslo University Hospital, Oslo, Norway. E-mail: ilangkob@medisin.uio.no
- A. V. Vasilakos is with the Department of Computer Science, University of Western Macedonia, Kozani, Greece. E-mail: vasilako@ath.forthnet.gr
- H. N. Lee is with the School of Information and Communications, Gwangju Institute of Science and Technology, Gwangju, South Korea. E-mail: heungno@gist.ac.kr

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1. Hereafter, SU implies transmitting SU.

2. Refer to [5] for the details.

thors have extended *Fisher model* for a market consisting of multi-sellers as well as multi-buyers, and divisible goods. Considering concave utility functions for buyers and linear utility functions for sellers, the authors have proved that we can achieve an equilibrium that clears the market by solving a convex optimization problem called *Eisenberg-Gale convex program*. However, in our spectrum market, the purchasing amount is restricted by other buyers as well as sellers unlike the model in [6]: the transmission rate of SU is restricted by the interference limits defined by other SUs as well as PUs. In order to derive an equilibrium with this restriction, the buyers (i.e., SUs) should pay additional charges to other buyers.

Including the interference limit among SUs into the market model in [6], we define the market equilibrium such that the following conditions hold:

- 1) The transmission power vector yielded by the market equilibrium maximizes the joint utility of SUs in the Pareto sense satisfying the constraints of budget, interference to PUs, and interference to other SUs.
- 2) The transmission power vector yielded by the market equilibrium maximizes the utility of every PU, and completely consumes the interference exactly up to the limits set by the PUs.
- 3) With the prices and charges yielded by the market equilibrium, the sum of the initial budget possessed by all SUs equals the sum of the profits made by all PUs plus the sum of the interference charges gathered by all the SUs.

We model our spectrum market using the Eisenberg-Gale convex program whose objective is to maximize the joint utility function of all SUs, which is given by log-sum-utility, over a convex region defined via a set of linear constraints, and prove that the convex program yields the equilibrium holding the above-mentioned conditions. Meanwhile, this convex program becomes *Nash bargaining problem* whose solution satisfies the weak Pareto optimality, and its Lagrangian dual variables turn out to be the prices and charges given by the equilibrium. Furthermore we show that the equilibrium satisfies the *core stability* of PUs. However, the utility functions in the convex program should be concave and homogeneous of degree one. Since we formulate the joint utility function of all SUs as a concave function, we apply a monotone-transformation that transforms a concave function into an equivalent function that is homogeneous of degree one with maintaining the concavity.

In order to find the Lagrangian dual variables, we should solve the system of the linear equations that consist of the Karush-Kuhn-Tucker (KKT) optimality conditions of the convex program. However, the system is generally inconsistent, and which implies that it is impossible to find the exact Lagrangian variables. For this reason, we solve the system with a certain small precision bound. In numerical experiments, we show that approximate Lagrangian variables are found with

quite small precision error, and yield an approximate equilibrium quite close to the exact one.

We also consider a distributed implementation for solving the convex program, which enables PUs and SUs to make their decisions autonomously as follows: given the price of each channel, (i) each SU computes her optimal transmission power vector using *best response dynamics*; (ii) using linear dynamics, each PU updates the price of her each subchannel in proportion to the interference invoked from the SUs, and, meanwhile, each SU updates the interference charge on each subchannel in proportion to the interference invoked from all the other SUs. (iii) repeat (i) and (ii) until the linear dynamics become stationary. We show that the linear dynamics are *asymptotically stable* with any initial points, and the solutions yielded at the stable state is equivalent to the solutions given by the convex program (i.e., *market equilibrium*). By numerical evaluations, we illustrate the market equilibrium given by the distributed algorithm is quite close to the equilibrium yielded by the convex program.

Recently, a distributed algorithm for finding the market equilibrium in Fisher model has been studied in [7]; the authors have applied proportional response dynamics and proved its convergence. To our best knowledge, there is no previous work that has developed a distributed implementation for finding the equilibrium in the extended model given in [6]. Furthermore, in this paper, we consider the situation where the purchasing amount is restricted by buyers (SUs) as well as sellers (PUs).

The rest of this paper is organized as follows: In Section 2 we summarize some important characteristics of the market model studied in [6], and survey some recent market-based approaches for CRNs. In Section 3, we give our spectrum market model. In Section 4, we define the market equilibrium in our spectrum market. In Section 5, we give the Eisenberg-Gale convex program for our spectrum market, and investigate its properties. In Section 6, we present the distributed algorithm for solving the convex program, and analyze its stability and asymptotic optimality. In Section 7, we give several results of the numerical evaluations. Finally, we conclude this paper in Section 8.

2 RELATED WORK

2.1 Market-based Approaches for CRNs

Recently, the market-based approach has been deployed to CRNs by several research efforts.

Xing et al. [8] have considered a spectrum market where different consumers evaluate the same supplier differently according to their applications and locations. Considering limited information, they have developed price dynamics with a stochastic learning algorithm in order to find the optimal price yielding maximum benefit of the suppliers. However they have not addressed the utility of the consumers.

Hong et al. [9] have proposed a fully distributed algorithm - no collaboration among SUs and PUs - that achieves a market equilibrium in multi-channel sharing CRN such that the supply of the spectrum equals to its demand, and the network of SU is stable. They have investigated and presented the convergence condition of the algorithm in terms of the channel gain. Unlike other approaches, they have considered the utilities of PUs as well as SUs. However, they have not investigated the social optimality of the market equilibrium.

Niyato et al. [10], [11] have modeled a multi-level bandwidth sharing in CRN into an interrelated market. They have proposed a price-demand decision algorithm that guarantees the convergence to a market equilibrium with which all the primary and secondary services are satisfied. However, they have assumed that the price-demand decision algorithm is directed by a central authority in each level, and have not considered the way of allocating the spectra to users in each level.

Xu et al. [12] have proposed a secondary network where SUs trade among themselves their channels purchased from PUs in the direction of asymptotic optimal spectrum utilization. To this end, they have devised dynamic double auction mechanism that is conducted by a centralized spectrum broker, and proved the truthfulness and asymptotic efficiency in the total social welfare.

Li et al. [13] have addressed a spectrum auction mechanism between SUs and spectrum owners without any central auctioneer. They have deployed an iterative matching algorithm that achieves the price set in core where no SUs and spectrum owners can negotiate to do better for both. However, they have assumed fixed transmission power, and there is no consideration of the market clearance.

In [14], the authors have handled a two-tier market: spectrum contracts from a PU to SUs in Tier-1, and spectrum redistribution among SUs to satisfy SUs dynamic traffic demands in Tier-2. They have applied Nash bargaining solution in Tier-1 market in order to achieve the fairness between the utility of the PU and the aggregate utility of all the SUs. For Tier-2 market, they have deployed random matching and bilateral bargaining. However, they have considered a single PU, and have not addressed the market clearance.

Xie et al. [15] have addressed the spectrum trading between wireless users - that can be regarded as SUs in a CRN - and a single price manager - that can be regarded as a PU or spectrum broker in a CRN, and investigated a market equilibrium where the market clears, and the budgets of the wireless users are completely consumed. Unlike our work, they have addressed the actual interference among the wireless users in the utility function. Then they have shown that the market equilibrium is given by the solution of a linear complementarity problem, and under the symmetric channel gain and low-rank conditions, they have proved that this problem becomes equivalent to the problem of finding KKT points of a quadratic pro-

gram. Furthermore, they have developed a decentralized tatonnement process that converges to the equilibrium. However, they have not included the manager's utility in the market equilibrium. Moreover, the KKT points of the quadratic program do not guarantee the optimality, and due to this reason, it is not verified whether the distributed algorithm (tatonnement process) converges to the optimal solution even asymptotically.

Koutsopoulos et al. [16] have surveyed various action mechanisms for spectrum allocation in CRN. They have indicated that auction mechanisms (including double auction mechanisms) involve a single seller and multiple buyers, and have no interaction among buyers.

2.2 Brief of The Extended Market Model

Fisher model considers a market consisting of multibuyers and divisible goods. In Fisher model, the budget for each buyer and the amount of each commodity need to be specified, and the utility functions of buyers are assumed to be concave. Then the market equilibrium in Fisher model is given by the price of each commodity that yields optimal utilities of buyers at least in Pareto sense and clears the market: there should be neither surplus nor deficiency in any of the commodities and the budgets [17]. In 1959, Eisenberg and Gale gave a convex program for computing market equilibrium for Fisher model of linear utility functions [18], and in 1961, Eisenberg generalized this to concave homogeneous functions of degree one [19]. In 1954, Nobel laureates Arrow and Debrue generalized Fisher model considering agents who come to the market with initial endowments of goods, and at any set of prices, want to sell all their goods and buy optimal bundles at these prices. The problem again is to find market clearing prices [20].

