Performance Analysis of LDPC-Coded Space-Time Modulation over MIMO Fading Channels

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***Abstract*— A closed-form upper bound on the error performance is proposed for LDPC-coded space-time modulation over MIMO block/slow fading channels based on the analysis framework developed for the fast fading case. This follows from the observation that the pairwise error probability (PEP) in all these fading cases is determined by a certain metric of codewords, with respect to which we can enumerate all distinct PEPs and thus concisely formulate the union bound. Simulation results indicate that the bound is useful to benchmark the performance of iterative decoding and detection algorithms.**

***Index Terms*— Maximum likelihood upper bounds, MIMO systems, LDPC codes, space-time codes.**

I. Introduction

UNION bound analysis has been used throughout the history of the coding theory. There are many concise-form union bound results and techniques which are mainly for the AWGN and the single-input and single-output (SISO) fading channels. In the light of capacity approaching turbo-like codes, various tight union bounds have also been developed for these channels [1]. These bounds can be easily applied to multi-input multi-output (MIMO) systems if an equivalent SISO channel model is available [2]. In a recent progress, the union bound analysis is applied to trellis-coded space-time MIMO modulations [3].

Low-density parity-check (LDPC) codes have been recently proposed to drive space-time modulations in a concatenated coding scheme for MIMO systems [4][5]. Currently, the error performance of this system is generally evaluated by tedious system simulation or by numerical techniques such as the EXIT chart [5] and density evolution techniques [6]. While these techniques are good and certainly appropriate for investigating the performance limits of iterative receivers with block lengths approaching infinite, they may be less suitable for use at a commonly used block length of up to few thousands. In addition, it is curious to know how the performance of the practical iterative decoding and detection receiver can be compared with that of an ideal maximum likelihood receiver.

With such motivations, we investigate the use of union bounds for the LDPC coded space-time modulation system in this paper. We note that there have been no (tight) union bound results for this system to date, to the best knowledge of the authors. The concatenation of a long block code with a short inner space-time code creates a rather new and interesting codeword enumeration problem in the union-bound analysis.

Manuscript received August 8, 2006. The associate editor coordinating the review of this letter and approving it for publication was Dr. Jaap van de Beek. This work was supported in part by ADCUS, Inc., Wexford, PA, and the University of Pittsburgh through the CRDF fund.

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Digital Object Identifier 10.1109/LCOMM.2007.061237.

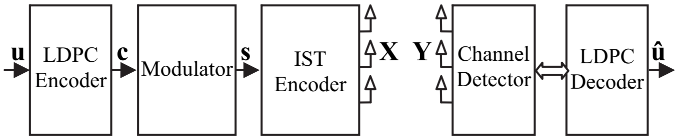

Fig. 1. MIMO system over fading channels.

In the sequel, we first give the system description. This is followed by the presentation of our union-bound analysis framework. The numerical evaluation of derived bounds is then compared with system simulation results. Concluding remarks are finally offered.

II. System of Interest

Consider the MIMO system with M-transmit and N-receive antennas that is illustrated in Fig. 1. A sequence $\mathbf{u}$ of K information bits is coded into an LDPC codeword $\mathbf{c}$ of length L. The modulator adapts a constellation of size 2^{K_b}; thus its output is a sequence of L/K_b symbols $\mathbf{s}$. Each group of Q symbols is thereafter encoded into one inner space-time (IST) block $\mathbf{S}$ of size $M \times T_s$. Assume L to be a multiple of QK_b, $D = L/QK_b$, for convenience. Each codeword $\mathbf{c}$ is therefore mapped onto an $M \times DT_s$ space-time word $\mathbf{X} = [\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_D]$ (i.e., a sequence of D IST blocks $\mathbf{S}_d$), which is transmitted over M antennas in $T = DT_s$ channel uses.

