Derivation of Black-Scholes equation for the Beginners of financial engineering

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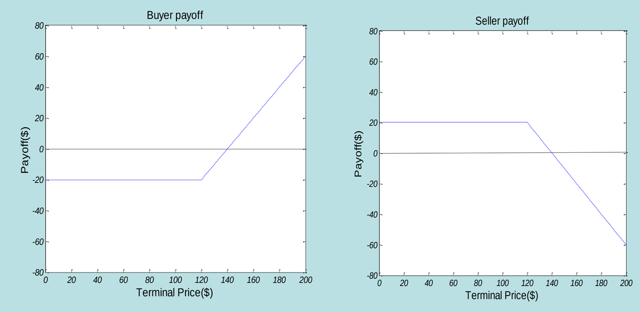
ABSTRACT

In this paper, our aim is to understand the fundamental concept of the Black-Scholes equation by carefully re-deriving it. We have constructed an asset model using the concept of stochastic processes. MATLAB based simulation program has been developed to facilitate the understanding of key concepts such as the Wiener processes and Generalized Wiener processes. Importantly, we have paid our attention to the Ito process and Ito's lemma which enable us to derive stochastic differential equations for modeling stock and option price variation. We have then constructed two portfolios, with which and utilizing the no-arbitrage principle, the Black-Scholes equations have been obtained. Two MATLAB based numerical examples for option pricing, one for the Monte Carlo and the other for the Binomial method, have been included. Our study is mostly aimed at an introductory level throughout the paper, say to help a junior year college student in a financial engineering program. The derivation process of our asset models, nevertheless, is deemed to give some substance which will be useful for analyzing, and comparing with, other asset variation and pricing models.

INTRODUCTION

Option: Derivative of underlying assets. It consists of two species; Call & Put Options.

Call/Put Option = Buyer of Call/Put has right to buy/sell certain financial instrument at designated price at designated date.



Buyer & Seller's Payoff Graph

Option Pricing Issue : Buyer and seller must agree with a Certain Option price; Otherwise, one have high possibility to lose.

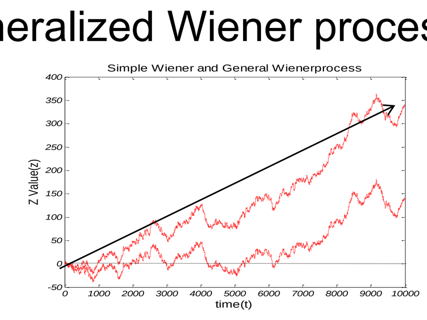
Ex. Stock market has high Volatility

MATERIALS AND METHODS

Wiener process

Wiener process W_t has properties :

1. After Δt $\Delta W_t = N(0,1^2)\sqrt{\Delta t}$ where $N(0,1^2)$ represents for standard normal distribution.
2. Any non-overlapping Interval of time, ΔW_t is **i.i.d** (Independent and identically distributed) process.
3. Generalized Wiener process $\Delta X = a\Delta t + b\Delta W_t$
a is so called drift rate and b is variance rate.



Above graph designates generalized Wiener process.

Ito process & Ito's lemma

Ito process is general version of Wiener process

General form : $\Delta X = a(x,t)\Delta t + b(x,t)\Delta W_t$

To Solve this **Stochastic Differential Equation**, he need to threat complex mathematical background so called Ito's Integrals methods.

***Ito's lemma** : When we have function $G(X,t)$ and X follow Ito's process $\Delta X = a(x,t)\Delta t + b(x,t)\Delta W_t$, Then $G(X,t)$ also follows Ito process which expressed as

$$\Delta G = \left(\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2\right)\Delta t + \frac{\partial G}{\partial x}b\Delta z.$$

Underlying Asset Modeling

Assume underlying asset has expected return rate u

$$S(t_{i+1}) = S(t_i)e^{udt} \qquad S(t_{i+1}) - S(t_i) = dS = S(t_i)udt$$

Add volatility which assumed to be proportional to asset

$$dS = S(t)udt + \sigma S(t)dz$$

Substitue $G = \ln S$, using Ito's lemma, we have

$$dG = \left(\frac{\partial G}{\partial S}\mu S + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial G}{\partial S}\sigma S dz \qquad \frac{\partial G}{\partial S} = \frac{1}{S}, \frac{\partial^2 G}{\partial S^2} = -\frac{1}{S^2}, \frac{\partial G}{\partial t} = 0$$

$$dG = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dz \qquad \ln S_T \sim N\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right]$$

Black- Scholes equation

Option Price function f is determined by underlying asset price S and time t.

