Fast communication

MB iterative decoding algorithm on systematic LDGM codes: Performance evaluation

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ABSTRACT

We investigate the performance of regular systematic low-density generator matrix (LDGM) codes under the majority rule based (MB) iterative decoding algorithm. We derive a recursive form which can be used to extract the error performance of the code. Based on the recursive expression, we derive a tight non-recursive lower bound. These results can serve as efficient tools to evaluate the performance of the code for different degrees.

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1. Introduction

Systematic low-density generator matrix (LDGM) codes with moderate code length are of interest not only because they can provide satisfying performance at moderate block length while maintaining low encoding and decoding complexities [1–4], but also because the systematic form of LDGM codes makes the code useful in new applications such as cooperative wireless multiple access relay network [5] and joint source-channel encoding systems [6].

In this paper, we are interested in the performance of the majority rule based (MB) iterative decoding algorithm for systematic LDGM codes. Although the MB algorithm is based on hard-decision and thus its performance cannot match those based on soft-decision, it has drawn significant interest in the past owing to its simplicity and low computation complexity, which allow fast decoding [4,7–9].

We investigate the asymptotic performance of the code consisting of regular systematic LDGM codes and the MB iterative decoding algorithm. By assuming infinite block length, we derive a recursive expression which predicts both the threshold and error floor behaviors of the code. Gallager has analyzed an MB iterative decoding for low-density parity-check (LDPC) codes; we build our analysis on systematic LDGM codes by extending his results reported in [9]. Based on the recursive expression, we further derive a non-recursive lower bound expression which is simply a function of the *degree of variable nodes*. We show that the bound is tight in simulation, and thus it can be useful to quickly assess the performance of the code for given degrees.

The rest of the paper is organized as follows. In Section 2, we briefly introduce the systematic LDGM codes and the majority rule based iterative decoding algorithm. The recursive expression and the lower bound expression are derived in Section 3. In Section 4, computer simulation results and analysis results are compared and discussion is provided. Finally, we make a conclusion in Section 5.

2. Systematic LDGM codes and MB algorithm

Similar to the well known LDPC codes, systematic LDGM codes can be represented by sparse matrices,

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see [10]. We briefly give the definition of systematic LDGM codes, and then introduce the majority rule based iterative decoding algorithm.

2.1. Systematic LDGM codes

Systematic LDGM codes are linear block codes with parity check matrix $\mathbf{H} = [\mathbf{P};\mathbf{I}]$, where $\mathbf{P}$ is an $(n-k)$ by k sparse matrix and $\mathbf{I}$ is the $(n-k)$ by $(n-k)$ identity matrix. The positive integer k denotes the number of input bits and n denotes the number of output bits of a systematic LDGM encoder. The matrix $\mathbf{P}$ of ones and zeros can be generated at random. A systematic LDGM code will be called *regular* if both the number of 1's in column in the $\mathbf{P}$ matrix and that in row stay fixed for all columns and rows. Though irregularity can provide performance improvement, regularity could lead to simplified modular implementation in hardware realization. We will study the regular version only in this paper. We denote the *degree* of a variable node as d_v, which is the number of ones in each column in the $\mathbf{P}$ matrix. Similarly, the degree of a check node, d_c, represents the number of ones in each row in the $\mathbf{H}$ matrix. The code can be completely specified by a bipartite graph [11] consisting of check nodes and variable nodes. Since a systematic codeword is composed of message bits and parity-check bits, the variable nodes can be further separated into *message-bit variable* (MV) *nodes* and *parity-check-bit variable* (PV) *nodes*. Based on the structure of the $\mathbf{H}$ matrix, the code rate R of (d_v, d_c)-regular systematic LDGM codes is given as $R = 1/(d_v/(d_c - 1) + 1)$.

It should be noticed that, when a code has a generator matrix with rows of constant weight, the code contains code words of the specified constant weight, and hence the minimum distance of the code could have been specified. In the case of regular systematic LDGM codes, the minimum distance of the code is not larger than d_v+1, i.e., the weight of the rows of the generator matrix.