In corresponding to each IST block $\mathbf{S}_d$ in $\mathbf{X}$, an $N \times T_s$ receive signal $\mathbf{R}_d$ is obtained,

$$\mathbf{R}_d = \sqrt{\rho_s}\mathbf{H}_d\mathbf{S}_d + \mathbf{Z}_d, d = 1, 2, \ldots, D, \quad (1)$$

where ρ_s is the average symbol energy at each transmit antenna, $\mathbf{Z}_d$ is the $N \times T_s$ matrix of independent complex white Gaussian noise with zero mean and variance N_0, and $\mathbf{H}_d$ is the $N \times M$ channel matrix whose n^{th}-row, m^{th}-column entry $\alpha^d_{n,m}$ follows an independent, identical Ricean distribution (Ricean factor K_R). We refer to it as the *fast, block*, or *slow* fading channel according to the following variation of respective assumptions: let the channel matrix $\mathbf{H}_d$ remain fixed within each channel use, within each IST block $\mathbf{S}_d$, or within the whole duration of the space-time word $\mathbf{X}$, respectively, while letting the channel vary independently from one channel use, one IST block or one space-time word, to another.

III. Union Bound Analysis Framework

In this section, we first present our union-bound analysis framework for the fast fading case. It will be extended to the block and slow fading cases later.

The sketch of our idea behind this analysis framework is given as follows. We first note that the pairwise error probability (PEP) can be determined by a certain metric of codewords. Identifying this metric and determining the average number of codewords having such a metric in an LDPC code are two key steps in the analysis. Then, the union bound is formulated by summing all distinct PEPs each of which is weighted by the number of codewords that share the identical metric. A brute-force enumeration and evaluation of these PEPs is computationally intensive. We propose a polynomial-expansion approach to avoid the brute-force enumeration for the *fast* and *block* fading channels, and a compact form of union bounds for each channel case is obtained.

Now, we move forward into details. There are $J = 2^{QK_b}$ distinct bit strings of length QK_b, $\mathbf{b}_{(1)}$, $\mathbf{b}_{(2)}$, ..., $\mathbf{b}_{(J)}$, each of which has a respective Hamming weight w_j and is mapped onto a distinct IST block $\mathbf{S}_{(j)}$, for j = 1, 2, ..., J. Assume $\mathbf{X} = [\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_D]$ is composed of a number δ_j of $\mathbf{S}_{(j)}$, and thus $\mathbf{c}$ by a number δ_j of $\mathbf{b}_{(j)}$. The vector $\underline{\delta} := (\delta_1, \delta_2, \ldots, \delta_J)$ is considered as a metric of $\mathbf{X}$ or $\mathbf{c}$. Thus, all LDPC codewords of metric $\underline{\delta}$ have the same weight $h = \sum \delta_j w_j$. Assume A_h as the average number of codewords of weight h in one LDPC code. Based on the statistical property of the ensemble of LDPC codes [7, Theorem 1], the average number $A_{h,\delta}$ of LDPC codewords $\mathbf{c}$ (or space-time words $\mathbf{X}$) that have the same metric $\underline{\delta}$ is given by

$$A_{h,\underline{\delta}} = A_h \left[\binom{L}{h}^{-1} \binom{D}{\delta_1,\ \delta_2,\ \ldots\ \delta_J} \right], \tag{2}$$

for any $\underline{\delta} \in \Omega_h$,

$$\Omega_h = \left\{ \underline{\delta} \middle| \delta_j \in \{0, 1, \ldots, D\}, \sum \delta_j = D, \sum \delta_j w_j = h \right\}, \tag{3}$$

where $\binom{\sum x_i}{x_1, x_2, \cdots x_n} = \frac{(\sum x_i)!}{\prod x_i!}$ denotes the multinomial coefficient, and the bracketed term in (2) is indeed the probability that a codeword of weight h has a metric $\underline{\delta}$.

An upper bound on the error performance can be obtained by considering $\mathbf{X}^* = [\mathbf{S}_{(*)}, \mathbf{S}_{(*)}, \ldots, \mathbf{S}_{(*)}]$ as the transmitted space-time word if $\mathbf{S}_{(*)}$ is the worst IST word in the sense that

$$\mathbf{S}_{(*)} = \arg\max_{\mathbf{S}} \sum_{j=1}^{J} \bar{P}^{IST}(\mathbf{S} \rightarrow \mathbf{S}_{(j)}), \tag{4}$$

