Ordering Aided Schnorr-Euchner Sphere Decoding with Statistical Pruning Based on IRA for MIMO Systems

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***Abstract*—A Schnorr-Euchner sphere decoder (SESD) with *increasing radii algorithm* (IRA) named as the IRA-SESD and two ordering preprocessing strategies are considered in this paper. Statistical constrain radii (SCRs) are obtained from probabilistic distribution of path metric in order to statistically prune branches. Ordering preprocessing schemes are jointly applied to further reduce computational complexity of the IRA-SESD. This ordering aided IRA-SESD presents near-ML performance with low complexity. The proposed scheme has been evaluated by computer simulations for uncoded multiple-input multiple-output (MIMO) systems.**

I. INTRODUCTION

The input-output relationship between the transmitted symbol vector and the received signal vector of MIMO system is described by

$$\tilde{\mathbf{y}} = \tilde{\mathbf{H}}\tilde{\mathbf{x}} + \tilde{\mathbf{z}} \tag{1}$$

where $\tilde{\mathbf{y}} = [\tilde{y}_1, \tilde{y}_2, ..., \tilde{y}_n]^T \in \mathcal{C}^n$ is the vector of received complex signals, $\tilde{\mathbf{x}} = [\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_m]^T \in \mathcal{Q}^m$ is the vector of transmitted symbols which are selected from constellation set $\mathcal{Q}^m$, $\mathrm{E}[\tilde{\mathbf{x}}\tilde{\mathbf{x}}^H] = (E_s/m)\mathbf{I}_m$, E_s is the total transmit energy in each channel use, $\mathbf{I}_m$ is the $m \times m$ identity matrix, $\tilde{\mathbf{z}} = [\tilde{z}_1, \tilde{z}_2, ..., \tilde{z}_n]^T$ is the complex additive white Gaussian noise (AWGN) vector of which elements are independent and identically distributed (i.i.d.) with zero mean and variance $\sigma^2 = N_0/2$ per dimension, and $\tilde{\mathbf{H}} \in \mathcal{C}^{n\times m}$ is a column full rank channel matrix whose elements are i.i.d. zero-mean complex Gaussian variables having 0.5 variance per dimension. Here, we assume that the channel information is perfectly known to the receiver.

The complex signal model in (1) can be transformed to its real-value representation such as

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{z} \tag{2}$$

where $\mathbf{x} = [x_1, x_2, ..., x_M]^T = [\mathrm{Re}(\tilde{\mathbf{x}})^T \mathrm{Im}(\tilde{\mathbf{x}})^T]^T$, $\mathbf{y} = [y_1, y_2, ..., y_N]^T = [\mathrm{Re}(\tilde{\mathbf{y}})^T \mathrm{Im}(\tilde{\mathbf{y}})^T]^T$, $\mathbf{z} = [z_1, z_2, ..., z_N]^T = [\mathrm{Re}(\tilde{\mathbf{z}})^T \mathrm{Im}(\tilde{\mathbf{z}})^T]^T$, and

$$\mathbf{H} = \begin{bmatrix} \mathrm{Re}(\tilde{\mathbf{H}}) & -\mathrm{Im}(\tilde{\mathbf{H}}) \\ \mathrm{Im}(\tilde{\mathbf{H}}) & \mathrm{Re}(\tilde{\mathbf{H}}) \end{bmatrix}.$$

Here, $\mathrm{Re}(\cdot)$ and $\mathrm{Im}(\cdot)$ denote the real and imaginary part of $(\cdot)$ respectively, and $\mathbf{H}$ is the real-valued MIMO channel matrix of N by M and is column full rank matrix of M where $N = 2n$, $M = 2m$. From the lattice theory, we can consider all candidates of $\mathbf{Hx}$ as the lattice $\Lambda(\mathbf{H})$ generated by $\mathbf{H}$.

The maximum likelihood (ML) detector tries to find the closest lattice point $\hat{\mathbf{x}}_{\mathrm{ML}}$ from the received vector $\mathbf{y}$ among the given lattice $\Lambda(\mathbf{H})$, equivalently it can be written by

$$\hat{\mathbf{x}}_{\mathrm{ML}} = \arg\min_{\mathbf{x}\in\mathcal{D}} \|\mathbf{y} - \mathbf{Hx}\|^2 \tag{3}$$

where $\mathcal{D} \subset \mathcal{Z}^M$ is a finite subset of the integer lattice, and $\mathcal{Z}^M$ is the M-dimensional integer lattice. Here, we assume that $\mathcal{D} = \mathcal{A}^M$ and $\mathcal{A}$ is a real-valued signal constellation set; $\mathcal{A} = \{-3, -1, 1, 3\}$ for 16-QAM. (3) is referred to as the *integer least-squares problem*. Although the ML detector is optimal for achieving the minimum error probability, (3) is solved by exhaustive search over multiple dimensions. It is well-known to be NP-hard. Therefore, the direct implementation of the ML detector is impractical in the sense of practical systems.