Jain et al. [6] have extended Fisher model considering the utilities that are homothetic, quasiconcave, and homogeneous functions of arbitrary degree; they also have included sellers' utilities into their model. They have applied the Eisenberg-Gale convex program with the buyers' utilities monotone-transformed. They show that the equilibrium price, given by the Lagrangian dual variables of the convex program, maximizes sellers' utilities as well as buyers' utilities, and clears the market.

In our work, we employ this extended model to our spectrum market since it deals with the market where the buyers are clearly distinguished from sellers, and considers the utilities of sellers as well as those of buyers. However, in our spectrum market, the purchasing amount is restricted by other buyers as well as sellers: that is, the amount of interference that can be purchased by each SU is restricted by the interference limits imposed by other SUs as well as PUs. In order to derive an equilibrium with this restriction, we consider that each buyer (i.e., SU) pays additional charges to all the other buyers. Furthermore, we envisage a distributed implementation of our market model.

We emphasize that the extended Fisher model is a unique market model that enables the following con-

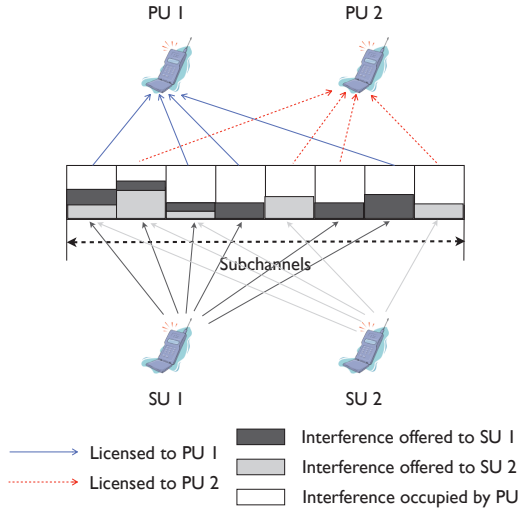


Fig. 1. Multi-channel sharing model with 2 PUs, 2 SUs, and 8 subchannels. Each subchannel is orthogonally allocated to a single PU, and can be shared with all the SUs unless the SUs invoke less interference than the PUs can tolerate.

siderations altogether: (i) strictly distinguished multiple sellers (PUs) from multiple buyers (SUs); (ii) guarantee of Pareto optimality for SUs' utility and the core-existence for PUs' utilities; (iii) guarantee of the market clearance; (iv) distributed implementation with no central authority; (v) providing the way of allocating the spectra to SUs.

3 SPECTRUM MARKET MODEL

We consider a CRN where all the PUs and SUs are located within a limited geographical region. Then we address the spectral resource on the frequency domain taking the multi-channel sharing into account - that is, the whole frequency range is split into multiple subchannels, and each subchannel can be shared by multiple users. In this work, we premise that each subchannel is exclusively licensed to a single PU; it however can be shared with multiple SUs concurrently unless the SUs invoke interference larger than certain limits. (Refer to Fig. 1).

Applying a market concept to our CRN scenario, the subchannels and interference are interpreted into the types of commodity and the quantity of each commodity, respectively. Upon the current prices, each SU decides subchannels and the amount of interference she would like to purchase; each PU updates the price on every her subchannel according to the interference invoked from the SUs.

Ideally, SUs' utility function should reflect actual interference from all the other SUs as well as interference from PUs, which however makes the problem nonconvex if the utility function is given as Shannon capacity [21]. Therefore, we consider the maximum allowable interference and reflect it to the utility function; the utility

function is given by least achievable transmission rate. Then we let SUs make and charge for every subchannel - on which she is transmitting - in proportion to the amount of interference from all the other SUs. Moreover we assume that the maximum allowable interference is given by a central authority in order to let SUs share the interference fairly.

The price as well as the interference charge are marked on every subchannel, and given as a *price per unit interference*.

As a supplier in general market has a limitation on the net supply of its commodity, we envisage that every PU has a limitation on the interference she can offer to SUs over every her subchannel. In addition, like consumers in general markets, every SU cannot spend more budget than she possesses on purchasing the interference from the PUs and paying the interference charges to all the other SUs.

Prior to giving the formal definition of the market equilibrium, we define the following denotations:

- \mathcal{I} : Set of transmitting SUs.
- \mathcal{L} : Set of PUs, and we let $m := |\mathcal{L}|$.
- \mathcal{J} : Set of subchannels, and we let $n := |\mathcal{J}|$
- $u_i : \mathbf{R}_+^n \rightarrow \mathbf{R}_+$: Utility function of SU $i \in \mathcal{I}$.
- $\mathbf{p}_i = [p_{i1}, \dots, p_{in}]^T$: Transmission power vector of SU i .
- p_{ij} : SU i 's transmission power on subchannel $j \in \mathcal{J}$.
- \mathcal{J}_l : Set of subchannels licensed to PU $l \in \mathcal{L}$.
- $v_l : \mathbf{R}^n \rightarrow \mathbf{R}$: Utility function of PU l .
- π_{lj} : Price marked by PU l on subchannel j .
- $\boldsymbol{\pi}_l = [\pi_{l1}, \dots, \pi_{lm}]^T$: Price vector of PU l .
- η_{ij} : Interference charge decided by SU i on subchannel j .
- $\boldsymbol{\eta}_i = [\eta_{i1}, \dots, \eta_{in}]^T$: Vector of interference charges decided by SU i .
- y_{lj} : Limit of allowable interference from all the SUs to PU l on subchannel j .
- γ_{ij} : Maximum allowable interference from all the other SUs to SU i on subchannel j .

Now we develop the spectrum market model with the following key considerations:

- 1) Each SU $i \in \mathcal{I}$ has a concave, scalable utility function u_i with respect to its transmission power vector \mathbf{p}_i . In this paper, the utility of SU is defined as the summation of least achievable rate on every subchannel, that is,

$$u_i(\mathbf{p}_i) = \sum_{j \in \mathcal{J}} B_j \log_2 \left(1 + \frac{p_{ij} G_{ij}}{N_0 + \gamma_{ij} + \Gamma_{ij}} \right) \quad (1)$$

where B_j is the bandwidth of subchannel $j \in \mathcal{J}$, G_{ij} is the channel gain on subchannel j between SU i and its target, Γ_{ij} is the interference invoked from the PU who owns subchannel j to SU i , and N_0 is the thermal noise.

- 2) SUs cannot purchase the interference larger than

the limits set by PUs. That is, $\forall l \in \mathcal{L}$ and $\forall j \in \mathcal{J}_l$,

$$\sum_{i \in \mathcal{I}} p_{ij} G_{ij}^l \leq y_{lj} \quad (2)$$

where G_{ij}^l is the channel gain on subchannel j between SU i and PU l . We assume that each PU has a minimum QoS requirement and sets the interference limit (i.e., y_{lj}) in order that the QoS may be guaranteed.

- 3) Each SU i has an initial endowment of budget $e_i > 0$, and the total budget spent on the purchase of subchannels and the interference charges cannot exceed e_i .
- 4) Each PU l offers her subchannels, i.e., subchannels in \mathcal{J}_l , to SUs with price π_{lj} , and its utility function v_l is defined as the net profit it makes, i.e.,

$$v_l = \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}_l} \pi_{lj} p_{ij} G_{ij}^l. \quad (3)$$

- 5) In every SU, the total interference on subchannel j from all the other SUs cannot exceed the maximum allowable interference: $\forall i \in \mathcal{I}$ and $\forall j \in \mathcal{J}$

$$\sum_{k \in \mathcal{I}, k \neq i} p_{kj} G_{kj}^i \leq \gamma_{ij} \quad (4)$$

where G_{kj}^i is the channel gain on subchannel j between SU k and the target of SU i . In this paper, we assume a symmetric channel gain among SUs: that is, $G_{kj}^i = G_{ij}^k$ where $i \neq k$.

The channel gain reflects the free space path loss defined by Friis transmission equation [22]:

$$G_{ij} = \frac{g_i g_k \lambda_j^2}{(4\pi)^2 d^2 L} \quad (5)$$

where g_i is the transmitter antenna gain, g_k is the receiver antenna gain, d is the transmitter-receiver separation distance in meters, L is the system loss factor not related to propagation, and λ_j is the wavelength in meters on subchannel j . Since the channel gain reflects the wave length of each subchannel, the higher frequency a subchannel has, the less amount of information a user can transmit on it given a fixed transmission power and subchannels with equal bandwidth [8]. In order to let every subchannel have equal opportunity of being purchased, we assume that the frequency range is divided into subchannels in the way that every subchannel yields similar transmission rate given a constant transmission power and separate distance.

