Wireless Communications

Module-1

Agenda

- Course Schedule
- Signal Space Representation & Optimal Receiver

E-mail

- My e-mail is

  heungno@gist.ac.kr
- I will use e-mail for sending lecture notes and special announcements.

Course Information

- Class hours: 9:00-10:30 am Monday, Wednesday
- Lecture room: B201
- Office hours:
  - 10:30am ~ 11:30am Monday,
  - 10:00am ~ 11:00am Tuesday.
  - Or make an appointment via e-mail.

Grade Distribution

- Two exams (Midterm#1: 20%, Final: 30%)
- Homework + Homework Grading (20%)
- Term Project (30%)
  - Binary modulation over AWGN channel simulation
  - Add LDPC encode/decode simulation
  - Add variations such as
    - Wireless network codes
    - Slepian Wolf distributed source coding
    - Compressive sensing

Tentative Schedule

1st week	General overview (Shannon's 1948 paper)	
2nd week	Optimal Transceiver	
3rd week	Gallager's Channel Coding Theorem	
4th week	Gallager's Channel Coding Theorem	
5th week	LDPC codes and probabilistic decoders	
6th week	Multipath fading channels/Diversity systems	
7th week	MIMO capacity theorems	
8th week	MIMO transceivers	Midterm 1
9th week	Design of LDPC and space-time codes and receivers	
10th week	Performance evaluation of MIMO transceivers	
11th week	Multi-user capacity/multi-user MIMO receivers	
12th week	Design of pre-coding MIMO signals	
13th week	Network codes	
14th week	Wireless network codes	
15th week	Overview	
16th week	Final project + Final Exam	Final project due

Homework, Class-Project Policies

- Discussion and exchange of ideas are strongly encouraged.
- You may submit your homework and project reports as a team of two persons.
- On each homework and class project set, a reviewer team will be assigned (will take turns).
- The job of each reviewer team is to
  - grade homework/project sets,
  - type up the best homework solution(rec. WORD with Mathtype),
  - get an approval of the solution manual from me, and
  - distribute the graded homework and solution to the students within a week.

Scope of this course

- In this course, we will learn wireless and MIMO networks with help of
  - Information Theory
  - Digital Communications Theory
  - Channel Coding Theory
- What's relevant are
  - Complexity of the system (Is the system implementable?)
  - Performance of the system (Probability of decision errors)
  - How far is it from the theoretical limit?

Scope of this course (2)

- Learn to apply the estimation/detection theory to communications problems.
- Learn to simulate communications systems for the purpose of evaluating a communications system.
- Be able to analyze the obtained simulation data and predict the performance of a given system, and provide a better design.
- Once we know how to predict/evaluate the performance of a communications system, we will use these knowledge and tool sets to design a better performing communications system.
- I say this is the way how the communications theory has been evolved.

Text Books

- Textbook: Proakis/Salehi, Digital Communications, 5th Edition, McGraw-Hill.

- Reference-1: Robert Gallager, Information Theory and Reliable Communication, John Wiley & Sons, Inc.   New York, NY, USA, 1968. ISBN:0471290483
- Reference-2: David Tse and P. Viswanath, Fundamentals of Wireless Communication, Cambridge Press, 2005.

  ISBN: 0521845270
- Reference-3: J. M. Wozencraft and I. M. Jacobs, Principles of Communication Engineering, Prospect Heights , Illinois, Waveland Press, 1990.

About Proakis/Salehi

- The book is like an encyclopedia of communications theory.
  - Covers a lot of topics
  - I certainly will not aim to cover all of them
  - I'll go over certain topics quickly to help your reading.
- Some homework problems will be taken from the book.

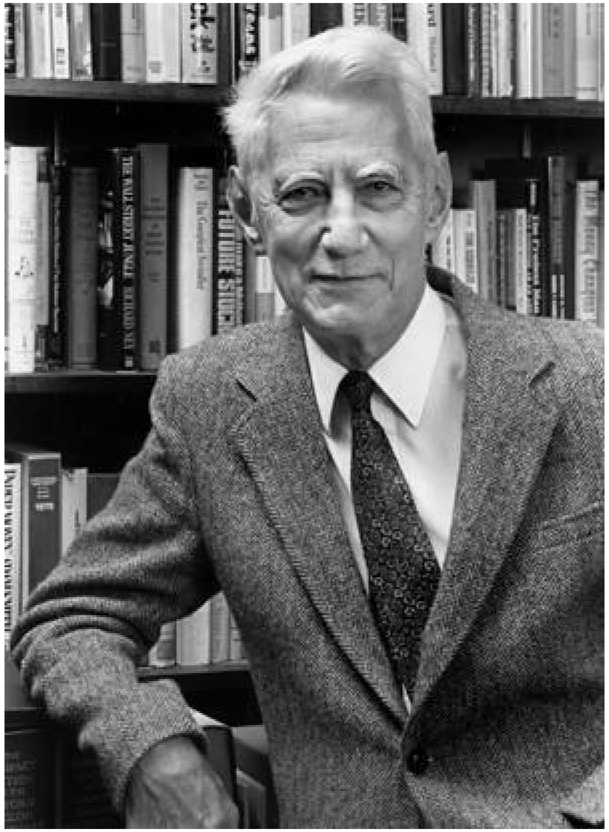
Interaction with other related areas

- Optimization
- Signal and Image Processing
- Estimation/Detection Theory
- Pattern Recognition
- Neural Network
- Artificial Intelligence
- Bio-informatics

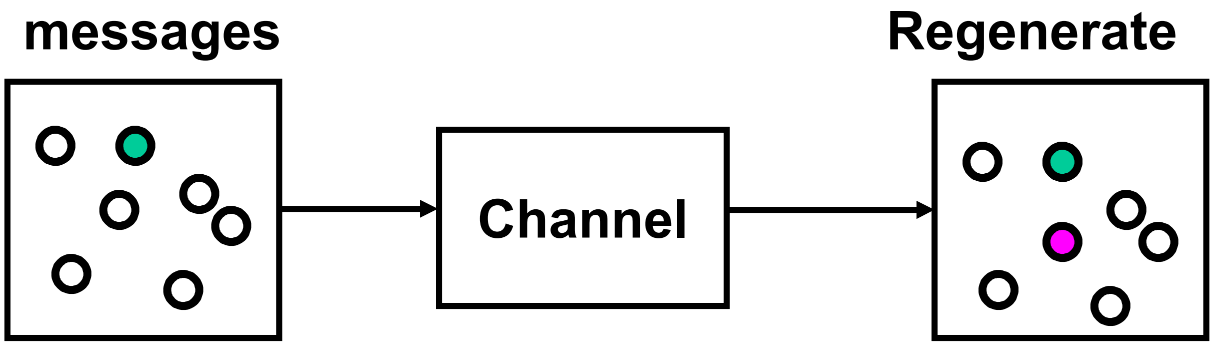
- Now, let's begin…

Claude E. Shannon (1916 -- 2001)