2.2. The majority-rule based iterative decoding algorithms

There are two steps in each iteration for the MB iterative decoding algorithm. The first step is done in a check node. The output binary message from the ith check node, toward the jth of its d_c variable nodes, is the result of the XOR operation on the rest of d_c-1 incoming binary messages. That is, $c_{i,j} = \oplus \sum_{k=1,(k \neq j)}^{d_c-1}(v_{k,j})$, where the summation is done in modular-2 addition and $v_{k,j}$ is the binary message from the kth variable to the ith check node. The second step is done in a variable node at which the majority rule is applied. Let $f_j, f_j \in \{0, 1\}$, denote the hard-decision binary value of the received signal for the jth bit transmission. The output binary message from the jth variable node, toward the ith of its d_v check nodes, is obtained from the rest of d_v-1 incoming messages, and is given by

$$v_{j,i} = \begin{cases} \tilde{f}_j & \text{if } \left(\sum_{k=1,(k\neq i)}^{d_v-1} XOR(f_j, c_{k,j}) \right) \geq m, \\ f_j & \text{o.w.} \end{cases} \quad (1)$$

That is, if m or more incoming messages are violated, then the message $v_{j,i}$ is the complement of f_j; otherwise, it holds the value of f_j. At the last iteration, the jth bit is decoded to be $\tilde{f}_j$ if $(\sum_{k=1}^{d_v} XOR(f_j, c_{k,j})) \geq m$; otherwise, the jth bit is decoded to be f_j. In the algorithm, the weight m is an integer between 0 and d_v. The weight m needs to be carefully chosen in each iteration as it affects the performance of the MB iterative decoding algorithm. From the above description, we note that the MB iterative decoding algorithm is extremely simple.

3. Error performance analysis

In this section, we derive the recursive expression (2) and the tight lower bound expression (11). These expressions serve as efficient tools to extract the performance of the codec for given degrees of systematic LDGM codes.

Due to the hard decision characteristic of the MB decoding algorithm, we may assume all the coded bits are transmitted through a binary symmetric channel with error probability P_0. Consider the error performance on an MV node. Assume infinite code length and unfold the MB iterative decoding onto a cycle free decoding tree. Then, the error probability for the message on the MV node after the ith iteration can be expressed by the recursive form

$$P_{i+1} = P_0(1 - f(m, P_i)) + (1 - P_0)(g(m, P_i)). \quad (2)$$

where

$$f(m,x) = \sum_{l=m}^{d_v-1} \binom{d_v - 1}{l} \left(\frac{1 + (1 - 2P_0)(1 - 2x)^{d_c-2}}{2} \right)^l$$
$$\times \left(\frac{1 - (1 - 2P_0)(1 - 2x)^{d_c-2}}{2} \right)^{d_v-1-l} \quad (3)$$

and

$$g(m,x) = \sum_{l=m}^{d_v-1} \binom{d_v - 1}{l} \left(\frac{1 - (1 - 2P_0)(1 - 2x)^{d_c-2}}{2} \right)^l$$
$$\times \left(\frac{1 + (1 - 2P_0)(1 - 2x)^{d_c-2}}{2} \right)^{d_v-1-l}. \quad (4)$$

The first term in (2) represents the probability of an event that the MV node was in the error state originally and the error correction mechanism of MB algorithm is not triggered because less than m extrinsic messages, out of d_v-1 total, are in violation. Thus the error in the variable node remains unchanged. The second term represents the probability of an event that the MV node was in the correct state but the error correction mechanism of MB algorithm is falsely triggered—because of m or more extrinsic messages in violation—and forces an error.

It is interesting to compare this recursion result (2) to Gallager's result on regular LDPC codes [9, p. 46]. The difference is that we have $(1 - 2P_0)(1 - 2x)^{d_c-2}$ in the recursion equation, instead of $(1 - 2x)^{d_c-1}$. This belongs to one of the characteristic results of the systematic LDGM codes. It is caused by the one and only one connection made by each PV node to the corresponding check node in the bipartite graph. This causes the error floor effect in systematic LDGM codes.