Additional assumptions are the following:

- 1) All PUs and SUs can access non-contiguous subchannels in parallel.
- 2) All SUs have the transmission power enough to fully utilize the interference offered by PUs.
- 3) The spectrum trade occurs on every predefined epoch, and all SUs perform their transmissions ceaselessly during the epoch.

- 4) All PUs perform their transmission ceaselessly without any changes in their transmission powers during the epoch.

4 MARKET EQUILIBRIUM

Henceforth, we let y_{lj} have a very large number for all $l \in \mathcal{L}$ and $j \in \mathcal{J}$ but $j \notin \mathcal{J}_l$. Based on the key considerations mentioned in Section 3, we define the equilibrium in the spectrum market following [6]: the equilibrium is defined as a pair of a nonnegative price vector $\pi = [\pi_1, \dots, \pi_m]^T$ and a vector of interference charges $\eta = [\eta_1, \dots, \eta_{|\mathcal{J}|}]^T$ at which there exists a transmission power vector p_i for each SU i such that the following conditions hold:

- 1) The vector p_i maximizes the utility of SU i given her initial endowment of budget e_i and the equilibrium π and η , that is, p_i maximizes u_i over all $p_i \in \mathbf{R}_+^n$ subject to

$$\sum_{l \in \mathcal{L}} \sum_{j \in \mathcal{J}} \pi_{lj} p_{ij} G_{ij}^l + \sum_{j \in \mathcal{J}} p_{ij} \sum_{k \in \mathcal{I}, k \neq i} \eta_{kj} G_{ij}^k \leq e_i, \quad (6)$$

and constraint (4).

- 2) For each PU l , the vector p_i maximizes the profit v_l subject to constraint (2).
- 3) The total interference offered by all PUs equals the total interference consumed by all SUs, that is, $\sum_{l \in \mathcal{L}} \sum_{j \in \mathcal{J}} y_{lj} = \sum_{i \in \mathcal{I}} \sum_{l \in \mathcal{L}} \sum_{j \in \mathcal{J}} p_{ij} G_{ij}^l$.
- 4) The sum of the initial budget possessed by SU i equals the sum of the prices paid to all PUs by SU i plus the sum of the interference charges paid to all the other SUs by SU i , that is, $e_i = \sum_{l \in \mathcal{L}} \sum_{j \in \mathcal{J}} \pi_{lj} p_{ij} G_{ij}^l + \sum_{j \in \mathcal{J}} p_{ij} \sum_{k \in \mathcal{I}, k \neq i} \eta_{kj} G_{ij}^k$.

Therefore, the market equilibrium in this model is also known as market clearing equilibrium since it clears not only all the commodities offered by suppliers but also all the initial budget possessed by consumers; that is, it clears all the budget possessed by SUs as well as all the interference offered by PUs.

However, if SU's maximum allowable interference, i.e., γ_{ij} for any i and j , is set significantly small or null in the worst case, the interference offered by PUs may not be consumed entirely. Then the market clearance cannot be guaranteed. We present this problem in section 5 in detail.

5 EISENBERG-GALE CONVEX PROGRAM

In this section, we develop a convex program, called Eisenberg-Gale convex program [6], [23], that yields the market equilibrium defined in the previous section. The convex program is to maximize the joint utility function

3. Over subchannels that a PU doesn't own, we set the interference limit very large numbers. Then we can drop the subscript l from \mathcal{J}_l . Surely, $\forall j \notin \mathcal{J}_l$, y_{lj} doesn't belong to the commodities of PU l .

of all SUs given by log-sum-utility over a convex region defined via a set of linear constraints.

The Eisenberg-Gale convex program can compute the market equilibrium only when the utility functions are homogeneous of degree one as well as concave. As given in (1), the utility function is concave but not homogeneous of degree one. Therefore, prior to presenting the convex program, we give a formal method of transforming a non-homogeneous function into an equivalent function homogeneous of degree one.

5.1 Monotone-Transformation of the Utility Function

Most of all, we need the following definition [24]:

Definition 1: Given a function $u : \mathbf{R}_+^n \rightarrow \mathbf{R}_+$:

- 1) u is *strictly monotonic* if for any $\mathbf{p}, \bar{\mathbf{p}} \in \mathbf{R}_+^n$, $\mathbf{p} > \bar{\mathbf{p}}$ implies that $u(\mathbf{p}) > u(\bar{\mathbf{p}})$;
- 2) Let u be a strictly monotonic function. Then u is *homothetic* if for any $\mathbf{p}, \bar{\mathbf{p}} \in \mathbf{R}_+^n$ and any $\alpha > 0$, $u(\mathbf{p}) \geq u(\bar{\mathbf{p}})$ iff $u(\alpha\mathbf{p}) \geq u(\alpha\bar{\mathbf{p}})$;
- 3) u is *homogeneous of degree one* if for any $\mathbf{p} \in \mathbf{R}_+^n$ and any $\alpha > 0$, $u(\alpha\mathbf{p}) = \alpha u(\mathbf{p})$.

It is not difficult to check whether u_i given by (1) is continuous, strictly monotonic and homothetic, but not homogeneous of degree one. Therefore we apply a *monotone-transformation* [6] that preserves strict monotonicity, concavity, and homotheticity. The monotone-transformation is given by the following theorem:

Theorem 1: Let $u : \mathbf{R}_+^n \rightarrow \mathbf{R}_+$ be a continuous, strict monotonic, concave, homothetic function. Then there is a monotone-transformation yielding a function $f : \mathbf{R}_+^n \rightarrow \mathbf{R}_+$ that is homogeneous of degree one, and preserves continuity, strict monotonicity, concavity, and homotheticity, and satisfies:

- 1) If $u(\mathbf{p}) = 0$, then $f(\mathbf{p}) = 0$.
- 2) If $u(\mathbf{p}) \neq 0$, then there exists a unique $\alpha \in \mathbf{R}_+$ such that $u(\mathbf{p}/\alpha) = 1$, and $f(\mathbf{p}) = \alpha$.

Consequently, the monotone-transformation can be done by finding α that satisfies the following nonlinear equation:

$$u(\mathbf{p}/\alpha) = 1. \quad (7)$$

5.2 Convex Program Yielding Market Equilibrium

By the monotone-transformation, we can transform u_i to f_i that is homogeneous of degree one. Then we develop the Eisenberg-Gale convex program that yields the market equilibrium as follows:

$$\text{maximize } \sum_{i \in \mathcal{I}} e_i \ln(f_i) \quad (8)$$

subject to

$$\sum_{k \in \mathcal{I}, k \neq i} p_{kj} G_{kj}^i \leq \gamma_{ij}, \quad \forall i \in \mathcal{I} \text{ and } \forall j \in \mathcal{J}; \quad (9)$$

$$\sum_{i \in \mathcal{I}} p_{ij} G_{ij}^l \leq y_{lj}, \quad \forall l \in \mathcal{L} \text{ and } \forall j \in \mathcal{J}. \quad (10)$$

As we have addressed in Section 4, if γ_{ij} is set too small, the market clearance cannot be guaranteed. Considering non-zero channel gains, we present the problem in detail:

- In case $\gamma_{ij} = 0$ for all i and j , all p_{ij} should be null. Then, the conditions for the market equilibrium (i.e., 3) and 4)) can never be met.
- In case γ_{ij} for all i and j is non-zero, but it has significantly small value, p_{ij} should have also small value in order to meet the constraint given in (9). Then the constraint (10) may not be tight always. The tightness of the constraint (10) is one of the condition for the market equilibrium, i.e., 3). If the tightness of the constraint (10) is not met, the price of the subchannels that are not entirely purchased by SUs is given as zero according to the KKT condition given in (15), and in turn the condition 4) may not hold either.