To reduce computational burden with the optimal ML performance, the Fincke-Pohst sphere decoder (FPSD) [1] was first researched for digital communications assuming fading channel model in [2]. In the later researches [3], [4], the SD was applied to complex-valued MIMO channels for lattice code decoding. In [4]–[6], the Schnorr-Euchner SD (SESD) was suggested as refined version of the FPSD due to intelligent enumeration of candidate symbols at each dimension. The SESD can be expected to further reduce computational complexity than the FPSD.

Basically, in conventional SDs, i.e., FPSD and SESD, the number of pruned branches may not always enough to noticeably reduce the complexity in initial search levels, and especially this problem becomes much worse for large dimensional systems. Hence, the statistical pruning strategies have been proposed to enhance detection efficiency. One of promising techniques is *increasing radii algorithm* (IRA) [7]. It uses probabilistic properties in order to strengthen pruning condition for tree search in SD, and the set of radii having increasing sequence was employed in IRA.

In this paper, the SESD with IRA (IRA-SESD) is considered, on the other hand the works of [7] were implemented based on the enumeration of FPSD. In addition, the ordering preprocessing is jointly applied to the IRA-SESD in order to further reduce complexity. Two kinds of ordering preprocessing are considered in this paper.

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The proposed ordering aided IRA-SESD can significantly reduce complexity with the near-ML performance overall signal to noise ratio (SNR) regime.

This paper is organized as follows: Section II reviews algorithm of the SD and enumeration method of SESD. In Section III, the ordering aided IRA-SESD are explained in detail. In Section IV, performance evaluation of proposed scheme is presented by computer simulations. Finally, we summarize and conclude this paper in Section V.

II. Sphere Decoding

In this section, we introduce the algorithm and formulations for SD and describe Schnorr-Euchner (SE) enumeration for candidate symbols in each level.

A. Sphere Decoding Algorithm

In order to achieve the ML solution, the SD tries to examine all $\mathbf{x}$ satisfying

$$\|\mathbf{y}-\mathbf{H}\mathbf{x}\|^2 \le C_0^2 \tag{4}$$

where C_0 is the radius of a hypersphere. The inequality (4) is referred to as the *sphere constraint*. The sphere constraint is the condition that the transmitted symbol vector lies within a hypersphere with a radius C_0. If the radius C_0 is sufficiently large enough to include the closest lattice point $\hat{\mathbf{x}}_{\mathrm{ML}}$, then the SD guarantees to provide the optimal ML solution. Fig. 1 illustrates the fundamental idea of SD for two dimensional case. The ML detector using exhaustive search tries to evaluate whole lattice points, whereas the search space of SD is limited to inside only a hypersphere of the radius C_0 in lattice space so that the required complexity to solve (3) is decreased.

It can be understood that the FPSD and the SESD merely adopt different enumeration method for candidate symbols at each dimension, hence the FPSD and the SESD are able to be formulated as similar manner. For the sake of brevity, we let $N=M$ for $\mathbf{H}$. The QR decomposition is applied to the channel matrix $\mathbf{H}$ in (4) such as

$$\mathbf{H}=\mathbf{Q}\mathbf{R} \tag{5}$$

where $\mathbf{Q}$ is an orthogonal matrix and $\mathbf{R}=[r_{i,j}]\in\mathcal{R}^{M\times M}$ is an upper triangular matrix with non-negative diagonal entries (Equivalently, the Cholesky decomposition of the Gram matrix, i.e., $\mathbf{G}=\mathbf{H}^T\mathbf{H}$ can be used). Then, (4) can be described as

$$\begin{aligned}&\|\mathbf{y}-\mathbf{Q}\mathbf{R}\mathbf{x}\|^2\\ &=\left\|\mathbf{Q}^T\mathbf{y}-\mathbf{R}\mathbf{x}\right\|^2=\left\|\mathbf{R}(\mathbf{R}^{-1}\mathbf{Q}^T\mathbf{y})-\mathbf{R}\mathbf{x}\right\|^2\\ &=\|\mathbf{R}(\hat{\mathbf{x}}-\mathbf{x})\|^2\le C_0^2\end{aligned} \tag{6}$$

where $\hat{\mathbf{x}}=\mathbf{R}^{-1}\mathbf{Q}^T\mathbf{y}=\mathbf{H}^{\dagger}\mathbf{y}$ is an unconstrained least-squares (LS) solution.

