Concatenation of LDPC codes with Golden Space-Time Block Codes over the Block Fading MIMO Channels: System Design and Performance Analysis

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***Abstract*- The use of low-density parity-check (LDPC) codes over the multi-input multi-output (MIMO) channels has shown the performance to be achieved to the capacity limit. For the 2 × 2 MIMO system, the Golden codes were recently developed as the optimal space-time codes with a full rate and a full diversity gain. In this paper, we give the detailed system design for the concatenated coding scheme, which consists of the LDPC codes as the *outer* code and the Golden space-time block codes (STBC) as the *inner* code. The outer binary block codeword transforms itself onto the sequence of the *internal* space-time (IST) symbols according to a fixed constellation map. The soft-input soft-output (SISO) message passing decoder of the LDPC codes can be combined with a SISO constellation de-mapper as the Golden STBC decoder. They exchange the extrinsic messages in turbo-iterations. We evaluate the error performance for different block lengths of the proposed system. For the Golden codes concatenated with the LDPC codes, the maximum-likelihood union bound analysis is used for performance benchmarking.**

I. Introduction

It has been shown that the LDPC codes [1] with the turbo-iterative decoder approach very closely to the Shannon-limit performance in the MIMO channels [2]. Elia *et. al.* [3] have recently shown that the Golden codes [4] obtain the optimal diversity-multiplexing tradeoff for a 2 × 2 MIMO system. It has a full diversity, full rate, and has a fixed minimum determinant, which does not depend on the constellation size. In order to approach the capacity limit of MIMO channels, the spatial multiplexing gain and temporal diversity gain have to be simultaneously considered.

The Golden codes have been used as the inner code in concatenated coded schemes. The following literature shows the application of the Golden codes. Luzzi *et. al.* [5] have considered the concatenation of the Golden codes with the Reed Solomon (RS) codes, a coded modulation scheme for 2 × 2 slow-fading MIMO channels. In [6], Lopez *et. al.* have presented the use of regular Serially-Concatenated Low-Density Generator Matrix codes, a concatenation with the Golden codes, for MIMO systems. In [7][8][9], the authors have shown that the concatenated coding scheme with the inner code as the Golden codes and the outer code as the trellis codes was developed for the slow and fast fading MIMO channels. In [9], Viterbo *et. al.* have done a performance analysis of the Golden space-time trellis coded modulation over block fading channels. The analysis is based on the use of the truncated union bound technique.

To the best of authors' knowledge, there has been no study on the concatenation of the Golden codes with the LDPC codes. We provide a detailed system design for the concatenated coding system, and evaluate the performance in different block lengths of the proposed system in the simulation. The receiver uses the turbo-iterative processor which exchanges the extrinsic messages between the LDPC decoder and the de-mapper of the Golden codes.

The maximum-likelihood (ML) union bound analysis is very useful to evaluate the error performance in the various coding schemes and channels [10][11][12]. But none of them are for the Golden codes as the inner code. In this paper, we use a union bound expression which can be derived for the concatenated coding scheme as the summation of all distinct pairwise error probabilities [13], each of which is weighted by the number of the corresponding codewords. Using the distance spectrum [15] of the regular LDPC codes based on the ensemble average over the Gallager's code [1], the union bound for the concatenation of the LDPC codes and the Golden space-time block codes can be easily computed. We compare the system simulation results with the union bounds at different block lengths.

The rest of the paper is organized as follows. In Section II, we give the description of the encoding process and the channel model, and the turbo-like decoder. In Section III, the efficient union bound expression is provided for the concatenated coded scheme. In Section IV, simulation results and the bounds are compared. Finally, we conclude in Section V.