- Math/EE Bachelor from UMich (1936)
- MSEE and Math Ph.D. from MIT (1940)
- A landmark paper "Mathematical Theory of Communications" (1948)
  - Founder of Information Theory
  - Fundamental limits on communications
  - Information quantified as a logarithmic measure
- For more info on him, make a visit to http://www.bell-labs.com/news/2001/february/26/1.html



Shannon's Perspective on Communications



- Communications: Transfer of information from a source to a receiver
- Messages (information) can have *meaning*; but they are irrelevant for the design of communications system.
- What's important then?
  - *A message is selected from a set of all possible messages and transmitted, and regenerated at the receiver*
  - The size of the message set is the amount of information
- *The capacity of a channel is the maximum size of message set that can be transferred over the channel and can be regenerated almost error-free at the receiver.*

Digital Communication

- It is to send a message index m (out of M total) over *the channel*
  - for the duration of time T , and
  - have an expectation that the same index m can be recovered almost error-free at the receiver.
- Transmission rate $R = \log_2(M)/T$ [bits/sec]
- *If* $R < C$, *then* almost error-free recovery can be achieved.
- We need to find a set of M waveforms to interface the channel.
  - An analog (physical) waveform shall be chosen to carry the messages. Why?
  - We may choose orthogonal waveforms
    - Pulse-position, frequency-position (OFDM), sin(x)/x, …, or any other orthogonal signal set
  - We need to find out how we can choose them.

Additive Noise Channel

❖ Let's consider a simple channel

$$r(t) = s(t) + n(t)$$

– received signal = signal + noise

Digital Representation

- Note that the simple ANC is based on continuous signals and noise
- We aim to replace the ANC with digits and vectors. Why?
  - Easier to deal with (digits rather than continuous waveforms)
    - Computer simulations without loss of information
    - Easier to do analysis
    - Easier to design
- How?
  - Via the use of *vector space* idea.

Hilbert Space

- A vector in the *vector space* (Hilbert space) can be added together and multiplied by scalars
  - Norm exists ($\sum_k |v_k|^2 < \infty$ or $\int_t |v(t)|^2 \, dt < \infty$)
  - Schwartz inequality holds ($|v_1 \cdot v_2| \leq ||v_1|| \, ||v_2||$ or $|\int_t f(t)g(t) \, dt| \leq \int_t |f(x)|^2 \, dt)^{1/2} (\int_t |g(t)|^2 \, dt)^{1/2}$)

- Extension of the ideas of *length* and *inner product* from vector space to signal space.

- Approximation of a function with a series of (finite number of) orthonormal functions.
  - the Karhunen-Loeve Expansion (pg. 76, Proakis/Salehi)

Transformation

- A set of *linearly independent* vectors which *spans* a vector space is called a *basis*.
- A coordinate system can be represented by a basis.
- There can be infinitely many coordinate systems in a *vector space*.
- Any vector with a finite norm can be represented by a linear combination of basis vectors.
- Any vector (a signal) with a finite norm can be represented by any coordinate system.
- Change of basis to represent a vector can be performed. This is called *transformation*.

Transformation (2)

- Example of transformation of a vector in different coordinate system
  - Representation of vectors in Cartesian coordinate systems
- Usually, we choose an orthonormal set of vectors as a basis
  - Norm of each basis vector is 1.
  - Inner product between any pair of basis vectors is 0.

Transformation (3)

- A vector (a signal) represented by $\{\mathbf{b}^1{}_k\}$ can be transformed into other coordinate system, or be represented by $\{\mathbf{b}^2{}_k\}$.
  - Let $\{\mathbf{b}^1{}_k\}$ and $\{\mathbf{b}^2{}_k\}$ be two different bases (*orthonormal*)
    - Let $\{\mathbf{b}^1{}_k\} = \{(1, 0, 0, \ldots), (0, 1, 0, \ldots), (0, 0, 1, \ldots), \ldots\}$ be Cartesian
  - Representation of a vector $\mathbf{x} = (x_1, x_2, \ldots)$ with the 1^{st} basis is

    $$\mathbf{x} = \sum_k (\mathbf{x} \cdot \mathbf{b}^1{}_k)\, \mathbf{b}^1{}_k = \sum_k v_k\, \mathbf{b}^1{}_k$$
  - Representation of the same vector with the 2^{nd} basis is

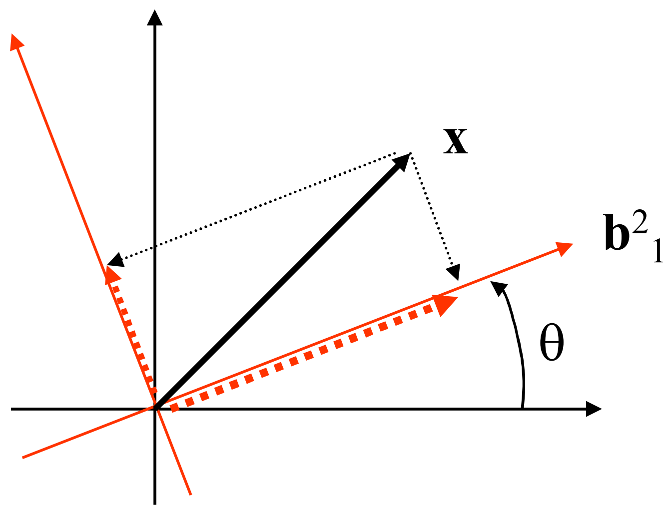
    $$\mathbf{x} = \sum_k (\mathbf{x} \cdot \mathbf{b}^2{}_k)\, \mathbf{b}^2{}_k$$

projection

k-th basis vector

Cartesian Coordinate

- $\mathbf{x} = (x_1, x_2) = (1, 1)$
- **Basis-1 =** $\{\mathbf{b}^1{}_1, \mathbf{b}^1{}_2\}$
  - $\mathbf{b}^1{}_1 = (1, 0)$
  - $\mathbf{b}^1{}_2 = (0, 1)$
  - $\mathbf{x} = x_1\, \mathbf{b}^1{}_1 + x_2\, \mathbf{b}^1{}_2$
- **Basis-2 =** $\{\mathbf{b}^2{}_1, \mathbf{b}^2{}_2\}$
  - $\mathbf{b}^2{}_1 = (\cos(\theta), \sin(\theta))$
  - $\mathbf{b}^2{}_2 = (-\sin(\theta), \cos(\theta))$
  - $\mathbf{x} = (\mathbf{x} \cdot \mathbf{b}^2{}_1)\, \mathbf{b}^2{}_1 + (\mathbf{x} \cdot \mathbf{b}^2{}_2)\, \mathbf{b}^2{}_2$
- Transform (changing the basis from 1 to 2)
- Inverse-Transform (change back to 1 from 2)