To prevent the above problem, we need to add the following necessity condition for the existence of the market equilibrium: every $\gamma_{ij} \forall i \in \mathcal{I}$ and $\forall j \in \mathcal{J}$ is non-zero and has a value that makes a solution vector \mathbf{p} satisfy $\sum_{i \in \mathcal{I}} p_{ij} G_{ij}^l = y_{lj} \quad \forall l \in \mathcal{L}$ and $\forall j \in \mathcal{J}$, and

$$\sum_{k \in \mathcal{I}, k \neq i} p_{kj} G_{kj}^i \leq \gamma_{ij} \quad \forall i \in \mathcal{I} \text{ and } \forall j \in \mathcal{J} \text{ always. Intuitively,}$$

SUs will always utilize the available interference entirely bounded by constraint (10) in order to maximize their collaborative utility given in (8) if constraint (10) is set tighter than constraint (9), and therefore the market clearance can be guaranteed. As an instance, we consider a spectrum market where the channel gains over all SUs, PUs and channels have an identical value, i.e., identical G_{ij}^l and G_{kj}^i for all $i, k \in \mathcal{I}$, $j \in \mathcal{J}$, and $l \in \mathcal{L}$. Additionally we let $\gamma_{ij} = c$ and $y_{lj} = f$ for all $i \in \mathcal{I}$, $j \in \mathcal{J}$, and $l \in \mathcal{L}$. Then, if $c \geq f$, constraint (10) over all l and j becomes always tight to achieve the maximum utility.

For all l and $j \notin \mathcal{J}_l$, we give a large value to y_{lj} . In this case, π_{lj} should be zero to satisfy the optimality condition given in (15), and this setting is reasonable. Then we draw the following remark from the convex program:

Remark 1: The solution of the above Eisenberg-Gale convex program (henceforth, convex program) is often regarded as *Nash bargaining solution* [25] with zero disagreement point. Therefore it has a unique solution vector that satisfies *Pareto optimality* due to the strict concavity of the objective function and linearity of the constraints [26].

Let \tilde{p}_{ij} denote the optimal solutions to the convex program. Notice that $f_i(\tilde{\mathbf{p}}_i) > 0$ for all i . Now we have the following KKT optimality conditions with the corresponding Lagrangian multipliers η_{ij} and π_{lj} :

$$\sum_{k \in \mathcal{I}, k \neq i} \tilde{p}_{kj} G_{kj}^i \leq \gamma_{ij}, \quad \forall i \in \mathcal{I} \text{ and } \forall j \in \mathcal{J}, \quad (13)$$

$$\tilde{p}_{ij} \left(\frac{e_i}{f_i(\tilde{\mathbf{p}}_i)} \frac{\partial f_i(\tilde{\mathbf{p}}_i)}{\partial p_{ij}} - \sum_{l \in \mathcal{L}} \pi_{lj} G_{ij}^l - \sum_{k \in \mathcal{I}, k \neq i} \eta_{kj} G_{kj}^i \right) = 0, \quad \forall i \in \mathcal{I} \text{ and } \forall j \in \mathcal{J} \quad (11)$$

$$\frac{e_i}{f_i(\tilde{\mathbf{p}}_i)} \frac{\partial f_i(\tilde{\mathbf{p}}_i)}{\partial p_{ij}} - \sum_{l \in \mathcal{L}} \pi_{lj} G_{ij}^l - \sum_{k \in \mathcal{I}, k \neq i} \eta_{kj} G_{kj}^i \leq 0, \quad \forall i \in \mathcal{I} \text{ and } \forall j \in \mathcal{J} \quad (12)$$

$$\sum_{i \in \mathcal{I}} \tilde{p}_{ij} G_{ij}^l \leq y_{lj}, \quad \forall l \in \mathcal{L} \text{ and } \forall j \in \mathcal{J}, \quad (14)$$

$$\pi_{lj} \left(\sum_{i \in \mathcal{I}} \tilde{p}_{ij} G_{ij}^l - y_{lj} \right) = 0, \quad \forall l \in \mathcal{L} \text{ and } \forall j \in \mathcal{J}, \quad (15)$$

$$\eta_{ij} \left(\sum_{k \in \mathcal{I}, k \neq i} \tilde{p}_{kj} G_{kj}^i - \gamma_{ij} \right) = 0, \quad \forall i \in \mathcal{I} \text{ and } \forall j \in \mathcal{J}, \quad (16)$$

and (11) and (12).

Subsequently, we establish the following linear program for each PU l :

LP1:

$$\text{maximize } \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \pi_{lj} p_{ij} G_{ij}^l \quad (17)$$

subject to

$$\sum_{i \in \mathcal{I}} p_{ij} G_{ij}^l \leq y_{lj}, \quad \forall j \in \mathcal{J}. \quad (18)$$

We prove that π_l and η_i is the equilibrium price, and $\tilde{\mathbf{p}}$ is the optimal solution of LP1 as well. This proof begins with Euler's theorem [24]:

Theorem 2 (Euler's theorem): Let $f(\mathbf{p})$ be a homogeneous function of degree 1 on \mathbf{R}_+^n . Then, for all \mathbf{p} ,

$$p_1 \frac{\partial f(\mathbf{p})}{\partial p_1} + p_2 \frac{\partial f(\mathbf{p})}{\partial p_2} + \dots + p_n \frac{\partial f(\mathbf{p})}{\partial p_n} = f(\mathbf{p}). \quad (19)$$

Then the following core theorem holds:

Theorem 3: The optimal solution to the convex program optimizes the utility of each SU i and the utility of each PU l , and the Lagrangian multipliers, π and η , are the equilibrium. In addition, the interference offered by all PUs is entirely exhausted by SUs, and the initial budget possessed by all SUs is fully spent and precisely equals to the total profit earned by the PUs plus the total interference charges. Namely, the market clears.

Proof: Summing (11) over $j \in \mathcal{J}$, we get

$$\begin{aligned} & \sum_{j \in \mathcal{J}} \frac{e_i}{f_i(\tilde{\mathbf{p}}_i)} \frac{\partial f_i(\tilde{\mathbf{p}}_i)}{\partial p_{ij}} \tilde{p}_{ij} - \sum_{j \in \mathcal{J}} \tilde{p}_{ij} \sum_{l \in \mathcal{L}} \pi_{lj} G_{ij}^l \\ & - \sum_{j \in \mathcal{J}} \tilde{p}_{ij} \sum_{k \in \mathcal{I}, k \neq i} \eta_{kj} G_{kj}^i = 0, \quad \forall i \in \mathcal{I}. \end{aligned} \quad (20)$$

By the Euler's theorem, the first term in the left side of (20) is reduced to e_i simply. Then, since we assume $G_{kj}^i = G_{ij}^k$, $\forall i, k \in \mathcal{I}$ and $\forall j \in \mathcal{J}$ where $i \neq k$,

$$\begin{aligned} e_i &= \sum_{j \in \mathcal{J}} \tilde{p}_{ij} \sum_{l \in \mathcal{L}} \pi_{lj} G_{ij}^l \\ &+ \sum_{j \in \mathcal{J}} \tilde{p}_{ij} \sum_{k \in \mathcal{I}, k \neq i} \eta_{kj} G_{ij}^k = 0, \quad \forall i \in \mathcal{I}, \end{aligned} \quad (21)$$

which implies that each SU spends her initial budget completely under the equilibrium π and η . The first term in the right side of (21) indicates the total price to be paid to all PUs by SU i , and the second term indicates the total interference charge to be paid to all other SUs by SU i .

We next consider the dual program of LP1 for each PU l with dual variables w :

LP2:

$$\text{minimize } \sum_{j \in \mathcal{J}} y_{lj} w_{lj} \quad (22)$$

subject to

$$w_{lj} G_{ij}^l = \pi_{lj} G_{ij}^l, \quad \forall i \in \mathcal{I} \text{ and } \forall j \in \mathcal{J}; \quad (23)$$

$$w_{lj} \geq 0, \quad \forall j \in \mathcal{J}. \quad (24)$$

Let $\tilde{\mathbf{p}}$ be the optimal solution of LP2. By the complementary slackness condition [27], the following equation holds as well:

$$w_{lj} \left(\sum_{i \in \mathcal{I}} \tilde{p}_{ij} G_{ij}^l - y_{lj} \right) = 0, \quad \forall l \in \mathcal{L} \text{ and } \forall j \in \mathcal{J}. \quad (25)$$

Assuming that $G_{ij}^l \neq 0$ for all i, j and l , (23) becomes $w_{lj} = \pi_{lj}$. Thus,

$$\pi_{lj} \left(\sum_{i \in \mathcal{I}} \tilde{p}_{ij} G_{ij}^l - y_{lj} \right) = 0, \quad \forall l \in \mathcal{L} \text{ and } \forall j \in \mathcal{J}. \quad (26)$$

We see that $\tilde{\mathbf{p}}$ also satisfies (26), and thus it can be the optimal solution of the LP2 as well. Hence, together with Remark 1, we draw a conclusion that all the PUs and SUs can achieve maximum utilities with respect to the equilibrium π and η .