The solution of (6) can be achieved by recursion of tree search approach, and it is processed from the M-th element x_M to 1-st element x_1 for $\mathbf{x}$, i.e., working backward. Due to the triangularity of $\mathbf{R}$, (6) can be expanded as

$$\begin{aligned}&\|\mathbf{R}(\hat{\mathbf{x}}-\mathbf{x})\|^2\\ &=r_{M,M}^2(\hat{x}_M-x_M)^2\\ &+r_{M-1,M-1}^2(\hat{x}_{M-1}-x_{M-1}+\tfrac{r_{M-1,M}}{r_{M-1,M-1}}(\hat{x}_M-x_M))^2\\ &+\ldots\\ &=B_M+B_{M-1}+\ldots+B_1\le C_0^2\end{aligned} \tag{7}$$

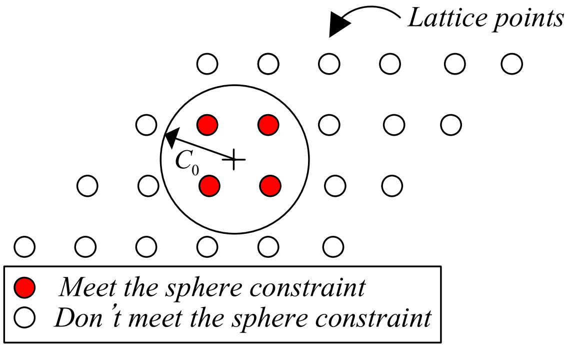

Fig. 1. Principle idea of the sphere decoder

where B_k represents the branch metric corresponding to level k (or depth $M-k+1$) in tree.

From (7), one can construct the tree with depth M, and thereby the SD algorithm can be understood as the solving of tree search problem. In the level k of the tree, the branch metric is described by

$$B_k=r_{k,k}^2\left[(\hat{x}_k-x_k)+\sum_{j=k+1}^{M}\frac{r_{k,j}}{r_{k,k}}(\hat{x}_j-x_j)\right]^2. \tag{8}$$

Here, B_k has arguments of $x_M,x_{M-1},\ldots,x_k$. Then, the path metric from root to level k is accumulation of the branch metric such as

$$P_k=\sum_{i=k}^{M}B_i. \tag{9}$$

Whenever a valid symbol vector satisfying the sphere constraint of (4) is found at bottom level $k=1$, the radius square C_0^2 is replaced by the path metric P_1 corresponding to searched symbol vector. Then, the search processing is restarted with new C_0^2. This search processing continues until there is no candidate symbol vector satisfying the sphere constraint. Thereby the search space of SD is iteratively restricted, and the ML solution could be obtained with substantially reduced complexity in SD.

B. Schnorr-Euchner Enumeration

The enumeration of candidate symbols at each level is based on the natural spanning called the Finke-Pohst (FP) enumeration in the FPSD [1], whereas the SESD leads to faster shrinkage of the search space by using the SE enumeration [5]. The SE enumeration spans candidate symbols in a zig-zag manner starting from the mid-point, i.e., the *Babai* point [6]. The candidate symbols are examined in an ascending order according to their branch metric in the SESD. Hence, when one candidate violates the sphere constraint, next candidates don't comply with sphere constraint, too. Thus, all of the next candidates can be pruned. This property is helpful to alleviate complexity. The first lattice point explored by the SESD with $C_0=\infty$ is Babai point which is referred to as the nulling and canceling (NC) point in the communication parlance [8].

III. Ordering Aided IRA-SESD

In this section, we propose IRA-SESD, and an ordering preprocessing is presented.

A. IRA-SESD

Recently, it was revealed that an average complexity of the SD is still exponential for large dimensional systems [9]. One reason of above results is that sphere constraint forced by a radius C_0 is relatively loose for initial levels. This problem becomes more serious for systems with large dimensions. Thus, the strategy to solve this difficulty is necessary to reduce computational redundancy.