Note that x stays the same, throughout!

Orthogonal Signals

- Functions $\psi_n(t)$ and $\psi_m(t)$, a < t < b, are orthogonal when

  $$\int_a^b \psi_n(t)\, \psi^*_m(t)\, dt = 0, \text{ for } m \neq n$$

  – Inner-product is zero

- Self inner-product is the energy K_n of the function

  $$\int_a^b \psi_n(t)\, \psi^*_n(t)\, dt = K_n$$

- A collection of orthogonal functions $\{\psi_n(t)\}$ is said to be an orthogonal set when the collection satisfies the following:

  $$\int_a^b \psi_n(t)\, \psi^*_m(t)\, dt = K_n\, \delta_{n,m}$$

Kronecker delta func
cf) Dirac delta function

Geometric View of Signals and Noise

- We aim to represent signals and noise with orthonormal signals.

- Suppose we have a collection of signals, $\{\psi_j(t), j \in 1,2, \ldots\}$, $0 \leq t \leq T$, *orthonormal* to each other.
  - *Orthonormality*: $\int_0^T \psi_j(t)\, \psi^*_k(t)\, dt = \delta(k - j)$

    where $\delta(k\text{-}j)$ is the Kronecker's delta function.
- The set of *orthonormal* signals can form a vector space.
- We can use the first N signals $\{\psi_j(t),\ j \in 1,2, \ldots, N\}$ as a basis for the signal space.

Representation of signals using $\{\psi_j(t),\ j \in 1,2, \ldots, N\}$

- A basis can be used to represent ANY signal in the space

  $$s(t) = \sum_{j=1}^{N} s_{\mathrm{j}}\, \psi_j(t),\ 0 \leq t \leq T$$

  where $s_j = \int_0^{\mathrm{T}} s(t)\, \psi^*_j(t)\, \mathrm{d}t$, for $1 \leq j \leq N$.
- Finally, a continuous signal $s(t)$ can be written as an N-tuple vector, i.e., $\mathbf{s} = (s_1, s_2, \ldots, s_N)$.

Energy Conservation

- The signal energy E

  $E = \int_0^T s^2(t)\, dt = \int_0^T [\sum_j s_j\, \psi_j(t)]\, [\sum_q s_q \psi_q(t)]^*\, dt$

  $= \sum_j \sum_q s_j\, s^*_q \int_0^T \psi_j(t)\, \psi^*_q(t)\, dt$

  $= \sum_{j=1}^N |s_j|^2$

Representation of noise using $\{\psi_j(t)\}$

- Compute the projection onto each basis signal

  $n_j = \int_0^T n(t)\, \psi_j(t)\, dt$

- Then, we note the noise have two parts

  $n(t) = n_i(t) + n_o(t)$

  where $n_i(t) = \sum_{j=1}^{N} n_j\, \psi_j(t)$

  (the noise which resides inside the signal-space)

  $n_o(t) = \sum_{j=N+1}^{\infty} n_j\, \psi_j(t)$

  (the noise orthogonal to, and thus resides outside, the signal space)

- Then, we can say, $n(t) \Rightarrow \mathbf{n} = (n_1, n_2, \ldots, n_N)$.
- The continuous noise can be represented by an N-tuple random noise vector $\mathbf{n}$.

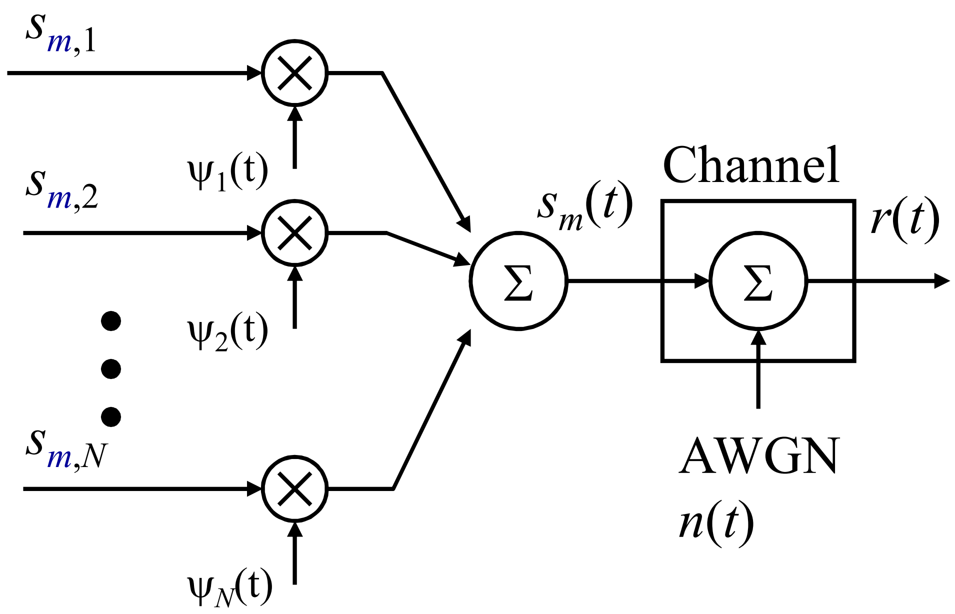
Additive White Gaussian Noise $n(t)$

- Suppose n(t) is Gaussian white noise with zero mean and double sided PSD $N_0/2$.
- Then, each projection coefficient n_j is a random variable