II. System Description

A. Encoder operation

The proposed concatenation MIMO system with N_t-transmit and N_r-receive antennas is shown in Fig. 1. A sequence $\mathbf{u}$ of K information bits is encoded into a binary codeword $\mathbf{c}$ of length N. The codeword $\mathbf{c}$ is then mapped onto an $[N_t \times T]$ space-time transmission (STT) matrix $\mathbf{X}$. The STT matrix $\mathbf{X}$ is composed of a sequence of the internal space-time (IST) symbols as shown in Fig. 1. There are D IST symbols S_d, $d = 1, 2, \ldots, D,$ corresponding to a single binary codeword

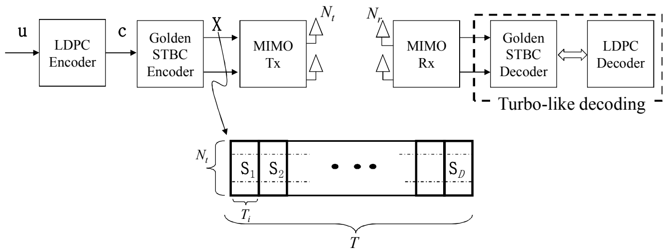

Fig. 1. The concatenated coding scheme of the LDPC codes and Golden space-time block codes for the MIMO system.

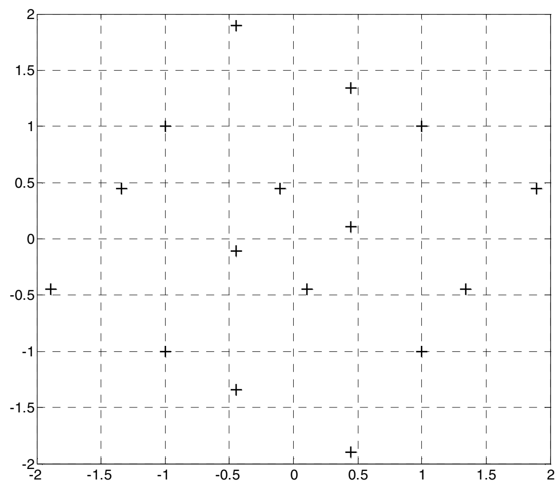

Fig. 2. Transformed constellation for 4-QAM by the Golden space-time codes.

c of length $N = DN_b$. Each IST symbol is $N_b = Q\log_2(M)$ bits/ IST symbol, Q is the number of the channel symbols, M is the constellation size. They compose a single STT matrix, i.e., $\mathbf{X} = [S_1\ \ S_2\ \ \ldots\ \ S_D]$, of size $[N_t \times T]$, where $T = DT_i$, T_i is the number of the channel usage per one IST symbol. The collection of STT matrices one-to-one corresponding to each outer binary codeword **c** is referred to as the space-time codes, in this case, we use the Golden codes. The Golden codes [4] take the four channel symbols and convert them to a 2 × 2 space-time matrix in the following,

$$\begin{bmatrix} S_a \\ S_b \\ S_c \\ S_d \end{bmatrix} \Rightarrow \mathbf{S} = \frac{1}{\sqrt{5}} \begin{bmatrix} \alpha\left(S_a + S_2\theta\right) & \alpha\left(S_c + S_d\theta\right) \\ i\alpha\left(S_c + S_d\bar{\theta}\right) & \alpha\left(S_a + S_b\bar{\theta}\right) \end{bmatrix},$$

where $\theta = \left(1+\sqrt{5}\right)/2$, $\bar{\theta} = \left(1-\sqrt{5}\right)/2 = \left(1-\theta\right)$, $\alpha = i\left(1-\theta\right)$, $\bar{\alpha} = 1 + i\left(1-\theta\right)$. Thus, there are 256 different IST symbols as assuming the use of the 4-QAM base constellation. The symbols per transmit antenna for 4-QAM are drawn from a coded constellation shown in Fig. 2.