The schedule of radii is employed for all level in the IRA to prune branches statistically. Hence, the IRA provides the forceful pruning condition and restricts large search space with high probability containing the optimal ML solution. When there are no feasible symbols within the reduced search space by a set of radii, the decoding is restarted with another set of larger radii until any candidate symbol vector is detected.

The proposed IRA-SESD also adopts the set of radii and is implemented by SE enumeration unlike IRA of [7] which is based on FP enumeration. [7] chooses a linear schedule as $r_i^2 = (\delta \log M + i)\sigma^2$ for $i = 1, ..., M$ to determine radii (see [7] for more detail). In the IRA-SESD, these radii are obtained directly from a statistical function, i.e., inverse cumulative distribution function (cdf) of $\chi_k^2(\sigma^2)$ and are named as *statistical constraint radii* (SCRs) in this paper. Here, $\chi_k^2(\sigma^2)$ stands for the Chi-square distribution with k degrees of freedom having probability density function (pdf) $f(x) = \frac{1}{2^{k/2}\Gamma(k/2)\sigma^k} x^{(k/2)-1} e^{-x/(2\sigma^2)}$ where $\Gamma(\cdot)$ denotes the Gamma function.

In the IRA-SESD, a set of the SCRs is denoted as

$$\mathbf{D} = \{D_1, ..., D_k, ..., D_M\} \tag{10}$$

where D_k is the SCR for level k in tree. For actual transmitted symbol vector $\mathbf{s}$, we can represents total path metric of $\mathbf{s}$ by (7), (9) such as

$$P_1 = \|\mathbf{R}(\hat{\mathbf{x}} - \mathbf{s})\|^2 = \left\|\mathbf{Q}^T\mathbf{z}\right\|^2 = \|\mathbf{v}\|^2 \tag{11}$$

where $\mathbf{v}$ is an i.i.d. noise vector with the same statistical properties of $\mathbf{z}$ (Because $\mathbf{Q}$ is an orthogonal matrix). Thus, the path metric for actual transmitted symbol vector $\mathbf{s}$ at level k is given by

$$\tilde{P}_k = \sum_{i=k}^{M} |v_i|^2 \tag{12}$$

where v_i is i-th component of $\mathbf{v}$. $\tilde{P}_k$ follows the Chi-square distribution with $M-k+1$ degree of freedom, i.e., $\chi_{M-k+1}^2(\sigma^2)$.

We denote pruning probability ε such that $\varepsilon = \Pr[\tilde{P}_k > D_k^2]$. ε is selected by user's requirement. And D_k satisfies

$$\Pr[P_k \leq D_k^2] \geq 1 - \varepsilon. \tag{13}$$

Hence, we can obtain the set $\mathbf{D}$ of SCRs from the inverse cumulative distribution function (cdf) of $\chi_{M-k+1}^2(\sigma^2)$ such as $D_k^2 = F^{-1}(1-\varepsilon; M-k+1)\sigma^2$ where $F^{-1}(\cdot;\cdot)$ is the inverse cdf of the standard Chi-square distribution, i.e., $\chi_{M-k+1}^2(\sigma^2 = 1)$. Before decoding algorithm starts, the set $\mathbf{D}$ is pre-calculated by using mathematical tools. During search processing, $\mathbf{D}$ is applied as look-up table.

B. Ordering Preprocessing for the IRA-SESD

The ordering preprocessing of the columns of channel matrix $\mathbf{H}$ is helpful to further reduce computational burden in SD, since it is expected to enhance the chance to find the ML solution during initial search period. Several ordering preprocessing schemes have been studied [4], [10]. The ordering preprocessing is computed only once at the starting point of each received frame in packet based communication systems. Accordingly, the complexity of ordering preprocessing is negligible under assumption of slow fading channel.

In this paper, two kinds of ordering preprocessing are applied for the ordering aided IRA-SESD. The first algorithm is known as column norm ordering [4]. This heuristic method is ordering the columns $\mathbf{h}_i$ of the channel matrix $\mathbf{H} = [\mathbf{h}_1, ..., \mathbf{h}_{M-1}, \mathbf{h}_M]$ according to their Euclidean norm $\|\mathbf{h}_i\|$ in an ascending manner. The ordering of $\mathbf{h}_i$ is processed by the permutation π so that

$$\left\|\mathbf{h}_{\pi(i)}\right\| \leq \left\|\mathbf{h}_{\pi(j)}\right\| \text{ for } i < j. \tag{14}$$

Then, the ordered channel matrix is represented as

$$\mathbf{H}' = \mathbf{H}\Pi \tag{15}$$

where Π is $M \times M$ permutation matrix such as

$$\Pi = \left[\begin{array}{cccc} \mathbf{e}_{\pi(1)}, & \mathbf{e}_{\pi(2)}, & \cdots, & \mathbf{e}_{\pi(M)} \end{array}\right] \tag{16}$$

where $\mathbf{e}_i$ is the column vector of which entries are 1 in only i-th position and are 0 in every other positions.