  $$n_j = \int_0^T n(t)\, \psi^*_j(t)\, \mathrm{d}t$$

- What kind of r.v.s are n_j?
  - Linear combination of Gaussian random variables is Gaussian.
  - Mean and covariance?
    - Mean = 0
    - Consider $E\{n_j\, n^*_k\} = E\{\int_0^T n(t_1)\, \psi^*_j(t_1)\, \mathrm{d}t_1 \int_0^T n^*(t_2)\, \psi_k(t_2)\, \mathrm{d}t_2\}$
      $= \int_0^T \int_0^T E\{n(t_1)\, n^*(t_2)\}\, \psi^*_j(t_1)\, \psi_k(t_2)\, \mathrm{d}t_2\, \mathrm{d}t_1$
      $= \int_0^T \int_0^T N_0/2\, \delta(t_2\text{-}t_1)\, \psi^*_j(t_1)\, \psi_k(t_2)\, \mathrm{d}t_2\, \mathrm{d}t_1$
      $= N_0/2 \int_0^T \psi^*_j(t_2)\, \psi_k(t_2)\, \mathrm{d}t_2 = N_0/2\, \delta_{k,j}$
    - Variance is $N_0/2$
  - Mutually uncorrelated Gaussian r.v.s

Representation of signals and noise



- From the development so far, we can say that

  $r(t) = s(t) + n(t)$

  $\Rightarrow \mathbf{r} = \mathbf{s} + \mathbf{n}$

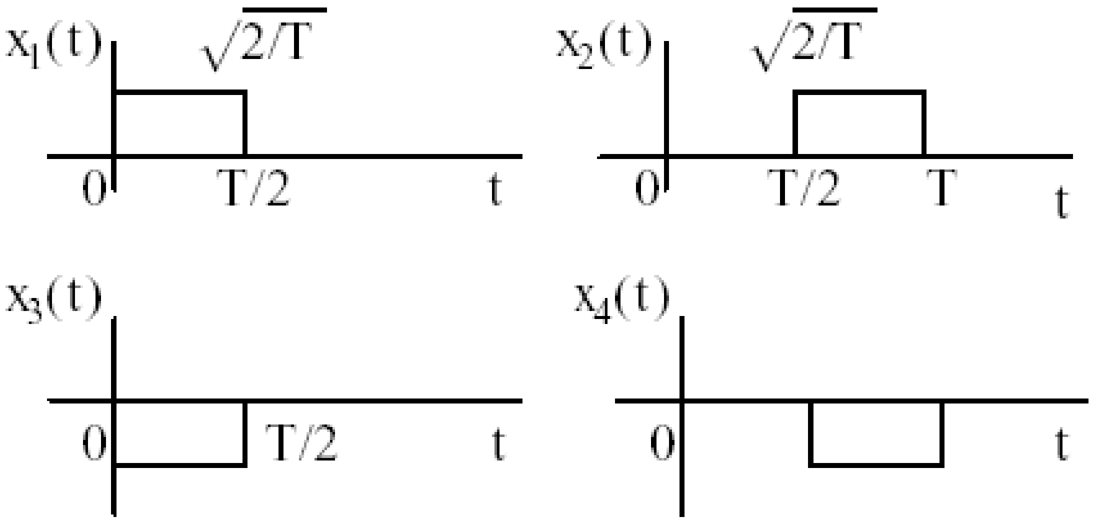
- Thus, the figure on the right makes sense!!!
  - Let's check

How to choose the basis and the signal set

- It depends on your resource and what you want
- Time division
- Frequency division
- Code-division
- Let's see some examples

Examples
(Signal Space Representation)

- ❖ Time Division Examples
  - It's simply because it's easier to illustrate the point
  - But, certainly you should not be limited by these simple cases
- ❖ Example 1:



$\phi_1(t) = x_1(t)$

$\phi_2(t) = x_2(t)$

Dimension = 2

$x_1 = (1\ 0)^t$

$x_2 = (0\ 1)^t$

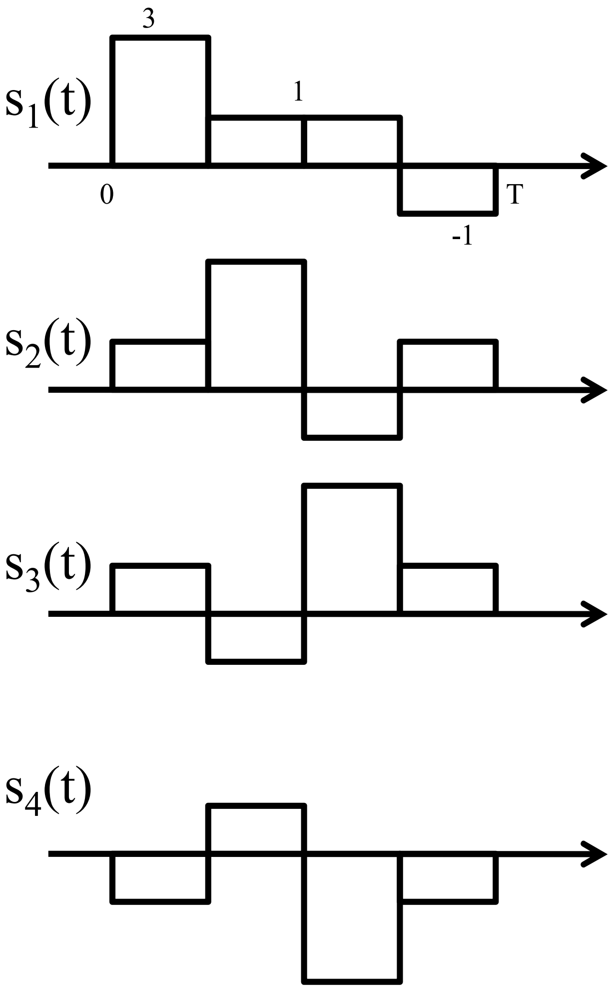
$x_3 = (-1\ 0)^t$

$x_4 = (0\ -1)^t$

Signal Space Representation

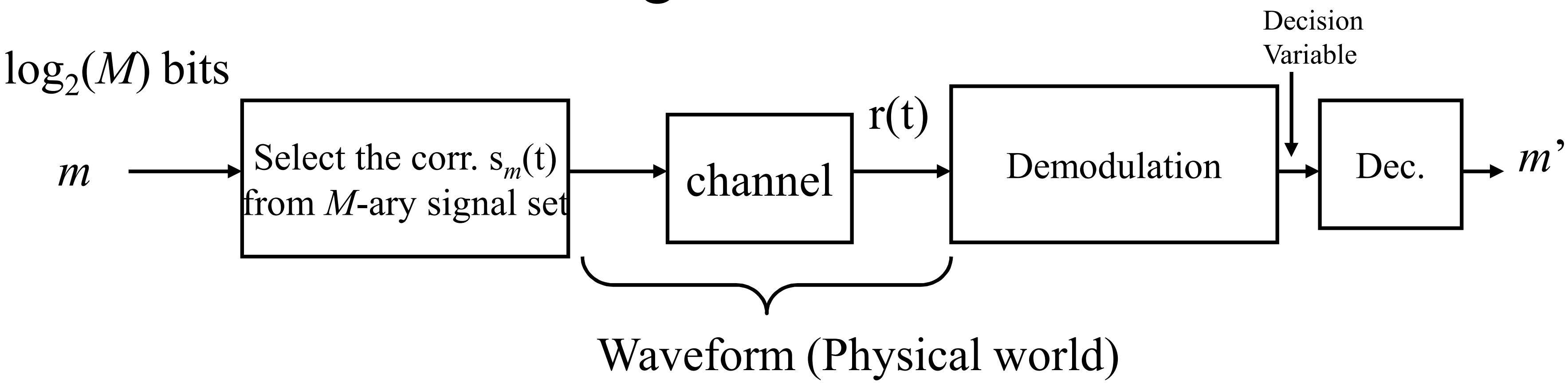
- We may attempt to draw signals in the signal space (it's doable up to three dimension)
  - Signal Space Plot, we call it.
  - When you draw, treat each basis vector as a coordinate.
- Can you compare M and N?

Example 2



- Find the dimensionality of the signal set
- Find a basis set
- Obtain a vector representation
- Draw signal space plot
- Compute the distances between signals

Big Picture View



- Communication over AWGN channel

$$r(t) = s_m(t) + n(t),\ 0 \le t \le T$$

- Choose a set of M distinct signal waveforms.
- From previous discussion (Sig. Space rep), we know we can represent the set of M messages using a set of basis functions.
- The size of the basis set $\{\psi_j(t), j = 1,2,\ldots,N\}$ required to span the M-ary signal set is the dimension N of the signal set.

Big Picture View (2)

- Modulation (wide sense) is to imply the mapping rule which takes a string of $k=\log_2(M)$ bits as input and transmits the corresponding signal $s_m(t)$ over the channel.
  - Transmission rate is k bit/T sec

- Demodulation (wide sense) is to imply the process of converting the analog waveform into a string of digits—a set of test statistics or decision variables
  - Use the matched filter (or correlator) for optimum performance.
  - This results in the receiver which is a *mirror image* of the transmitter.

- The decision device makes the final decision m' on the message.

- Minimize the probability of error, $P(e) = P(m \neq m')$.

Optimum Decision Rules

- Let's consider the optimal decision receiver.
- The receiver has the received vector
  - $\mathbf{r} = \mathbf{s}_m + \mathbf{n}$,
  - $\mathbf{n}$ is a multivariate Gaussian with $p(\mathbf{n}) = (2\pi\sigma^2)^{-N/2} \exp[-\|\mathbf{n}\|^2/2\sigma^2]$.
  - Each of the marginal distr. $p(n_i)$ is Gaussian with zero mean and variance σ^2.
- Maximum *a-posteriori* (MAP) criterion:
  Find the message index m, among 1, 2,…, M, that maximizes the posterior probability

$$P(\mathbf{s}_m \mid \mathbf{r}) = P(\mathbf{s}_m, \mathbf{r})/P(\mathbf{r}) \propto P(\mathbf{r} \mid \mathbf{s}_m)\, P(\mathbf{s}_m)$$

$$\text{Posteriors} \propto \text{Likelihoods} \times \text{Priors}$$

- When $P(\mathbf{s}_m) = 1/M$ (equally likely), maximization on the posterior probability is equivalent to maximization on the likelihood function.
  - Implies that Maximum A Posteriori decision = Maximum Likelihood decision

MAP = ML Detection, with equally likely input
= Min. E. D. Rule, with Gaussian noise

- The MLD rule for AWGN
- $P(\mathbf{r} \mid \mathbf{s}_m) = P(\mathbf{n} = \mathbf{s}_m - \mathbf{r}) = (2\pi\sigma^2)^{-N/2} \exp[- \|\mathbf{r} - \mathbf{s}_m\|^2 / 2\sigma^2]$

Norm/Euclidean distance
Between **r** and hypothesis $\mathbf{s}_m$

- Log is monotone increasing function
- Use $\log[\Pr(\mathbf{r} \mid \mathbf{s}_m)] \propto - \| \mathbf{r} - \mathbf{s}_m\|^2 / 2\sigma^2$
- Finally, we note that

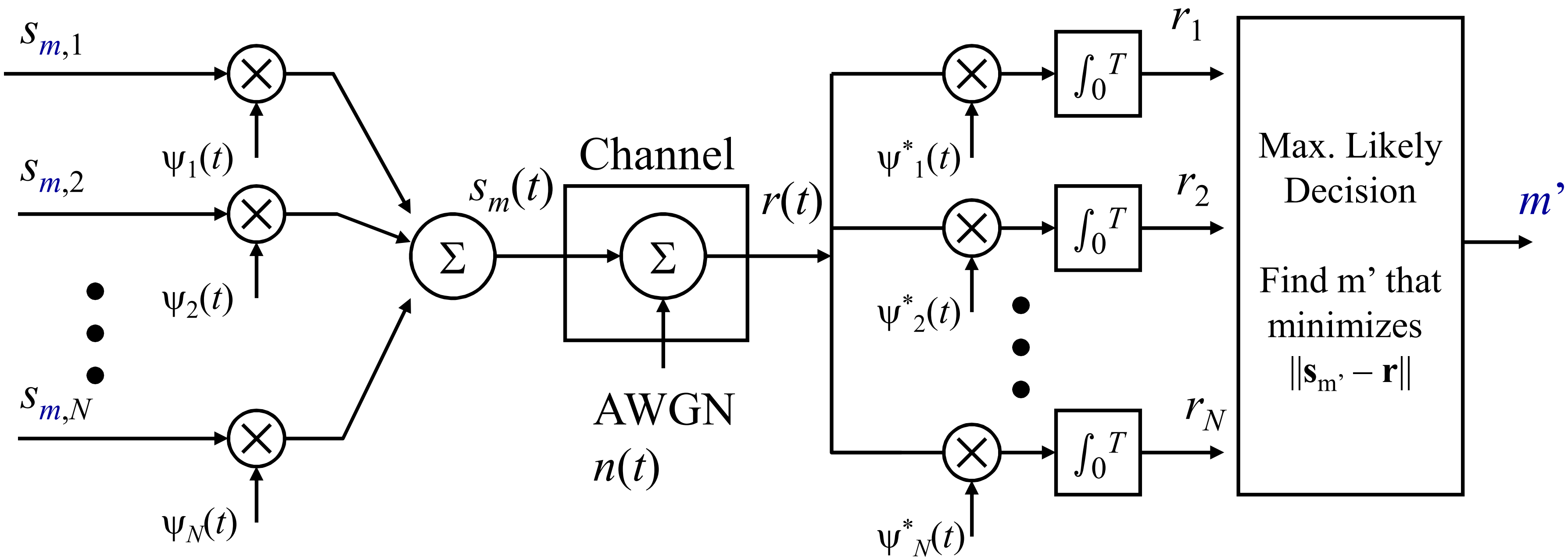
  Maximum Likelihood Detection rule = Minimum Euclidean Distance Rule