For a given channel error probability P_0, the weight m and the degrees d_v and d_c determine the behavior of the recursive process (2) and hence the error performance. The optimal weight m which minimizes P_{i+1} in (2) for the ith iteration can be found by exhaustively searching for the integer between 0 and d_v, or by solving the smallest integer m which satisfies the following inequality [9]:

$$\frac{1-P_0}{P_0} \le \left(\frac{1+(1-2P_0)(1-2P_i)^{d_c-2}}{1-(1-2P_0)(1-2P_i)^{d_c-2}} \right)^{2m-d_v+1}. \tag{5}$$

In the following, we derive the lower bound expression based on the recursive expression (2). Taking the partial derivative of (2) with respect to P_i, we obtain

$$\frac{\partial P_{i+1}}{\partial P_i} = -P_0 \frac{\partial f}{\partial P_i} + (1-P_0) \frac{\partial g}{\partial P_i}, \tag{6}$$

where

$$\frac{\partial f}{\partial P_i} = \binom{d_v-1}{m} (m\xi^{+^{m-1}} \xi^{-^{d_v-1-m}} \eta^-) \tag{7}$$

and

$$\frac{\partial g}{\partial P_i} = \binom{d_v-1}{m} (m\xi^{-^{m-1}} \xi^{+^{d_v-1-m}} \eta^+). \tag{8}$$

In (7) and (8), the notations $\xi^+, \xi^-, \eta^+,$ and η^- are defined as $\xi^+ = (1+(1-2P_0)(1-2P_i)^{d_c-2})/2$, $\xi^- = (1-(1-2P_0)(1-2P_i)^{d_c-2})/2$, $\eta^+ = (d_c-2)(1-2P_o)(1-2P_i)^{d_c-3}$, and $\eta^- = -(d_c-2)(1-2P_o)(1-2P_i)^{d_c-3}$.

Without loss of generality, we may assume that P_0 and P_i are restricted in the interval [0, 0.5]. Then, we observe that (6) is always non-negative, i.e., $\partial P_{i+1}/\partial P_i \ge 0$. This shows that (2) is a monotone increasing function of P_i. Therefore, by substituting $P_i = 0$ into (2), we can obtain a lower bound expression of P_{i+1}.

The weight m used in the lower bound expression can be determined via (5). By substituting $P_i = 0$ into (5), we have

$$\frac{1-P_0}{P_0} \le \left(\frac{1+(1-2P_0)}{1-(1-2P_0)} \right)^{2m-d_v+1}. \tag{9}$$

If P_0 is restricted within the interval [0, 0.5], then $(1-P_0)/P_0$ is not less than 1. The inequality is satisfied if and only if the exponent of the right hand side is greater than 1, i.e., $2m-d_v+1 \ge 1$. The smallest integer that satisfies this inequality is $m = \lceil d_v/2 \rceil$, where $\lceil \cdot \rceil$ is the ceiling operation. Notice that, at the last iteration, the number of available extrinsic messages for an MV node is d_v, instead of d_v-1. Hence, we choose the weight m^* for a given (d_v, d_c) regular systematic LDGM as

$$m^* = \left\lceil \frac{d_v+1}{2} \right\rceil. \tag{10}$$

Therefore, the lower bound expression, which is only a function of the degree of variable nodes d_v, is given by

$$P_{LB} = P_0 \left(1 - \sum_{l=m_*}^{d_v} \binom{d_v}{l}^l (1-P_0)^l (P_0)^{d_v-l} \right)$$
$$+ (1-P_0) \left(\sum_{l=m^*}^{d_v} \binom{d_v}{l}^l (P_0)^l (1-P_0)^{d_v-l} \right). \tag{11}$$

We note that the results (2) and (11) are based on the cycle-free assumption, i.e., infinite code length. Thus, they render the best performance for given degrees of systematic LDGM codes. The tightness of the lower bound is illustrated through simulation in Section 4.