where $\bar{P}^{IST}(\mathbf{S} \rightarrow \mathbf{S}_{(j)})$ denotes the PEP between two IST blocks $\mathbf{S}$ and $\mathbf{S}_{(j)}$. Take the fast fading case as an example; the PEP between $\mathbf{X}^*$ and any codeword $\mathbf{X}_{h,\delta} = [\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_D]$ having a metric $\underline{\delta}$ can be written into a product form [7]:

$$P(\mathbf{X}^* \rightarrow \mathbf{X}_{h,\underline{\delta}}) \leq \prod_{j=1}^{J} \bar{P}^{IST}(\mathbf{S}_{(*)} \rightarrow \mathbf{S}_{(j)})^{\delta_j} =: \prod_{j=1}^{J} \beta_j^{\delta_j}, \tag{5}$$

which, at a given signal-to-noise ratio, is completely determined by the metric $\underline{\delta}$ of $\mathbf{X}_{h,\delta}$. Thus, we have

Theorem 1: *In the case of fast fading, a union upper bound on the bit error probability of the system in Fig. 1 is given by*

$$P_b \leq \sum_{h=1}^{L} \frac{h}{L} \sum_{\underline{\delta} \in \Omega_h} A_{h,\underline{\delta}} P(\mathbf{X}^* \rightarrow \mathbf{X}_{h,\underline{\delta}}) \tag{6}$$

$$= \sum_{h=1}^{L} \frac{h}{L} \binom{L}{h}^{-1} A_h \phi_h, \tag{7}$$

where ϕ_h's are the coefficients of a polynomial expansion:

$$\left(\sum_{j=1}^{J} \beta_j x^{w_j} \right)^D = \sum_{h=0}^{L} \phi_h x^h. \tag{8}$$

The union bound on the right hand of (6) is merely a summation of all distinct PEPs, each of which is weighted by the number $A_{h,\delta}$ of codewords that have the identical metric $\underline{\delta}$. This is because the PEP is determined by the metric $\underline{\delta}$. As proved later, this property is satisfied in the block and slow fading cases as well.

Furthermore, benefiting from the product form of the PEP in (5), we can make use of (7) and (8) to evaluate the union bound and the computation complexity merely lies in the polynomial expansion (it is trivial to compute the PEP β_j between two IST codewords $\mathbf{S}_{(*)}$ and $\mathbf{S}_{(j)}$). As it will become clear below, a similar product-form PEP exists in the case of block fading and thus the polynomial-expansion method applies.

IV. Analysis on the Block Fading Case

Consider the system in Fig. 1 which operates over block fading channels. Conditioned on the channel realization $\mathbf{H}$, the PEP between any two space-time words $\mathbf{X} = [\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_D]$ and $\mathbf{X}' = [\mathbf{S}'_1, \mathbf{S}'_2, \ldots, \mathbf{S}'_D]$ is given by

$$P(\mathbf{X} \rightarrow \mathbf{X}'|\mathbf{H}) \leq \exp\left(-\frac{\rho_s}{4} d^2(\mathbf{X}, \mathbf{X}'|\mathbf{H}) \right), \tag{9}$$

where $d^2(\mathbf{X}, \mathbf{X}'|\mathbf{H})$ is the squared Euclidean distance between $\mathbf{X}$ and $\mathbf{X}'$ conditioned on $\mathbf{H}$,

$$\begin{aligned} d^2(\mathbf{X}, \mathbf{X}'|\mathbf{H}) &= \sum_{n=1}^{N} \sum_{t=1}^{T} \left| \sum_{m=1}^{M} \alpha_{n,m}(t)(\mathbf{X}_{m,t} - \mathbf{X}'_{m,t}) \right|^2 \\ &= \sum_{n=1}^{N} \sum_{d=1}^{D} \sum_{t=1}^{T_s} \left| \sum_{m=1}^{M} \alpha_{n,m}^d (\mathbf{S}_{d;m,t} - \mathbf{S}'_{d;m,t}) \right|^2, \end{aligned} \tag{10}$$

where $\mathbf{X}_{m,t}$ and $\mathbf{X'}_{m,t}$ are the m^{th}-row, t^{th}-column elements of $\mathbf{X}$ and $\mathbf{X}'$, respectively, and $\alpha_{n,m}(t)$ is the fading gain from the m^{th} transmit to the n^{th} receive antenna during the t^{th} channel use. The second equality follows from the block fading assumption, i.e., $\alpha_{n,m}(t) = \alpha_{n,m}^d$ during the d^{th} IST block.