Decision Cells and Decision Boundary

- Example) a binary signal set

- Example) 4-ary signal set

Optimal Transceiver



- Use a basis $\{\psi_j,\ j=1, 2, \ldots, N\}$ to modulate and demodulate a message m.
- $s_m(t) \leftrightarrow \mathbf{s}_m=(s_{m,1}, s_{m,2}, \ldots, s_{m,N})$
- $n(t) \leftrightarrow \mathbf{n} = (n_1, n_2, \ldots, n_N)$ $\quad$ $n_1, n_2, \ldots$ i.i.d. Gaussian $\sim \mathcal{N}(0, \sigma^2)$
- $r(t) \leftrightarrow \mathbf{r} =(r_1, r_2, \ldots, r_N)$ $\quad$ These are called sufficient statistics (test variables, decision variables)

Inner Product Receivers

- Note that at each receiver branch, the receiver calculates the inner-product.
- After obtaining **r**, it's easy to calculate the distance $\|\mathbf{s}_{m'} - \mathbf{r}\|$ for each candidate *m*'.
- ML decision is the Minimum Distance decision.
- Thus, the inner product receiver is the ML sense optimal receiver.

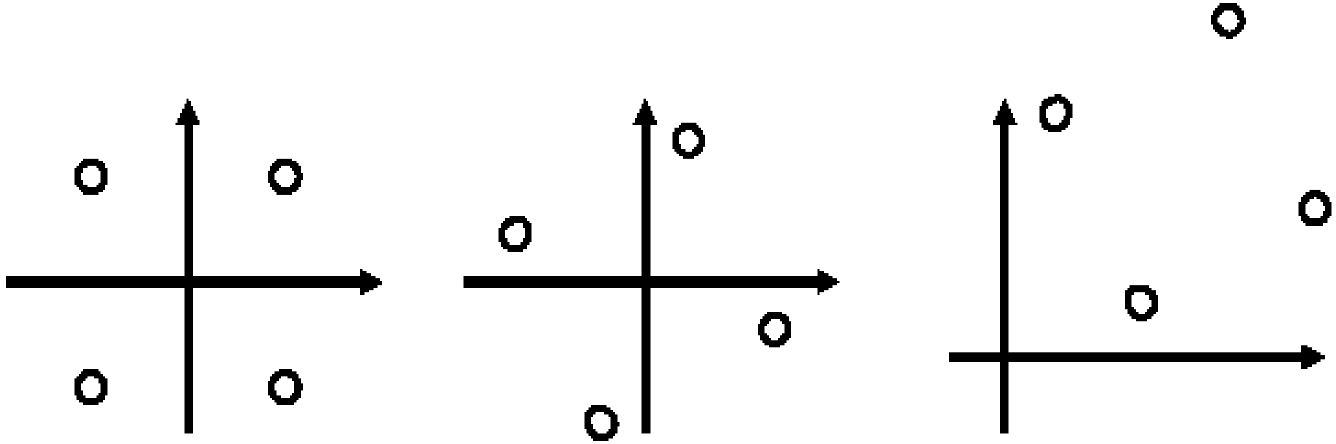
- We refer to the inner product operator as the *correlator*.
- The received signal gets *correlated* with each basis vector.

Signal Design Criteria

- Bandwidth efficient design
- Power efficient design
- A signal requires resources
  - Signals occupy time and frequency
    - Use as little *frequency* bandwidth as possible
    - Use as little *time* as possible
    - Fundamental limit—Time-frequency uncertainty
    - Thus, need a balance
- Another resource is power
  - Use as little power as possible

Good/Bad Constellations

(Signal Space Representation)



Example with 4-QAM

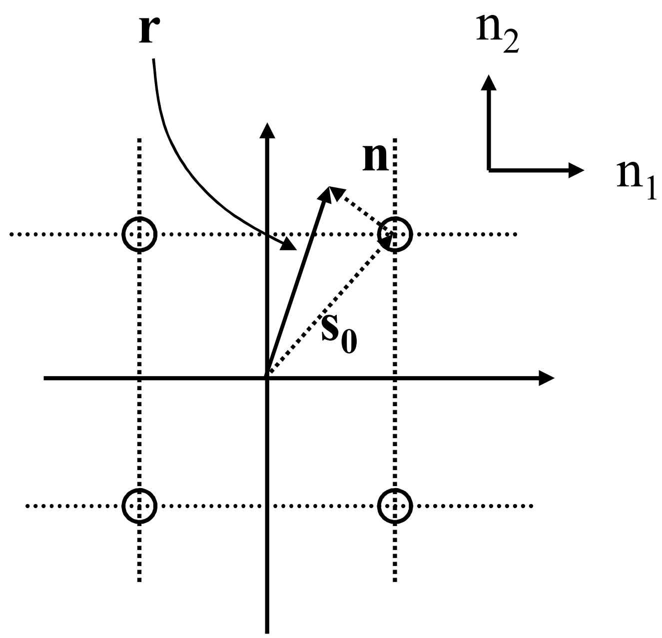
- Let's consider a 4-ary signal space plot again.
- 4-QAM constellation is a two dimensional signal set.
- Basis vectors of this space are