Finally we show that the market clears. Summing (21) over all $i \in \mathcal{I}$, we get

$$\begin{aligned} \sum_{i \in \mathcal{I}} e_i &= \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \tilde{p}_{ij} \sum_{l \in \mathcal{L}} \pi_{lj} G_{ij}^l \\ &+ \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \tilde{p}_{ij} \sum_{k \in \mathcal{I}, k \neq i} \eta_{kj} G_{ij}^k, \end{aligned} \quad (27)$$

and by (15),

$$\begin{aligned} \sum_{i \in \mathcal{I}} e_i &= \sum_{j \in \mathcal{J}} \sum_{l \in \mathcal{L}} \pi_{lj} y_{lj} \\ &+ \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \tilde{p}_{ij} \sum_{k \in \mathcal{I}, k \neq i} \eta_{kj} G_{ij}^k. \end{aligned} \quad (28)$$

Eventually, we conclude that the equilibrium given by the Lagrangian multipliers clears the market. \square

The equilibrium can be computed by solving the system of the linear equations (11), (12), (15), and (16) using the optimal solutions of the convex program. It is known that the system of the linear equations has a unique solution if it is consistent [17]. However, the system is normally inconsistent since there are more equations than the unknown variables. Thus we need to provide a certain precision bound on each linear equation in order to achieve an approximate equilibrium at least. The precision bound is dependent on the instance of the problem such as the sizes of PUs, SUs, subchannels, and non-zero solutions. In Section 7, we evaluate numerically the quality of the solution varying the precision bound given a CRN instance.

Besides, in order to solve the system of the linear equations, we need to compute the partial derivatives of the monotone-transformed function (required in (11) and (12)), and it is given by the following lemma [6]:

Lemma 1: If we let $f(\mathbf{p}) = \alpha$, then the partial derivatives to p_j of f is given by

$$\frac{\partial f(\mathbf{p})}{\partial p_j} = \alpha \frac{\partial u(\mathbf{p}/\alpha)/\partial p_j}{\nabla u(\mathbf{p}/\alpha)^T \mathbf{p}}. \quad (29)$$

5.3 Core Stability of the Production Vector

In this section, we show that the solution $\tilde{\mathbf{p}}$ yielded by the convex program is in the *core* of NTU (nontransferable utility) game in the context of cooperative game theory [28].

In a cooperative game, the existence of non-empty core guarantees that no player will break away from the grand coalition (i.e., the cooperation of all the players) since the payoffs achieved by the cooperation within any subcoalition are not larger than the payoff yielded by the cooperation of the grand coalition. Therefore, if the solution $\tilde{\mathbf{p}}$ is in the core of the spectrum market, we guarantee that no PU will leave this market.

Denoting the core of the spectrum market as $C(V)$, the core of an NTU game is defined as:

Definition 2: For a $S \subseteq \mathcal{L}$, let $V(S) = \{v_l(\mathbf{p}) : l \in S\}$. Then the core of the spectrum market, $C(V)$ is defined as the set of all undominated imputations, i.e., $\tilde{\mathbf{p}} \in C(V)$, if and only if there is neither $S \subset \mathcal{L}$, $S \neq \emptyset$, nor \mathbf{p} such that $v_l(\mathbf{p}) > v_l(\tilde{\mathbf{p}})$ for all $l \in S$.

Theorem 4: The solution vector $\tilde{\mathbf{p}}$ of the convex program belongs to $C(V)$.

Proof: In this proof, we apply the proof by contradiction. Thus this proof begins with supposing $\tilde{\mathbf{p}} \notin C(V)$. Then there exists $S \subset \mathcal{L}$, $S \neq \emptyset$, and some allocation $\hat{\mathbf{p}}$ such that

$$v_l(\hat{\mathbf{p}}) > v_l(\tilde{\mathbf{p}}) \quad (30)$$

for all $l \in S$. That is,

$$\sum_{j \in \mathcal{J}} \pi_{lj} \sum_{i \in \mathcal{I}} \hat{p}_{ij} G_{ij}^l > \sum_{j \in \mathcal{J}} \pi_{lj} \sum_{i \in \mathcal{I}} \tilde{p}_{ij} G_{ij}^l, \quad \forall l \in S. \quad (31)$$

Then, by (15),

$$\sum_{j \in \mathcal{J}} \pi_{lj} \sum_{i \in \mathcal{I}} \hat{p}_{ij} G_{ij}^l > \sum_{j \in \mathcal{J}} \pi_{lj} y_{lj}, \quad \forall l \in S \quad (32)$$

In order to satisfy (32), the following relation should be satisfied in any l and j :

$$\sum_{i \in \mathcal{I}} \hat{p}_{ij} G_{ij}^l > \sum_{j \in \mathcal{J}} y_{lj}, \quad \forall l \in S. \quad (33)$$

However, the inequality (33) violates the constraint of the total interference. This contradiction proves that $\tilde{\mathbf{p}}$ is undominated, and therefore belongs to $C(V)$. \square

Consequently, it is guaranteed that the interference solutions given by the convex program let no PU break away from the spectrum market.

6 DISTRIBUTED ALGORITHM

In this section, we develop a distributed approach whose stationary point is asymptotically equivalent to the optimal solution given by the convex program.

6.1 The Algorithm

A natural class of dynamics in multiplayer noncooperative system is the *best-response dynamics* where each player updates her strategy to maximize her utility given the strategies of other players [29]. In this algorithm, each SU deploys the best response dynamics to maximize her individual utility given the vectors of price and interference charges. That is, the best response of SU i is given by

$$\beta_i(\pi, \eta) = \arg \max_{\mathbf{p}_i: \text{s.t. (6)}} e_i \ln f_i(\mathbf{p}_i) \quad (34)$$

Accordingly, the algorithm with the best-response dynamics is given as follows:

- 1: Initialize $\pi(0)$ and $\eta(0)$;
- 2: $t \leftarrow 0$;
- 3: **loop**
- 4: **for** each $i \in \mathcal{I}$ **do**
- 5: Find the best response $\beta_i(t)$ given $\pi(t)$ and $\eta(t)$;
- 6: **end for**
- 7: **for** each $l \in \mathcal{L}$ and $j \in \mathcal{J}$ **do**
- 8: **if** $j \notin \mathcal{J}$ **then**
- 9: $\pi_{lj} = 0$;
- 10: **else**
- 11: Update the price such that

$$\dot{\pi}_{lj} = \alpha \left(\sum_{i \in \mathcal{I}} p_{ij}(t) G_{ij}^l - y_{lj} \right)_{\pi_{lj}}^+; \quad (35)$$

- 12: **end if**
- 13: **end for**
- 14: **for** each $i \in \mathcal{I}$ and $j \in \mathcal{J}$ **do**
- 15: **if** $p_{ij}(t) = 0$ **then**
- 16: $\eta_{ij} = 0$;


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17:   else
18:     Adjust the interference charge by

$$\dot{\eta}_{ij} = \alpha \left( \sum_{k \in \mathcal{I}, k \neq i} p_{kj}(t) G_{kj}^l - \gamma_{ij} \right)_{\eta_{ij}}^+ \quad (36)$$