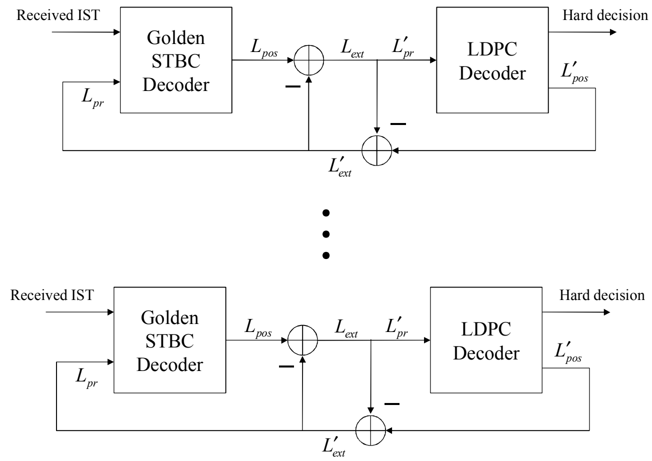

Fig. 3. The turbo-like decoder of the concatenated coding scheme for the MIMO system.

B. MIMO Channel

We assume the transmission of STT matrix **X** over the flat fading (N_t, N_r) MIMO channel in T channel usages. Then, the $[N_r \times T_i]$ receive signal $\mathbf{R}_d$ during the d^{th} BCU (Block Channel Use) is given by,

$$\mathbf{R}_d = \sqrt{\rho_s}\,\mathbf{H}_d\mathbf{S}_d + \mathbf{Z}_d, \quad d = 1, 2, \cdots, D \tag{1}$$

where ρ_s is defined as the average symbol energy, $\mathbf{Z}_d$ denotes the $[N_r \times T_i]$ matrix of independent complex white Gaussian noise with zero mean and variance N_0. $\mathbf{H}_d$ is the $[N_r \times N_t]$ flat fading MIMO channel matrix. Fading coefficients at each row and column of $\mathbf{H}_d$ are mutually independent and identically distributed complex Gaussian variables which can be generated by two independent real Gaussian distribution. For the assumption of the block fading channel, the channel matrix is held fixed during each block of T_i channel uses.

C. Turbo-like decoder

In this paper, we use the turbo-like decoding process to decode the received signal. It consists of the two iterations; the *super-iteration* for the de-mapping of the Golden space-time block code and the *internal-iteration* for the LDPC decoder as shown in Fig. 1. Basically, the super-iteration decoding is based on the maximum *a posteriori* (MAP) log-likelihood ratio (LLR) generation. As shown in Fig. 1, the N_b bits which can be carried by T_i channel uses compose the each space-time symbol. For an arbitrary coded bit c_k, for $k \in \{1, 2, \ldots, N_b\}$, a maximum likelihood ratio can be generated as follow:

$$L_{pos}(c_k) = \log \frac{\Pr\{c_k = 1 | \mathbf{R}_d\}}{\Pr\{c_k = 0 | \mathbf{R}_d\}} = \log \frac{\sum_{\mathbf{S}_d : c_k = 1} \Pr\{\mathbf{R}_d | \mathbf{S}_d\} \Pr\{\mathbf{S}_d\}}{\sum_{\mathbf{S}_d : c_k = 0} \Pr\{\mathbf{R}_d | \mathbf{S}_d\} \Pr\{\mathbf{S}_d\}}$$
$$= L_{pr}(c_k) + \log \frac{\sum_{\mathbf{S}_d : c_k = 1} \exp\left\{ -\left\| \mathbf{R}_d - \sqrt{\rho_s} \mathbf{H}_d \mathbf{S}_d \right\|^2 + \sum_{i \neq k, i=1}^{N_b} L_{pr}(c_i) \right\}}{\sum_{\mathbf{S}_d : c_k = 0} \exp\left\{ -\left\| \mathbf{R}_d - \sqrt{\rho_s} \mathbf{H}_d \mathbf{S}_d \right\|^2 + \sum_{i \neq k, i=1}^{N_b} L_{pr}(c_i) \right\}} \tag{2}$$
$$= L_{pr}(c_k) + L_{ext}(c_k).$$

Let $L_{ext}(c_k)$ and $L_{pr}(c_k)$ denote the extrinsic messages, which can be exchanged the information between the de-mapper and the LDPC decoder. This iteration is called the super-iteration in this paper.