  $\psi_1(t)$ = sqrt($2/T$) cos($w_c t$),

  $\psi_2(t)$ = - sqrt($2/T$) sin($w_c t$),

  for $0 \leq t \leq T$
- The input/output signals can be represented with 2-tuples

  r = **s** + **n.**



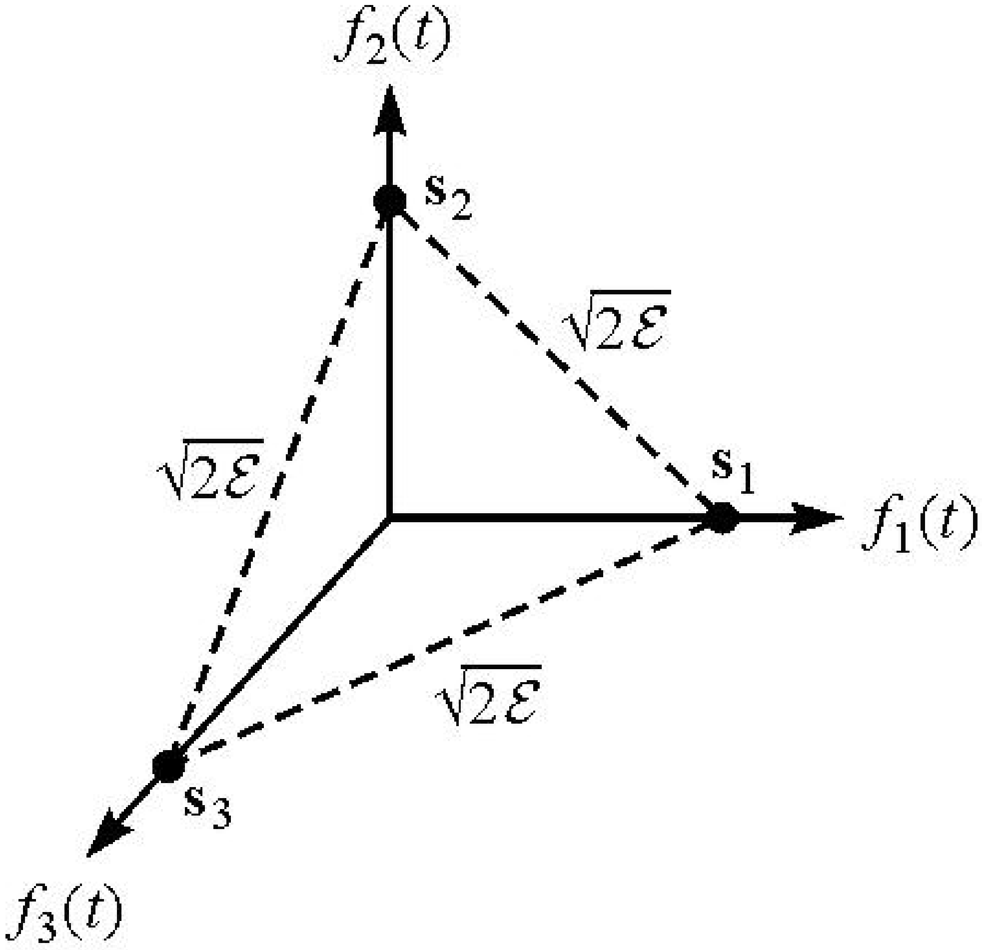
Example with M-ary FSK signals

- M-ary FSK signals, for $0 \le m \le M$-1

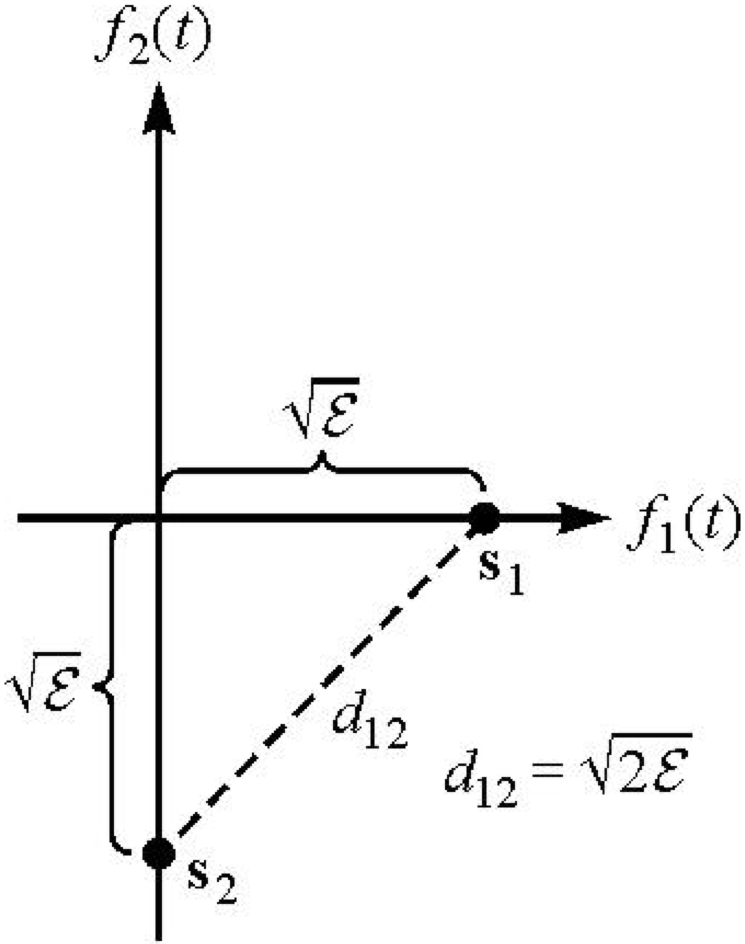
$$s_m(t) = \mathrm{Re}\{g_m(t)\exp(j\,2\pi f_c\, t)\}$$
$$= \sqrt{\tfrac{2E_s}{T}}\cos[2\pi f_c\, t + 2\,\pi\, m\,\Delta f\, t], \qquad 0 \le t \le T,$$
$$g_m(t) = \sqrt{\tfrac{2E_s}{T}}\exp(j\,2\pi\, m\,\Delta f\, t), \qquad 0 \le t \le T,$$

where E_s is the energy of the signal.

- The minimum frequency separation between adjacent signals, for orthogonality, is $1/2T$.