Using the pairwise error analysis framework in [8], or using a moment generating function based approach, the pairwise error probability (9) averaged over the Ricean block fading $\mathbf{H}$ can be obtained as follows:

$$\begin{aligned} P(\mathbf{X} \rightarrow \mathbf{X}') &\leq \prod_{d=1}^{D} \prod_{n=1}^{N} \prod_{m=1}^{M} \frac{1}{1+\frac{\rho_s}{4}\lambda_m^d} \exp\left(-\frac{K_R \frac{\rho_s}{4} \lambda_m^d}{1+\frac{\rho_s}{4}\lambda_m^d} \right) \\ &=: \prod_{d=1}^{D} \bar{P}^{IST}(\mathbf{S}_d \rightarrow \mathbf{S}'_d), \end{aligned} \tag{11}$$

where λ_m^d's are the eigenvalues of $A_d = (\mathbf{S}_d - \mathbf{S}'_d)(\mathbf{S}_d - \mathbf{S}'_d)^H$.

According to (11), the PEP between $\mathbf{X}^* = [\mathbf{S}_{(*)}, \mathbf{S}_{(*)}, \ldots, \mathbf{S}_{(*)}]$ and any other space-time words $\mathbf{X}_{h,\delta} = [\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_D]$ of metric $\underline{\delta}$ is given by

$$\begin{aligned} P\left(\mathbf{X}^* \rightarrow \mathbf{X}_{h,\underline{\delta}}\right) &\leq \prod_{d=1}^{D} \bar{P}^{IST}(\mathbf{S}_{(*)} \rightarrow \mathbf{S}_d) \\ &= \prod_{j=1}^{J} \bar{P}^{IST}(\mathbf{S}_{(*)} \rightarrow \mathbf{S}_{(j)})^{\delta_j} := \prod_{j=1}^{J} \beta_j^{\delta_j}, \end{aligned} \tag{12}$$

where the second line is obtained by grouping the like terms under each power exponent δ_j and then defining β_j as

$$\beta_j = \prod_{n=1}^{N} \prod_{m=1}^{M} \frac{1}{1+\frac{\rho_s}{4}\lambda_m^{(j)}} \exp\left(-\frac{K_R \frac{\rho_s}{4} \lambda_m^{(j)}}{1+\frac{\rho_s}{4}\lambda_m^{(j)}} \right), \tag{13}$$

where $\lambda_m^{(j)}$'s are the eigenvalues of $A_{(j)} = (\mathbf{S}_{(*)} - \mathbf{S}_{(j)})(\mathbf{S}_{(*)} - \mathbf{S}_{(j)})^H$.

The PEP in (12) has the same product form as (5). The union bound derived for the fast fading case is thus applicable.

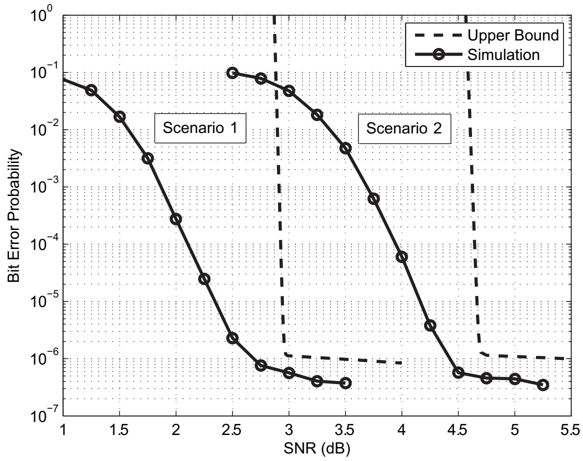

Fig. 2. Simulation results v.s. upper bounds. Scenario 1: M=N=4, 4PSK, the space-time code in [9, Eq. (38)]; Scenario 2: M=N=2, 8PSK, Alamouti code.