Let the internal-iteration denote the message-passing algorithm of the LDPC decoder, which was developed in [1]. At the end of a fixed number of iteration as the internal-iteration, the *a posteriori* LLRs as $L_{ap}(c_k)$ for the all of bits can be generated as shown in Fig. 3. The prior LLRs eliminated from the extrinsic messages forward to the de-mapper. Each IST symbol can be finally estimated by the de-mapper after the run of the super-iteration and internal-iteration.

III. Union Bound for the Concatenated Coding System

The aim of this section is to show the union bound in the concatenation coded system. We apply the union bound technique proposed in [10][11] to the concatenation coded system of the LDPC codes and the Golden codes. The union bound is derived by using the distance spectrum [15]. Let $A_{\mathbf{d}(h)}$ denote the distance spectrum with compared to the average number of codewords A_h with Hamming weight h. In this paper, the number of the codewords in the regular LDPC codes [1] is used as the ensemble average.

Using upper bound to the word error probability and the distance spectrum, the union bound is given by,

$$P_w \le \sum_{\mathbf{X}' \neq \mathbf{X}^*} P^B(\mathbf{X}^* \to \mathbf{X}') = \sum_h \sum_{\mathbf{d}(h) \in \Omega_h} A_{\mathbf{d}(h)} \prod_{j=1}^{J} \left(\beta_j^B\right)^{d_j}, \tag{3}$$

where the second equation is obtained by enumerating all codeword $\mathbf{X}'$ according to their distance profile $\mathbf{d}(h)$ [10] and Hamming weight h. And β_j^B is defined as the pairwise error probability for the IST symbols defined in [13],

$$\beta_j^B := P^B(\mathbf{S}_{(*)} \to \mathbf{S}_{(j)}) = \prod_{n=1}^{N_r} \prod_{m=1}^{N_t} \frac{1}{1 + \frac{\rho_s}{4} \lambda_m^{(j)}} \exp\left(- \frac{F_r \frac{\rho_s}{4} \lambda_m^{(j)}}{1 + \frac{\rho_s}{4} \lambda_m^{(j)}} \right). \tag{4}$$

Then, making use of the distance spectrum, the R.H.S. of (3) can be rewritten as

$$P_w \le \sum_h A_h \binom{N}{h}^{-1} \underbrace{\sum_{\mathbf{d}(h) \in \Omega_h} \binom{D}{\mathbf{d}(h)} \prod_{j=1}^{J} \left(\beta_j^B\right)^{d_j}}_{=:\phi(h)}. \tag{5}$$

We now define a sequence of utility coefficients $\phi(h)$, i.e.,

$$\phi(h) := \sum_{\mathbf{d}(h) \in \Omega_h} \binom{D}{\mathbf{d}(h)} \prod_{j=1}^{J} \left(\beta_j^B\right)^{d_j}. \tag{6}$$

Then, the union bound becomes

$$P_w \le \sum_h A_h \binom{N}{h}^{-1} \phi(h). \tag{7}$$

The calculation of $\phi(h)$ using (7) is not tractable because the combinatorial search with respect to each distance profile, i.e., $\mathbf{d}(h) \in \Omega_h$. Notice that Ω_h is the collection of all distance profiles for a Hamming weight h. The cardinality of Ω_h grows prohibitively large when h approaches $N/2$, and thus a brute-force partitioning would take large amount of computation. In order to avoid these complex computations, we use the fast algorithm [14] which eliminates the needs for explicit enumeration of the codewords via the following polynomial expansion approach. Namely, we utilize the dummy variable z, and the following power expansion:

$$\left(\sum_{j=1}^{J} \beta_j^B z^{w_j} \right)^D = \sum_{h=0}^{N} \sum_{\mathbf{d}(h) \in \Omega_h} \binom{D}{\mathbf{d}(h)} \prod_{j=1}^{J} \left(\beta_j^B\right)^{d_j} z^h = \sum_{h=0}^{N} \phi(h) z^h. \tag{8}$$

Once we find $\phi(h)$, the union bound can be easily obtained. Further details are given in [14].