19:   end if
20: end for
21: if  $\tilde{\pi}_{lj} < \epsilon$  and  $\dot{\eta}_{ij} < \epsilon$  for all  $l \in \mathcal{L}$  and  $j \in \mathcal{J}$  then
22:   Terminate the loop;
23: else
24:    $t \leftarrow t + 1$ ;
25:   Continue the loop;
26: end if
27: end loop

```

In this algorithm, α and ϵ indicate the adjustment speed of the linear dynamics and the termination condition of the algorithm, respectively. If we set the adjustment speed too small, the algorithm requires more iterations until it converges. On the other hand, setting it too large can make the algorithm oscillate. There is no formal way of finding an adequate value for the adjustment speed except trial-and-error. In this paper, an adequate adjustment speed is found by testing the algorithm with various adjustment speed. $(a)_b^+$ implies $\max(a, 0)$ if $b = 0$, and equal to a if $b > 0$. In every time t , by (34), each SU finds the transmission power vector that is her best response to the current price and interference charges. When each SU computes its best response dynamics, she should know the amount of interference from PUs as well as the channel gain to her target. By (35), each PU updates the price on each her subchannel according to the interference from all SUs. By (36), each SU updates her interference charge on each subchannel according to the interference from other SUs. We note that PUs and SUs update their interference charges with only the amount of interference they observe. Then the price updated by each PU (i.e., π) is delivered to all SUs, and the price updated by each SU (i.e., η) is delivered to other SUs. These price lists are the only feedbacks that should be delivered.

6.2 Stability Analysis

In this subsection we prove the asymptotic stability of the linear dynamics given by (35) and (36). To this end, we develop the following theorem:

Theorem 5: The linear dynamics given by (35) and (36) are globally asymptotically stable.

Proof: See Appendix A. \square

We see that, as the distributed algorithm proceeds, SUs' utilities approach to those yielded by the convex program.

6.3 Investigation of Asymptotic Equivalence to the Convex Program

We have shown the linear dynamics are asymptotically stable. Then, assuming the system of the linear equations

(11), (12), (15), and (16) is consistent, we can prove that the distributed algorithm yields the solutions *asymptotically equivalent* to the solutions of the convex program. That is, the solutions yielded at $t = \infty$ is equivalent to the solutions of the convex program. However, as discussed in Section 5.2, it is not easy to validate the equivalence because of the inconsistency in the system of the linear equations. Nonetheless, by numerical experiments, we illustrate that the utilities of SUs determined by the distributed algorithm are almost identical to those yielded by the convex program in spite of the inconsistency, and which will be explained in Section 7. We also show that the equilibrium yielded by the distributed algorithm also meets the KKT conditions within a small tolerance.

We give the following theorem:

Theorem 6: Postulating that the system of the linear equations (11), (12), (15), and (16) are consistent, the solutions and equilibrium obtained by the distributed algorithm are asymptotically equivalent to those yielded by the convex program and its KKT conditions.

Proof: We denote the best response of SU i at $t = \infty$ as $\tilde{\beta}_i$, the price made by PU l on subchannel j at $t = \infty$ as $\tilde{\pi}_{lj}$, and the interference charge made by SU i on subchannel j at $t = \infty$ as $\tilde{\eta}_{ij}$, then the KKT conditions (37) ~ (39) hold with Lagrangian multiplier $\kappa_i \geq 0$.

$$\sum_{j \in \mathcal{J}} \sum_{l \in \mathcal{L}} \tilde{\pi}_{lj} \tilde{\beta}_{ij} G_{ij}^l + \sum_{j \in \mathcal{J}} \tilde{\beta}_{ij} \sum_{k \in \mathcal{I}, k \neq i} \tilde{\eta}_{lj} G_{ij}^k \leq e_i \quad (40)$$

Summing (37) over all $j \in \mathcal{J}$, we get (41), and, by Euler's theorem,

$$e_i = \kappa_i \left(\sum_{j \in \mathcal{J}} \sum_{l \in \mathcal{L}} \tilde{\pi}_{lj} \tilde{\beta}_{ij} G_{ij}^l + \sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{I}, k \neq i} \tilde{\eta}_{lj} \tilde{\beta}_{ij} G_{ij}^k \right). \quad (42)$$

Substituting e_i in (39) with (42), we get (43).

For all $i \in \mathcal{I}$, at least one of $\tilde{\mathbf{p}}_i$ should be non-zero in order to make $f_i(\tilde{\beta}_i) > 0$. Moreover, due to the strict monotonicity of f_i , the partial derivatives of f_i should be always positive. Therefore at least one element of $\tilde{\pi}$ and $\tilde{\eta}$ should be non-zero in order to make (37) hold. We also set all the channel gains non-zeros. Subsequently, in order to make both (37) and (43) hold for all i , κ_i should be 1. Then, the KKT condition (37) and (38) turn out to be identical with (11) and (12), respectively, if we let $\tilde{\beta}_i = \tilde{\mathbf{p}}_i$ for all i , $\tilde{\pi} = \pi$, and $\tilde{\eta} = \eta$.

Since the two linear dynamics (i.e., (35) and (36)) are stable, there exist some $l \in \mathcal{L}$ and $j \in \mathcal{J}$ such that

$$\sum_{i \in \mathcal{I}} \tilde{\beta}_{ij}(t) G_{ij}^l = y_{lj} \quad (44)$$

if $\tilde{\pi}_{lj} > 0$, or

$$\sum_{i \in \mathcal{I}} \tilde{\beta}_{ij}(t) G_{ij}^l < y_{lj} \quad (45)$$

if $\tilde{\pi}_{lj} = 0$; in a similar way, there exist some $i \in \mathcal{I}$ and $j \in \mathcal{J}$ such that

$$\sum_{k \in \mathcal{I}, k \neq i} \tilde{\beta}_{ij}(t) G_{kj}^i = \gamma_{ij} \quad (46)$$

$$\tilde{\beta}_{ij} \left(\frac{e_i}{f_i(\tilde{\beta}_i)} \frac{\partial f_i(\tilde{\beta}_i)}{\partial p_{ij}} - \kappa_i \left(\sum_{l \in \mathcal{L}} \tilde{\pi}_{lj} G_{ij}^l + \sum_{k \in \mathcal{I}, k \neq i} \tilde{\eta}_{ij} G_{ij}^k \right) \right) = 0, \forall j \in \mathcal{J} \quad (37)$$

$$\frac{e_i}{f_i(\tilde{\beta}_i)} \frac{\partial f_i(\tilde{\beta}_i)}{\partial p_{ij}} - \kappa_i \left(\sum_{l \in \mathcal{L}} \tilde{\pi}_{lj} G_{ij}^l + \sum_{k \in \mathcal{I}, k \neq i} \tilde{\eta}_{ij} G_{ij}^k \right) \leq 0, \forall j \in \mathcal{J} \quad (38)$$

$$\kappa_i \left(\sum_{j \in \mathcal{J}} \tilde{\beta}_{ij} \sum_{l \in \mathcal{L}} \tilde{\pi}_{lj} G_{ij}^l + \sum_{j \in \mathcal{J}} \tilde{\beta}_{ij} \sum_{k \in \mathcal{I}, k \neq i} \tilde{\eta}_{lj} G_{ij}^k - e_i \right) = 0 \quad (39)$$

$$\sum_{j \in \mathcal{J}} \frac{e_i}{f_i(\tilde{\beta}_i)} \frac{\partial f_i(\tilde{\beta}_i)}{\partial p_{ij}} \tilde{\beta}_{ij} = \kappa_i \left(\sum_{j \in \mathcal{J}} \sum_{l \in \mathcal{L}} \tilde{\pi}_{lj} \tilde{\beta}_{ij} G_{ij}^l + \sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{I}, k \neq i} \tilde{\eta}_{lj} \tilde{\beta}_{ij} G_{ij}^k \right) \quad (41)$$

if $\tilde{\eta}_{ij} > 0$, and

$$\sum_{k \in \mathcal{I}, k \neq i} \tilde{\beta}_{ij}(t) G_{kj}^i < \gamma_{ij} \quad (47)$$

if $\tilde{\eta}_{ij} = 0$. We notice that (44) and (45) are equivalent to the KKT condition (13), and (46) and (47) are equivalent to the KKT condition (16) if we let $\tilde{\beta}_i = \tilde{\mathbf{p}}_i$ for all i , $\tilde{\pi} = \pi$, and $\tilde{\eta} = \eta$. Finally, (44) ~ (47) satisfy both (13) and (14) as well if we let $\tilde{\beta}_i = \tilde{\mathbf{p}}_i$ for all i .