Theorem 2: *In the case of block fading, a union upper bound on the bit error probability of the system in Fig. 1 is given by (6) − (8).*

V. Analysis on the Slow Fading Case

In the case of slow fading, the channel matrix is fixed during the transmission of an entire LDPC codeword. This is a special case of block fading by considering the space-time word $\mathbf{X}$ itself as a "super" IST block. Thus the PEP between $\mathbf{X}^*$ and $\mathbf{X}_{h,\delta}$ can be obtained from (11) by setting $D = 1$, i.e.,

$$P(\mathbf{X}^* \rightarrow \mathbf{X}_{h,\underline{\delta}}) \leq \prod_{n=1}^{N} \prod_{m=1}^{M} \frac{1}{1+\frac{\rho_s}{4}\lambda_m} \exp\left(-\frac{K_R \frac{\rho_s}{4}\lambda_m}{1+\frac{\rho_s}{4}\lambda_m}\right), \tag{14}$$

where λ_m's are the eigenvalues of $A = (\mathbf{X}^* - \mathbf{X}'_{h,\delta})(\mathbf{X}^* - \mathbf{X}'_{h,\delta})^H$.

Lemma: *The matrix A is solely determined by the metric $\underline{\delta}$ of $X_{h,\delta}$., and so is the PEP of (14).*

Proof: It is equivalent to showing that $A_1 = (\mathbf{X}^* - \mathbf{X}_1)(\mathbf{X}^* - \mathbf{X}_1)^H$ is equal to $A_2 = (\mathbf{X}^* - \mathbf{X}_2)(\mathbf{X}^* - \mathbf{X}_2)^H$ if $\mathbf{X}_1$ and $\mathbf{X}_2$ have the same metric $\underline{\delta}$.

The same $\underline{\delta}$ implies $\mathbf{X}_1$ and $\mathbf{X}_2$ can be regarded as a block permutation of each other (the order of IST blocks within the space-time word is changed). Hence, $(\mathbf{X}^* - \mathbf{X}_1)$ is a column permutation of $(\mathbf{X}^* - \mathbf{X}_2)$, and we have $A_1 = A_2$. END

According to Lemma, the PEP of (14) is determined by the metric $\underline{\delta}$. Thus, a union bound similar to (6) can be obtained:

Theorem 3: *In the case of slow fading, a union upper bound on the bit error probability of the system in Fig. 1 is given by*

$$P_b \leq \sum_{h=1}^{L} \frac{h}{L} \binom{L}{h}^{-1} A_h \sum_{\underline{\delta} \in \Omega_h} \binom{D}{\delta_1, \delta_2, \ldots, \delta_J} P(\mathbf{X}^* \rightarrow \mathbf{X}_{h,\underline{\delta}}). \tag{15}$$

Due to the lack of a product-form PEP, the union bound in (15) cannot be simplified to the forms given in (7) and (8). A straightforward method is to enumerate all elements $\underline{\delta}$ in Ω_h and then calculate the summation in (15). This is a tedious task we want to avoid, even for LDPC codes of moderate block lengths; but we are not able to do so for this slow fading case.

VI. Results

The derived upper bounds are compared with simulation results in Fig. 2. Due to the space limit, we only present the results for the most general block fading case. We consider Gallager's (3000, 3, 6) LDPC code [10] as the outer codes to drive Alamouti code [11] or the space-time block code in [9, Eq. (38)], as inner space-time schemes. In simulations, the detector and the decoder exchange extrinsic information during three iterations, while the LDPC decoder runs its own message passing decoding operation [10] for twenty iterations. To average the performance over the code ensemble, we randomly generate 5,000 LDPC codes and use each of them for ten codeword transmissions; i.e., the error probability is averaged over 50,000 randomly selected transmit codewords. As is shown, in both investigated scenarios, the SNR gap between the bound and the simulation result is about 0.5 dB at the waterfall region. Also, the bound has a good prediction on the behavior of the error floor. These indicate that the bound is promising to serve as a benchmarking tool for the performance of the practical iterative decoding and detection receivers.

VII. Conclusion

In this paper, we present a union-bound analysis framework for LDPC coded space-time MIMO modulation systems over fading channels. Other than the fast fading case, we show the applicability of the framework to the block and slow fading cases. Referring to the recent development of upper bounds for single-input single-output channels (see [1] and references therein), we expect the union bound derived in this paper can serve as a basis for potential tighter bounding techniques for LDPC coded space-time MIMO modulation systems.

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