4. Simulation results and discussion

Assuming BPSK modulation over AWGN channels, the error probability of the equivalent binary symmetric channel (BSC) for the MB algorithm is obtained by $P_0 = 0.5erfc(\sqrt{RE_b/N_0})$, where R is the code rate, E_b is the energy per bit, and N_0 is the one sided power spectral density of the noise. We assess the best possible performance of the code by using the recursion form (2) in the following manner. While numerically evaluating the recursion expression (2), we test out all the possible choices of m in each iteration and then select the best value of m that results in the lowest error probability at the end of each iteration. In addition, we let a large number of iterations (more than 50) to ensure the convergence of the recursive form (2).

Fig. 1(a) shows the BER curves for rate half systematic LDGM codes with degree (8, 9), (9, 10), (10, 11), (11, 12), (12, 13), and (13, 14). Fig. 1(b) shows the BER curves for rate around 1/3 systematic LDGM codes with degree (9,6), (10,6), (11,6), and (11,7). The dashed curves in the figure are obtained from the recursive method (2), whereas the solid curves are obtained from the non-recursive lower bound (11). We note that the lower bound is asymptotically tight with respect to channel signal to noise ratio (SNR). This is expected because, at high SNR, P_i in (2) can evolve to a value very close to zero, and hence the assumption $P_i = 0$ we made to derive the lower bound becomes more valid. We also note that the lower bound expression predicts the performance well in the entire error floor region. Defining threshold as the SNR the waterfall starts, moreover, we note that a code with small degrees exhibits a high error floor but a small threshold; whereas a code with large degrees shows a low error floor but a larger threshold. Considering the trade-off relation between the error floor and the threshold behavior, the best degree d_v for systematic LDGM codes under MB iterative decoding algorithm can be selected. For example, we may select it to be 10 based on our results. Systematic LDGM codes with other degrees are not good, since they exhibit either a high error floor or a large threshold. In addition, we observe that, for rate half systematic LDGM codes, the curves of (d_v, d_v+1) and (d_v+1, d_v+2) converge asymptotically for even d_v. This is one of the characteristic behaviors of systematic LDGM codes which was also reported in [2].

Fig. 2 shows the BER curves obtained from the Monte Carlo computer simulation. Ten iterations are used in the MB algorithm. Two randomly constructed (8, 9) and (9, 10) systematic LDGM codes with length 6000 are used in simulation.