For the reasons stated above, the stable solutions and the equilibrium yielded by the distributed algorithm satisfies the KKT condition of the convex program. Since the equilibrium as well as the solutions of the convex program are unique, we confirm $\tilde{\beta}_i = \tilde{\mathbf{p}}_i$ for all i , $\tilde{\pi} = \pi$, and $\tilde{\eta} = \eta$. Consequently, we conclude that the solutions and equilibrium yielded by the distributed algorithm are asymptotically equivalent to those given by the convex program and its KKT conditions. \square

7 NUMERICAL EVALUATIONS

7.1 Experimental Setup

We generate a CRN within a 500×500 square, and consider the frequency range of 54-862 MHz TV band following the IEEE 802.22 standard [30]. As mentioned in Section 3, we divide the frequency range into subchannels in the way that every subchannel yields equal transmission rate as possible given a constant transmission power. We vary the sizes of SUs, PUs, and subchannels according to the type of the experiment we perform. Besides, we use Interior Point Optimizer (IPOPT) [31] for solving the convex program and the best response dynamics, and GNU Linear Programming Kit (GLPK) [32] for solving the system of the linear equations. Additional experimental parameters are the following:

- e_i : randomly chosen from (0, 1.0]
- $y_{lj} : \forall l \in \mathcal{L}$ and $\forall j \in \mathcal{J}$, 1e-08
- $\gamma_{ij} : \forall i \in \mathcal{I}$ and $\forall j \in \mathcal{J}$, 1e-08
- Initial π_{lj} for the distributed algorithm: $\forall l \in \mathcal{L}$ and $\forall j \in \mathcal{J}$, 6e06
- Initial η_{ij} for the distributed algorithm: $\forall i \in \mathcal{I}$ and $\forall j \in \mathcal{J}$, 6e06
- Antenna gain: 1.0 for both transmitter side and receiver side
- System loss factor: 1.0
- Speed of light: 3e08 m/s
- Thermal noise: 1e-10
- PU's transmission power: 0.1W for all PUs

7.2 Illustration of the Strict Monotonicity

First we illustrate the strict monotonicity of the monotone-transformed utility function, i.e., f_i . To this end, we consider two subchannels and compute the function values of f_i varying the transmission power on each subchannel. Fig. 2b shows the results. We plot also the function values of the original utility function, i.e., u_i in Fig. 2a. As shown in Fig. 2b, the function value increases strictly monotonically as the transmission power on each subchannel increases, and which illustrates the strict monotonicity.

7.3 Illustration and Evaluation of Equilibrium Price

Next we measure the transition of the total interference demand according to the change of the prices given by PUs when the function (8) is maximized subject to the budget constraint (i.e., (6)). For these experiments, we locate 3 SUs and 2 PUs accommodating 2 subchannels⁴, and ignore the limit of the interference from other SUs, that is ignore (9). Thus all the budget possessed by the SUs will be paid to the PUs. Setting $y_{lj} = 8e-08$

4. We let each PU have one subchannel.

$$\kappa_i (1 - \kappa_i) \left(\sum_{j \in \mathcal{J}} \sum_{l \in \mathcal{L}} \tilde{\pi}_{lj} \tilde{\beta}_{ij} G_{ij}^l + \sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{I}, k \neq i} \tilde{\eta}_{lj} \tilde{\beta}_{ij} G_{ij}^k \right) = 0 \quad (43)$$

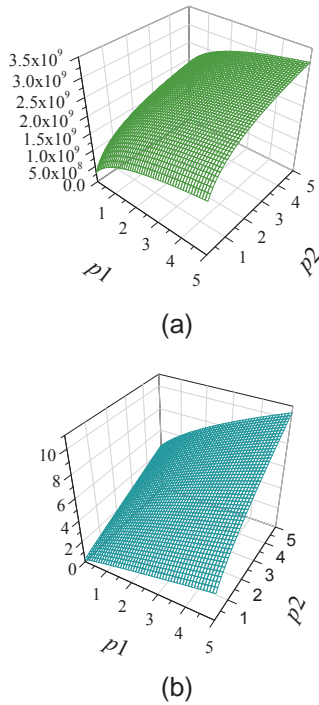


Fig. 2. Illustrations of the strict monotonicity of u_i and f_i .

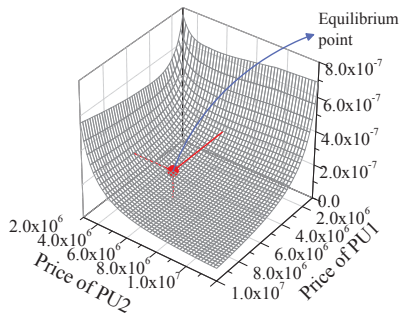


Fig. 3. The transition of the total interference demand according to the prices when we locate 3 SUs, 2 PUs, and 8 subchannels. This figure also plots the equilibrium point; the price pair given by the equilibrium point is the equilibrium price.

for all l, j , the measured results are shown in Fig. 3, and it is observed that the demand decreases as the prices increase. By the definition, the equilibrium price is obtained when the total interference demand equals $2 \times \sum_{l,j} y_{lj}$, that is, $1.6e-07$.

The next set of experiments are done in order to evaluate the precision of the equilibrium price obtained by solving the system of the linear equations (11) ~ (16) under various precision bound. For this set of

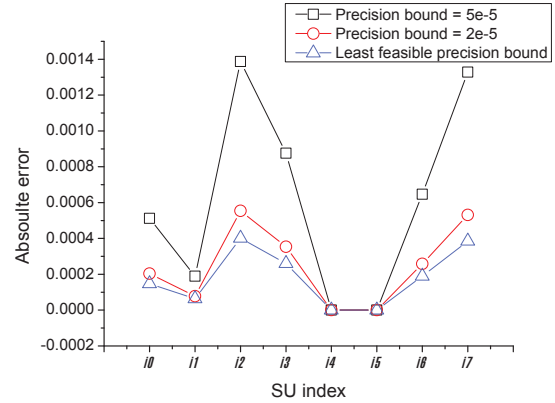


Fig. 4. The absolute gap between the initial budget of each SU and her payment decided by the KKT conditions of the convex program.

experiments, we locate 8 SUs and 8 PUs, and arrange 32 subchannels. Then we measure the absolute gap between the initial budget and the payment of each SU under three different precision bounds. Fig. 4 shows the results. If the precision bound is made lower than $1.4508e-5$, then the system becomes inconsistent. As shown in the graph, the absolute gaps are measured at most around $4.0e-4$ with the smallest feasible precision bound.

7.4 Evaluation of the Distributed Algorithm

In this subsection, we illustrate the convergence process of the distributed algorithm, and evaluate it in terms of convergence speed and solution quality.

7.4.1 Illustration of the Convergence Process

First we illustrate the convergence process to the equilibrium point with the distributed algorithm. In this set of experiments, we consider 3 SUs, 3 PUs and 8 subchannels, and apply the constant adaptive size of $1e13$. In Fig. 5, we plot the convergence trajectories as the iteration proceeds. Each axis on each graph indicates the utility of each SU (Fig. 5a), the utility of each PU (Fig. 5b), the total price gathered by each PU (Fig. 5c), and the total interference charge gathered by each SU (Fig. 5d). It is observed that, as the algorithm approaches to the equilibrium point, the amount of update in each iteration decreases, and which illustrates the asymptotic stability of the algorithm.

7.4.2 Illustration of the Convergence Speed and the Solution Quality

Next we evaluate the distributed algorithm in terms of the convergence speed and the solution quality. In these

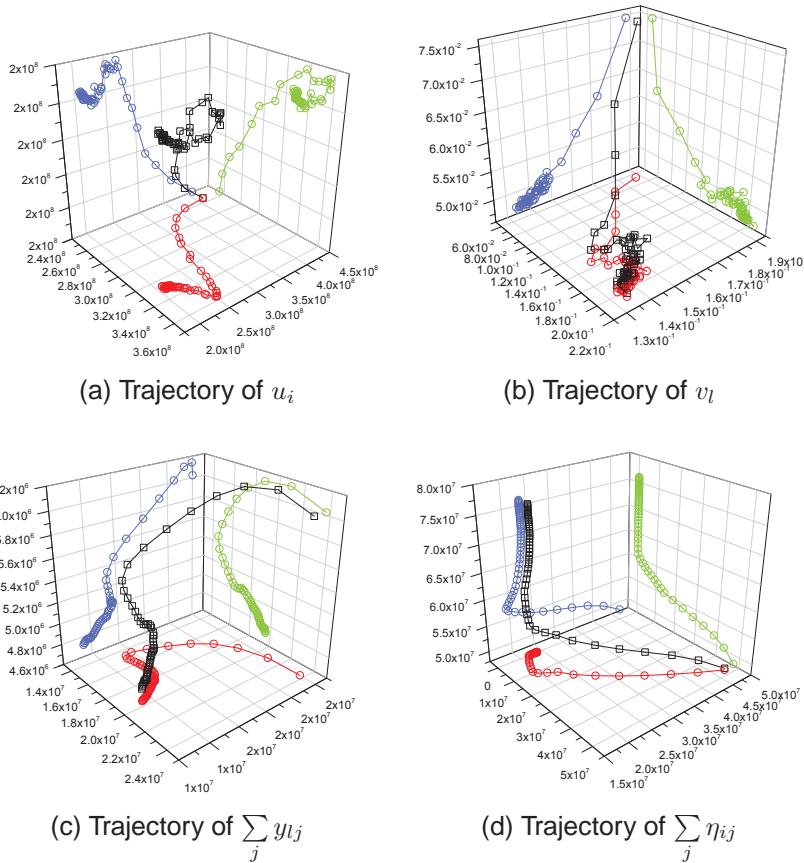


Fig. 5. The trajectories of the transitions of the utilities of SUs and PUs, prices, and interference charges as the distributed algorithm proceeds. We also plot the projection of each trajectory to the surfaces.

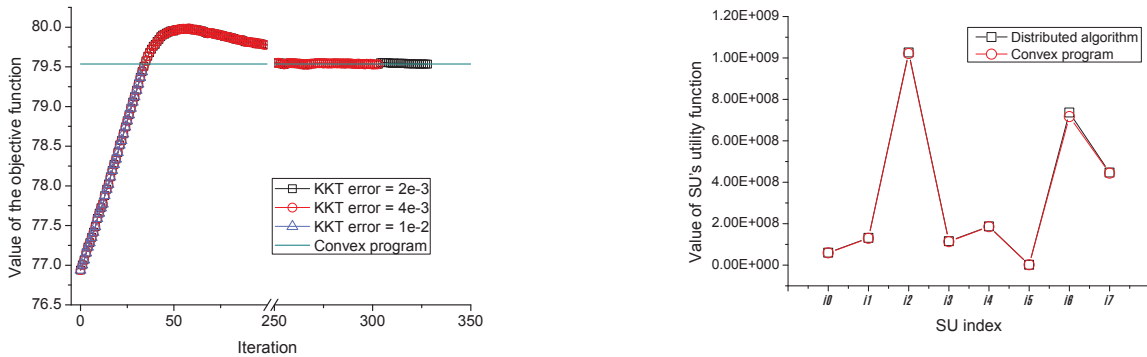


Fig. 6. Graphical presentation of the number of iterations with various termination conditions (that is, KKT error). The solid green line indicates the optimal value of the objective function yielded by the convex program.

experiments, 8 SUs and 8 PUs are located accommodating 32 subchannels. First we measure the number of iterations required to reach the termination condition. Here we define the termination condition as the errors in the KKT optimality conditions. The results are shown in Fig. 6 where we plot the objective function value of (8) on every iteration. In addition, the speed of adjustment is set

Fig. 7. The comparison of SUs utility.

2e13. As shown in the graph, as the KKT error is larger, the distributed algorithm converges faster. The function value yielded by the convex program is 79.53536. With KTT error of 1e-2, the algorithm terminates after 33 iterations, and the value of the objective function measured 79.43998. With KKT error of 4e-3, the algorithm stops at 188th iteration, and the value is measured 79.59514. With KKT error of 2e-3, the algorithm terminates at 303rd iteration with the function value of 79.53107. Therefore, we notice the tradeoff between the convergence speed and the solution quality.