IV. System Simulation and Performance Analysis

In this paper, the super-iterative decoding algorithm like the turbo decoding for the concatenated coding schemes follows the maximum a posterior (MAP) with the log-likelihood ratio (LLR) generation methods. In this case, the Gallager's code ($d_c = 3$, $d_r = 6$), where d_c and d_r are the number of the nonzero in the column and row for the parity-check matrix respectively, is used and the turbo-receiver uses 5 super-iterations and 40 internal-iterations as shown in Fig. 3. The turbo-like decoder of the concatenated coding scheme for the MIMO system.

In order to generate the IST symbols, we take Q channel-symbols, which are formed by 4-QAM modulation. We deal with Q = 4, $N_t = 2$, and $T_i = 2$. The number of bits N_b that each IST symbol carries is $N_b = Q\log_2(M) = 8$ bits/IST symbol, thus there are $J = 2^8 = 256$ IST symbols in the set. We consider the outer code of length 120, 240, 360, and 600 for the Gallager's code (3, 6).

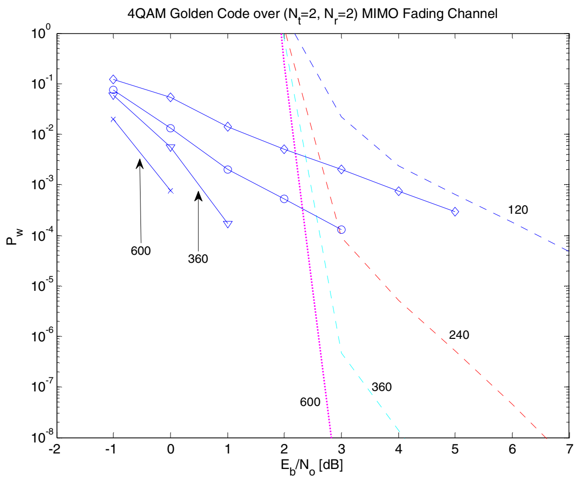

Fig. 4. The union bound on word error probability for 4-QAM Golden code over the block fading (2, 2) MIMO channel. Numbers shown inside the figure are the block lengths of the (3, 6) Gallager code. Union bounds are shown as the dashed lines; while the simulation results for block length 120, 240, 360, and 600 are shown as solid lines with the plotting symbols.

Fig. 4 shows the simulation results and the union bound for the Golden codes with the length of 120, 240, 360, and 600 in the block fading MIMO channels. The union bound is based on the ensemble average considering the all possible codes of the Gallager's code (d_c=3, d_r=6). We see that the water-fall behavior occurs in the larger length of the codeword via the union bound without the system simulation. The error floor which is the one of the performance evaluation in the LDPC codes can be predicted from the union bound at each block length of the codeword. It is clear that as increasing the length of the codeword, the average behavior of the specific code *concentrates* around the union bound [16].

V. CONCLUSION

We have proposed a concatenated coding system in which the LDPC codes are used as the outer code and the Golden codes as the inner code. The motivation for this combination was that the Golden codes are full rate fully diversity STBC and thus we can expect optimal performance in many channel conditions. We have considered the performance of the coding system in the MIMO block fading channels. We have provided a turbo-iterative receiver and shown its performance via extensive computer system simulation. Verification of the system simulation results are done with the union bounds for the concatenated coded system. We have shown that the system simulation results match with the union bounds. In particular, the union bounds and the simulation results agree in both the error floor regions for various block lengths. For a large block length, the water-fall behavior becomes more eminent which is also well predicted by the union bound.

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