So, using Ito's lemma we have $\Delta f = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t + \frac{\partial f}{\partial S}\sigma S \Delta z$ …(1)

And we also assume that out underlying asset is following $dS = S(t)udt + \sigma S(t)dz$ …(2)

Consider **two portfolio**:
Portfolio A : sell one Option, Buy $\frac{\partial f}{\partial S}$ amount of underlying asset.

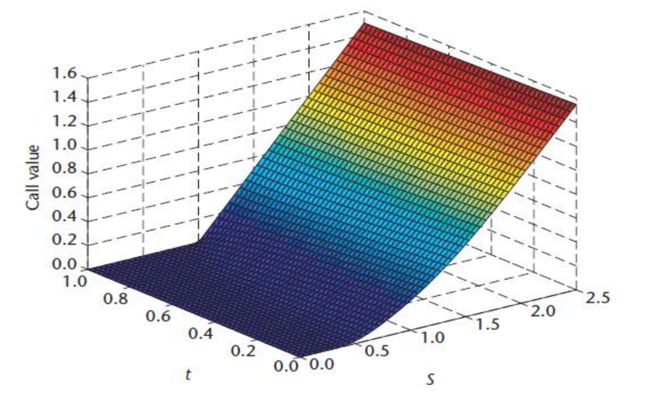
Portfolio B : Keep same Value of Portfolio A in the risk-free instrument.

Porrtfolio A,B both have same value at time t=0,
$\Pi = -f + \frac{\partial f}{\partial S}S$ …(3)

$$\Delta \Pi = -\Delta f + \frac{\partial f}{\partial S}\Delta S$$

$$\Delta \Pi = \left(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t \quad \text{…(4)}$$

Considering **no-arbitrageurs principle** we must have $\Delta \Pi = r\Pi\Delta t$ …(5)



Black- schoels equation

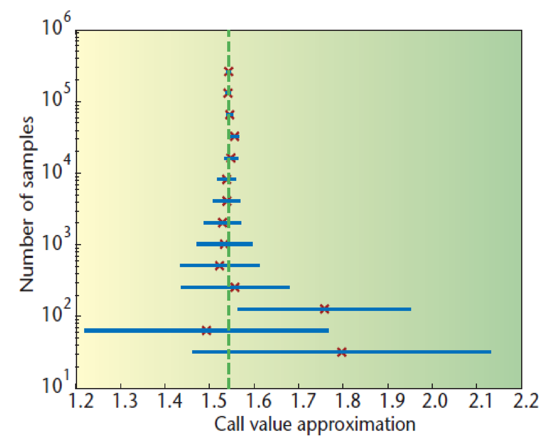
$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 = rf.$$

Discussion for Numerical Methods

Monte carlo method : we can pricing option by using simulation on MATLAB

for $i = 1$ to M
  set $S_i = S_0 e^{(r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\xi_i}$
  set $P_i = e^{-rT}\max(S_i - E, 0)$
end
set $P_{\text{mean}} = \frac{1}{M}\sum_{i=1}^{M} P_i$

set $P_{\text{var}} = \frac{1}{M-1}\sum_{i=1}^{M}(P_i - P_{\text{mean}})^2$



Binomial methods : we can pricing option by assuming that Volatility part for each process is binomial process

$$C_n^i = e^{-r\Delta t}\left(pC_{n+1}^{i+1} + (1-p)C_n^{i+1}\right), 0 \le n \le i, 0 \le i \le M-1.$$

RESULTS

In this paper, I derived Black-Scholes equation. Firstly, I introduced the concept of Wiener process and Generalized Wiener process. Although the derived real underlying model was Ito process, this concept was very helpful for understanding the system of underlying asset model. And I introduced about Ito's lemma. This lemma was very powerful tool for analyzing Ito's process. The issue here was that the function of Ito process random variable also follows kind of Ito process. I mainly used Ito process and Ito's lemma for deriving underlying asset model. And finally, I introduced Option Pricing function and two portfolios. By using Ito's lemma and no-arbitrageurs principle, I made Black-Scholes equation. This black-Scholes equation is kind of initial value partial stochastic differential equation problem. And has no explicit form for general initial condition. Although it is difficult to solve and hard to analyzing it, the pathway of constructing this equation teaches us system modeling methods and interpretation method about modeled system.

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