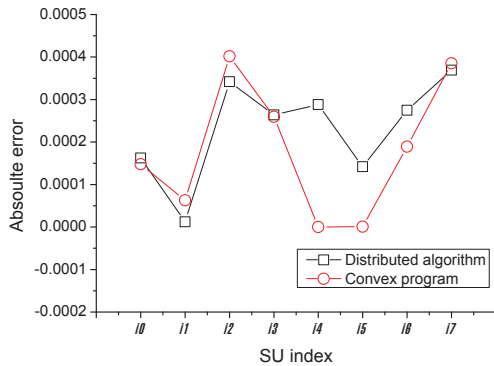


Fig. 8. The comparison of the absolute gaps between the initial budget of each SU and her total payment.

Furthermore, we measure the utilities of the SUs obtained by the distributed algorithm and those yielded by the convex program. Fig. 7 plots the results. The largest difference is measured around $3e6$ at i6, but in percentage, it is 0.67%.

We also measure the absolute gaps between the initial budget of each SU and her payment yielded by the distributed algorithm as we have done with the KKT conditions of the convex program in Section 7.3. The measured results are plotted in Fig. 8 together with the results with the KKT conditions. We see that the distributed algorithm yields larger gaps than the KKT conditions, and the largest absolute gap is measured $3.69e-4$ at most.

8 CONCLUDING REMARK

In this paper, we consider a market equilibrium in multi-channel sharing CRN. PUs and SUs act as suppliers and purchasers, respectively: PUs offer their subchannels to SUs with bounding the total amount of interference invoked from SUs transmissions, and SUs purchase the offered subchannels observing their budget constraints and the interference bounds given by the PUs. Moreover, we consider that each SU sets the constraint of interference from other SUs. Accordingly, SUs pay the interference charges to other SUs she interferes. The utility functions of SUs and PUs are given as the least achievable transmission rates and the net profits, respectively.

The market equilibrium not only optimizes all traders' (PUs and SUs) utilities but also achieves the market clearance. We show that the market equilibrium is yielded by solving the optimization problem called Eisenberg-Gale convex program, and the equilibrium price is given by the Lagrangian dual variables of the convex program. The convex program yields the equilibrium only when the utility functions of SUs are homogeneous of degree one. Therefore, we apply a monotone-transformation to SUs utility functions with maintaining the strict monotonicity and concavity.

We also develop a distributed algorithm with which the traders can reach the market equilibrium without any central authority. However, it is impossible to yield the exact equilibrium price since the system of the linear equations - that are composed of the KKT conditions of the convex program - are normally inconsistent, and the convergence behavior of the distributed algorithm is asymptotic. For these reasons, we provide the system of the linear equations with a certain precision bound that makes the system consistent, and give a termination threshold to the distributed algorithm.

By the numerical experiments, we illustrate the strict monotonicity of the monotone-transformed function, and present graphically the existence of the equilibrium and the convergence process of the distributed algorithm. We also measure the absolute errors in the solutions and KKT conditions, which is yielded by the precision bound in the system of the linear equations and the asymptotic optimality of the distributed algorithm. The measured results show that the solutions achieved by the distributed algorithm are quite close to those of the convex program.

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Sang-Seon Byun received the BS and MS degrees in computer science from Korea University, Seoul, Korea, in 1996 and 2002, respectively. He received the Ph.D. degree in computer science at Korea University, Seoul, Korea, in 2007. He was an assistant research professor of Graduate School of Embedded Software at Korea University in 2007. From 2007 to 2011, he has served as a postdoc and research scientist of Department Electronics and Telecommunications at Norwegian University of Science and

Technology (NTNU), Trondheim, Norway. Currently, he is a research professor of School of Information and Communication in Gwangju Institute Science and Technology (GIST), Gwangju, Korea. His current research interests are in the fields of Cognitive Radio Networks and Software-Defined Radio Architecture.



Ilanko Balasingham received MSc and PhD degrees from the Department of Telecommunications, Norwegian University of Science and Technology (NTNU), Trondheim, Norway in 1993 and 1998, respectively, both in signal processing. He performed his Masters degree thesis at the Department of Electrical and Computer Engineering, the University of California Santa Barbara, USA. From 1998 to 2002, he worked as a Research Scientist at Fast Search and Transfer ASA, Oslo, Norway. Since 2002 he has been

with the Intervention Centre, Oslo University Hospital, Oslo, Norway as a Sr. Research Scientist, where he heads the Wireless Sensor Network Research Group. He was appointed as a Professor in Signal Processing in Medical Applications at NTNU in 2006. His research interests include super robust short range communications for both in-body and on-body sensors, body area sensor network, microwave short range sensing of vital signs, and short range localization and tracking sensors, catheters, and micro robots. He is the General Chair of the 2012 Bodynets conference in Norway and serves an Area Editor of the Elsevier Nano Communication Networks.



Athanasios V. Vasilakos is currently Professor at the Dept. of Computer and Telecommunications Engineering, University of Western Macedonia, Greece and visiting Professor at the Graduate Programme of the Dept. of Electrical and Computer Engineering, National Technical University of Athens (NTUA). He has authored or co-authored over 200 technical papers in major international journals and conferences. He is author/coauthor of five books, 20 book chapters in the areas of communications. He served as

general chair, TPC chair and symposium chair for many international conferences. He served or is serving as an Editor or/and Guest Editor for many technical journals, i.e. IEEE TNSM, IEEE TSMC-part B, IEEE TITB, JSAC. He is founding Editor-in-chief of the journals: International Journal of Adaptive and Autonomous Communications Systems (IJAACS, <http://www.inderscience.com/ijaacs>), International Journal of Arts and Technology (IJART, <http://www.inderscience.com/ijart>). He is chairman of the European Alliance for Innovation.



Heung-No Lee graduated from UCLA obtaining Ph.D. in Electrical Engineering in 1999. He received B.S. and M.S. degrees in Electrical Engineering there as well, in 1993 and 1994 respectively. He then moved to HRL Laboratory, Malibu, California, and worked there as a Research Staff Member from 1999 to 2002. He was then appointed as Assistant Professor at the University of Pittsburgh, Pittsburgh, Pennsylvania, in 2002 where he stayed until 2008. He then moved to Gwangju Institute of Science and Technology

(GIST) in 2009. General areas of his research lie in Information Theory, Signal Processing Theory, and Communications/Networking Theory, and their application to Wireless Communications and Networking, Compressive Sensing, Future Internet, and Brain Computer Interface. He has served as the Lead Guest Editor for EURASIP Journal on Wireless Communications and Networking in 2010-2011. He has also served as a member of technical program committees for several renowned IEEE conferences, including the IEEE International Conference on Communications, and Globecom.