To draw best threshold behavior while maintaining a low error floor, we use the following strategy for selecting
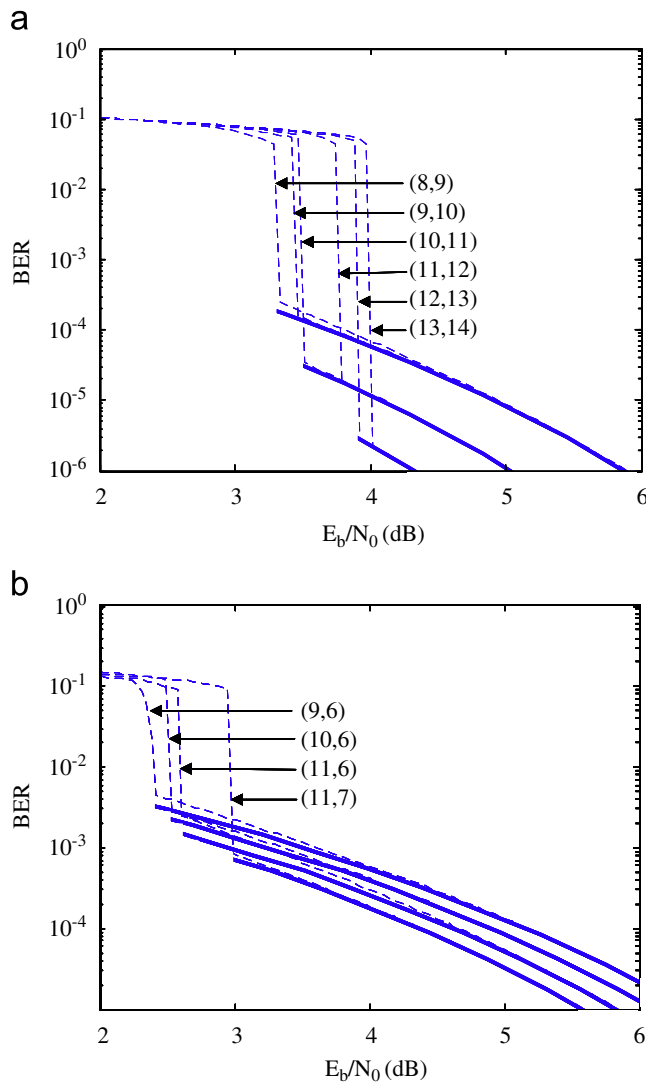
Fig. 1. BER performance of (a) rate half systematic LDGM codes and (b) rate around $\frac{1}{3}$ systematic LDGM codes. In the figure, the dashed curves represent the BER obtained from the recursive expression (2) with dynamic optimized weight m; the solid curves represent the BER obtained by the non-recursive lower-bound expression (11).
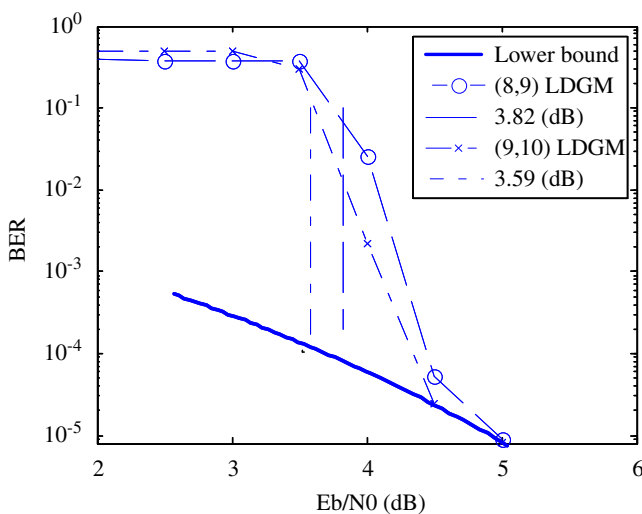

Fig. 2. Comparison between the Monte Carlo computer simulation results and the analytic results.

the weight m. Initially, we choose the weight m which has the smallest threshold. This initial weight is used all the way through the last iteration, and at the last iteration the weight calculated from (10) is used to push down the error floor. For the (8, 9) systematic LDGM code, the weight of the smallest threshold is $m = 5$ and the weight calculated from (10) is also $m = 5$. Hence, we select the weight to be 5 throughout the iterations. For the (9, 10) systematic LDGM codes, the weight of the smallest threshold is $m = 6$, whereas the weight calculated from (10) is $m = 5$. We choose $m = 6$ for the iterations all the way until the last one, and then choose $m = 5$ for the last iteration.

The simulation results show that both the (8, 9) systematic LDGM code and the (9, 10) systematic LDGM code not only can achieve the lower bound, but also can achieve the thresholds. In other words, the derived recursive expression and non-recursive lower bound are tight and can successfully serve as efficient tools to access the error performance of the codec.

5. Conclusion

Systematic LDGM codes and the majority-rule based iterative decoding algorithm may be of interest for communications system engineers because they render simple encoding and decoding complexities. The codec exhibits two eminent error performance behaviors, the threshold and the error floor. We have provided the analytic expression (2) for efficiently assessing the best possible performance of the codec for given degrees of systematic LDGM codes. Furthermore, a simple tight lower bound expression (11) has been derived, which can be readily evaluated once the degree of variable nodes is given.

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