Another ordering preprocessing is the V-BLAST-ZF ordering [10]. In this scheme, the ordering preprocessing tries to maximize the minimum post-detection SNR over all detection order. As a result, choosing the stream of best SNR at each order leads to the optimum ordering in maximin sense. The ordering of $\mathbf{h}_i$ is done iteratively according to the Euclidean norm of the row vectors of $\mathbf{H}^\dagger$ in descending fashion. The detail description of ordering procedure can be referred in [10].

IV. Simulations

The performance of ordering aided IRA-SESD has been investigated by using Monte Carlo simulations. In our simulations, the 4×4 uncoded spatially multiplexed (SM)-MIMO system with 16-QAM modulation is considered (After converting to the real-valued system, $M = N = 8$). Hence, the spectral efficiency of corresponding system is 16 bits/s/Hz. We have $E_b/N_0 = E_s/N_0 + 10\log_{10}\frac{N}{MS}$ where E_b denotes average power per one bit, and S is number of bits per one transmitted symbol. The MIMO channel model is assumed to a slow flat Rayleigh fading channel. The number of channel generation is at least 10000, and the channel matrix is fixed for 100 symbol times.

The BER and the average floating point operations (FLOPs) which are counted by *MATLAB* are presented in order to evaluate the decoding accuracy and complexity. The counting of average FLOPs includes only the part of search processing except ordering preprocessing, QR decomposition, and matrix inversion of $\mathbf{H}$, since computational complexity of search processing is the most dominant in the sense of total complexity. The initial

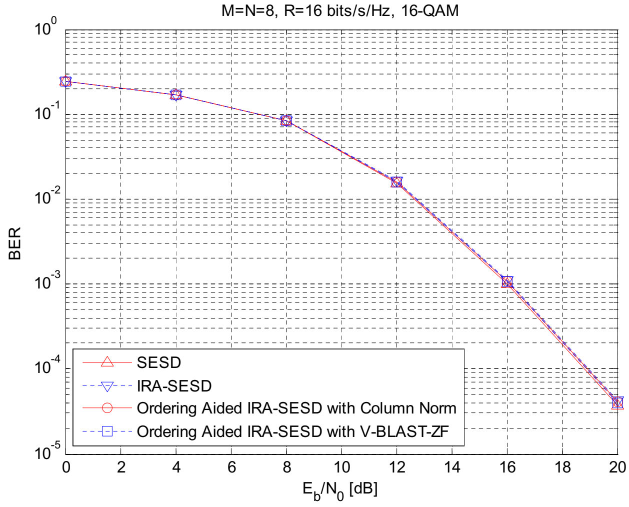

Fig. 2. The BER curves of SDs for uncoded SM-MIMO system with 4 transmit antennas and 4 receive antennas with 16-QAM ($M = 2m$)

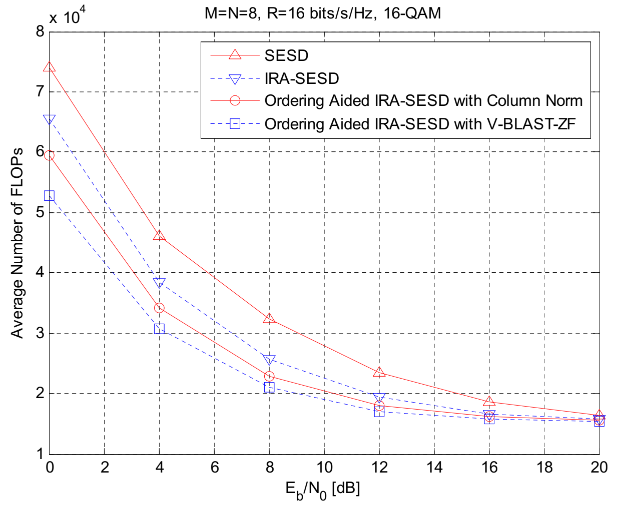

Fig. 3. The average FLOPs of SDs for uncoded SM-MIMO system with 4 transmit antennas and 4 receive antennas with 16-QAM ($M = 2m$)

radius C_0 is set to infinity for all considered SDs. For the ordering aided IRA-SESD and IRA-SESD, the set of pruning probability ε is selected as $\{0.1, 0.01, 0.001, 0\}$, and the corresponding SCRs are pre-computed for all SNR region. We simulated the SESD, the IRA-SESD, and two ordering aided IRA-SESDs with column norm ordering and V-BLAST-ZF ordering.

