Conditions on Recovery of Sparse Signals for Compressed Sensing over Finite Fields[1)]

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Abstract

We consider a framework of compressed sensing systems over finite fields for discrete signals such as bit streams for data storage and pixel images. In this paper, we aim to investigate the number of measurements needed in the compressed sensing system over finite fields. We show that the necessary and sufficient conditions for perfect recovery of sparse signals are strictly identical as the size of signals increases as well as the size of the finite fields are varied.

I. Introduction

Over the past few years, the idea of compressed sensing (CS) has attracted considerable interest in researchers from the fields of signal processing and information theory. One of interesting discovery is that a sparse signal can be recovered from a small number of linear projection measurements. The reconstruction of a sparse signal is performed through optimization techniques such as linear programming and greedy algorithms [1]-[3].

In this paper, we aim to theoretically investigate the recovery performance of a sparse signal over finite fields. Using the upper and lower bounds, we investigate the requirement of the perfect recovery of sparse signals such as the necessary and sufficient conditions in terms of the size of signals, the number of measurements, and the number of nonzero entries in sparse signals as well as the size of finite fields.

This paper is partly presented in the original letter [5] which more describes the derivation on detail results and specific discussions.

II. Compressed Sensing System

We describe the following system model in the finite field of the size q as F_q: Let $x \in F_q^N$ be a signal vector of length N with sparsity k_1, which indicates the number of nonzero entries from 0 to K in x, $k_1 \in \{0,1,...,K\}$, and let $A \in F_q^{M \times N}$ be an $M \times N$ sensing matrix with $N > M$. The sensed signal y is given as

$$y = Ax. \qquad (1)$$

Let Ω denote the set of all signals of length N considered less than sparsity K. And the size of the set Ω is given by $|\Omega| = \sum_{k_1=0}^{K} \binom{N}{k_1} (q-1)^{k_1}$, where $|\cdot|$ denotes the cardinality of the set. The sparse signal is randomly and uniformly selected from the set Ω. We assume that the elements of the sensing matrix A are independent and identically distributed (i.i.d.), so that

$$\Pr\{A_{ij} = \alpha\} = q^{-1}, \qquad (2)$$

where A_{ij} denotes the element of the i-th row and j-th column of the sensing matrix where $i = 1,2,...,M$ and $j = 1,2,...,N$, and α denotes the

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dummy variable such as $\alpha \in F_q$. All theoretical analysis is carried out without considering measurement noise.

III. Conditions on Perfect Recovery

In this section, we derive the necessary and sufficient conditions on unique recovery of sparse signals with high probability. We assume that the decoder in our scheme decides the candidate signal when it finds a sparsest feasible solution $\bar{x}$ satisfying the condition (1) as follows,

$$\min \|\bar{x}\|_0 \text{ subject to } y = A\bar{x}. \qquad (3)$$

where $\bar{x}$ is a feasible solution, and $\bar{x} \in \Omega$ under the condition as $\|\bar{x}\|_0 = k_2 \leq k_1$. We define an occurrence of error when a vector $\bar{x}$ is estimated by satisfying $y = A\bar{x}$ for $x \neq \bar{x}$. With these assumptions, we investigate the probability of error for unique recovery of sparse signals.

The error event ε is defined as follows,

$$\varepsilon := \{(A,\bar{x}) : x \neq \bar{x}, A\bar{x} = y, \bar{x} \in \Omega\}. \qquad (4)$$

Since a signal is randomly chosen from the set Ω, the error probability is given by

$$\Pr\{\varepsilon\} = \frac{1}{|\Omega|} \sum_{x,\bar{x} \in \Omega} \Pr\{Ax = A\bar{x}\}, \qquad (5)$$

where $\Pr\{Ax = A\bar{x}\}$ is the probability that a candidate $\bar{x}$ is a feasible signal. It is to be noted that since the elements of the sensing matrix are i.i.d., then, the probability is

$$\Pr\{A(x-\bar{x}) = 0\} = \prod_{i=1}^{M} \Pr\{A_i(x-\bar{x}) = 0\}, \qquad (6)$$

where A_i denotes the i-th row of A. We compute the probability as follows,

$$\Pr\{A_i(x-\bar{x}) = 0\} = q^{-1}, \qquad (7)$$

where the equality is from the fact that the elements of uniformly random sensing matrices defined in (2) are i.i.d.. Using the upper bounds, (5) can be rewritten as

$$\begin{aligned} \Pr\{\varepsilon\} &= \frac{1}{|\Omega|} \sum_{x,\bar{x} \in \Omega} q^{-M} \qquad (8) \\ &\leq K 2^{NH_b(K/N) + Klog_2(q-1) - Mlog_2 q}, \end{aligned}$$

where $H_b(\bullet)$ denotes the binary entropy. Consequently, from the condition that the exponent of (8) remains negative so that the probability of error goes to 0 as $N \to \infty$, we can derive the following upper bound on M,

$$M \geq \frac{NH_b(K/N) + Klog_2(q-1)}{\log_2 q}. \qquad (9)$$

Next, we derive the necessary condition for the unique recovery of a sparse signal by using the sensing matrix and the sensed signal. Using the Fano's inequality [4], the probability of error $\Pr\{\varepsilon\}$ is bounded as follows,

$$\Pr\{\varepsilon\} \geq \frac{H(x|y,A) - 1}{\log_q |\Omega|}, \qquad (10)$$

where $H(\bullet)$ denotes the entropy. According to the definition of conditional entropy, assuming that A is independent of x, we can rewrite (10) as follows,

$$\begin{aligned} \Pr\{\varepsilon\} &\geq \frac{H(x) - H(y|A) - 1}{\log_q |\Omega|} \qquad (12) \\ &\geq^{(a)} \frac{H(x) - M - 1}{\log_q |\Omega|}, \end{aligned}$$

where the inequality (a) is from the following, $H(y|A) \leq H(y) \leq M$. A vanishing probability of error requires that

$$M \geq \log_q \left[(q-1)^K \binom{N}{K} \right] - 1. \qquad (13)$$

We observe the limit conditions for the recovery of sparse signals over finite fields. In fact, for large values for N, both conditions (9) and (13) are identical.

IV. Conclusions

In conclusion, we evaluated the recovery performance of statistically and randomly chosen sensing matrices to obtain the compressed signal. For uniformly random sensing matrices, we obtained the necessary and sufficient conditions for a unique recovery of a sparse signal with high probability.

References

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