Fig. 2 presents BER curves of considered SDs. The SESD shows the exact optimal ML performance, and the IRA-SESD and ordering aided IRA-SESDs also present BER curves vary close to that of the SESD. Thus, we can notice that proposed schemes provide near-ML performance. Also, among the IRA-SESD and two ordering aided IRA-SESDs, the difference of BER curves is negligible.

Fig. 3 represents the average FLOPs of considered SDs in order to compare computational complexity. It can be seen that the ordering aided IRA-SESD with V-BLAST-ZF preprocessing presents the lowest complexity among simulated SDs. The reduction effect of complexity becomes weak in ordering aided IRA-SESDs, since the first symbol searched by the SD tends to be the ML solution with the high probability. The ordering aided IRA-SESD with V-BLAST-ZF requires about 63% average FLOPs of SESD at $E_b/N_0 = 8$dB. As seen in Fig. 3, the ordering preprocessing is helpful for statistical pruning strategies such as IRA-SESD. If the dimension of MIMO system becomes large, then we can expect much more computational benefit in ordering aided IRA-SESDs.

V. Conclusion

In this paper, we have been studied for the IRA-SESD and the ordering aided IRA-SESD in SM-MIMO system. In the ordering aided IRA-SESD, the SCRs can be pre-computed in offline in order to prune branches statistically. Also, the ordering aided IRA-SESD takes advantage of ordering preprocessing simultaneously. From simulation results, it can be noticed that the ordering aided IRA-SESD is helpful to reduce computational complexity with near-ML performance. Especially, it would be promising technique for large dimensional MIMO systems. Furthermore, one can handle the trade-off between the BER performance and the detection complexity by controlling pruning probability ε in proposed scheme.

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References

[1] U. Fincke and M. Pohst, Improved methods for calculating vectors of short length in a lattice, including a complexity analysis, *Mathematics of Computation*, vol. 44, pp. 463-471, Apr. 1985.

[2] E. Viterbo and J. Boutros, A universal lattice code decoder for fading channel, *IEEE Trans. Inform. Theory*, vol. 45, pp. 1639-1642, July 1999.

[3] M. O. Damen, A. Chkeif, and J. C. Belfiore, Lattice code decoder for spacetime codes, *IEEE Communications Letters*, vol. 4, no. 5, pp. 161-163, May 2000.

[4] M. O. Damen, H. Gamel, and G. Caire, On maximum-likelihood detection and the search for the closest lattice point, *IEEE Trans. Inf. Theory*, vol. 49, pp. 2389-2402, Oct. 2003.

[5] C. P. Schnorr and M. Euchner, Lattice basis reduction: Improved practical algorithms and solving subset sum problems, *Math. Programming*, vol. 66, pp. 181-191, 1994.

[6] E. Agrell, T. Eriksson, A. Vardy, and K. Zeger, Closest point search in lattices, *IEEE Trans. Inform. Theory*, vol. 48, pp. 2201-2214, Aug. 2002.

[7] R. Gowaikar and B. Hassibi, Efficient statistical pruning for maximum likelihood decoding, *in Proc. IEEE ICASSP*, 2003, vol. 5, pp. 49-52.

[8] C. J. Foschini, G. D. Golden, R. A. Valenzuela, and P. W. Wolniansky, “Simplified processing for high spectral efficiency wireless communication employing multi-antenna arrays,” *IEEE J. Select. Areas Commun.*, vol. 17, pp. 1841-1852, Nov. 1999.

[9] J. Jaldén and B. Ottersten, On the complexity of sphere decoding in digital communications, *IEEE Trans. Signal Processing*, vol. 53, no. 4, pp. 1474-1484, Apr. 2005.

[10] P. W. Wolniansky, G. J. Foschini, G. D. Golden, and R. A. Valenzuela, “V-BLAST: An architecture for realising very high data rates over the rich-scattering wireless channel,” in *Proc. ISSSE*, 1998, pp. 295-300.