$f_2(t)$
$\mathbf{s}_2$
$\sqrt{2\mathcal{E}}$
$\sqrt{2\mathcal{E}}$
$\mathbf{s}_1$
$f_1(t)$
$\sqrt{2\mathcal{E}}$
$\mathbf{s}_3$
$f_3(t)$

$M = N = 3$


$f_2(t)$
$\sqrt{\mathcal{E}}$
$f_1(t)$
$\mathbf{s}_1$
$\sqrt{\mathcal{E}}$
d_{12}
$d_{12} = \sqrt{2\mathcal{E}}$
$\mathbf{s}_2$

$M = N = 2$

Matched filters, why?

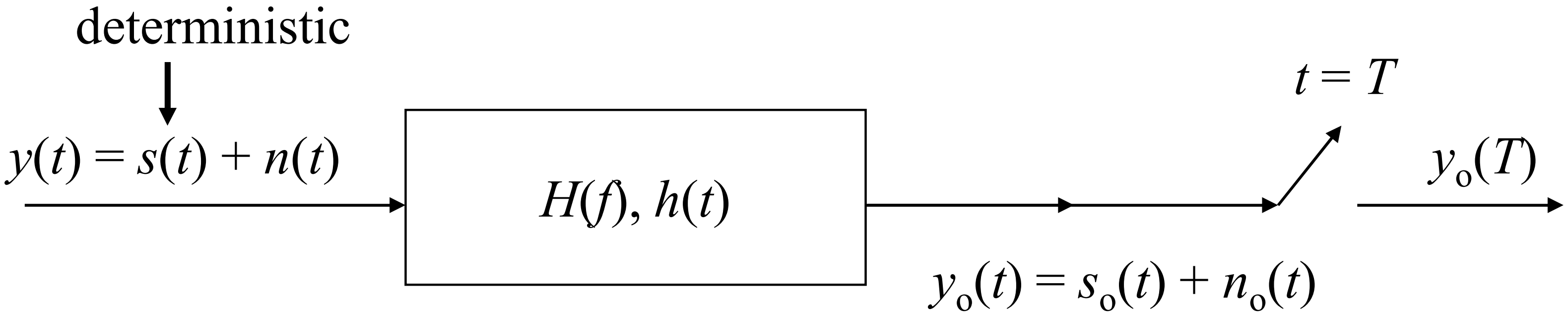
- We use the inner-product receiver (or the correlator) for the optimal receiver.
- The correlator output can also be obtained from using matched filters.
- Thus, the optimal receiver can also be realized with matched filter bank.
- Matched filter is sometimes easier to implement than the correlator.
- The following discussion shows that matched filter maximizes the SNR when it is sampled at the right moment.

Schwarz's Inequality

- Schwartz inequality
- $|\mathbf{v}_1 \cdot \mathbf{v}_2| \leq ||\mathbf{v}_1|| \; ||\mathbf{v}_2||$ with equality iff $\mathbf{v}_2 = c \; \mathbf{v}_1$

- $|\int_t f(t) \, g^*(t) \, \mathrm{d}t| \;\leq\; (\int_t |f(t)|^2 \, \mathrm{d}t)^{1/2} \, (\int_t |g(t)|^2 \, \mathrm{d}t)^{1/2}$ with equality if and only if $f(t) = K \, g(t)$

Matched Filter Maximizes $(S/N)_{out}$



- $y(t)$ is the received signal. We know that the signal $s(t)$ has a duration $0 \leq t \leq T$, and the PSD of the noise $n(t)$ is $P_n(f)$.
- Find the best filter $H(f)$ so that $(S/N)_{out} := |s_o(t)|^2 / E\{n^2_o(t)\}$ is maximum at $t=T$.
- The signal part: $s_o(t=T) = \int H(f)S(f) \exp(j2\pi f T)\, df$.
- The noise part: $n_o(t=T) = \int H(f)\, N(f) \exp(j2\pi f T)\, df$.

Matched Filter Maximizes $(S/N)_{out}$

- $E\{n^*_o(T)\, n_o(T)\} = \int |H(f)|^2 P_n(f)\, df$
- Thus, the output SNR is

$$
\begin{aligned}
(S/N)_{out} &= \frac{\left| \int_{-\infty}^{\infty} H(f) S(f) e^{j2\pi fT} df \right|^2}{\int_{-\infty}^{\infty} |H(f)|^2 P_n(f) df} \\
&\leq \frac{\int_{-\infty}^{\infty} |H(f)|^2 P_n(f) df \ \int_{-\infty}^{\infty} \frac{|S(f)|^2}{P_n(f)} df}{\int_{-\infty}^{\infty} |H(f)|^2 P_n(f) df}
\end{aligned}
$$

  - where the second line is due to Schwartz' inequality
  - Equality is achieved iff $A(f) = K\, B(f)$
    - $A(f) = H(f)\, P_n(f)^{1/2}$
    - $B(f) = S^*(f)\, \exp(-j2\pi fT)/P_n(f)^{1/2}$

Matched Filter Maximizes $(S/N)_{out}$

- $(S/N)_{out} \le \int |S(f)|^2/P_n(f)\, df$
- The equality is attained when

  $$H(f) = K\, S^*(f)\, e^{-j2\pi fT}/P_n(f)$$

- When the noise is white, then $P_n(f) = N_o/2$, we have

  $H(f) = S^*(f)\, e^{-j2\pi fT}$

  – $(S/N)_{out} \le \frac{\int |S(f)|^2 df}{N_o/2} = \frac{E_s}{N_o/2}$

Closer Look at SNR_o

- When the noise is white, $H(f) = S^*(f)\, e^{-j2\pi fT}$
- The variance of the noise $n_o(T)$ is

  $E\{n_o^*(T)n_o(T)\}$

  $= E(\int\int H^*(f_1)H(f_2)\, N(f_2)\, N(f_1) \exp(j2\pi\, f_1T) \exp(-j2\pi\, f_2T)\, df_1\, df_2)$

  $= \int\int H^*(f_1)H(f_2)\, E(N^*(f_2)\, N(f_1)) \exp(j2\pi\, f_1T) \exp(j\text{-}2\pi\, f_2T)\, df_1\, df_2$

  $= \int\int H^*(f_1)H(f_2)\, (N_o/2)\, \delta(f_2\text{-}f_1) \exp(j2\pi(f_2 - f_1)T)\, df_1\, df_2$

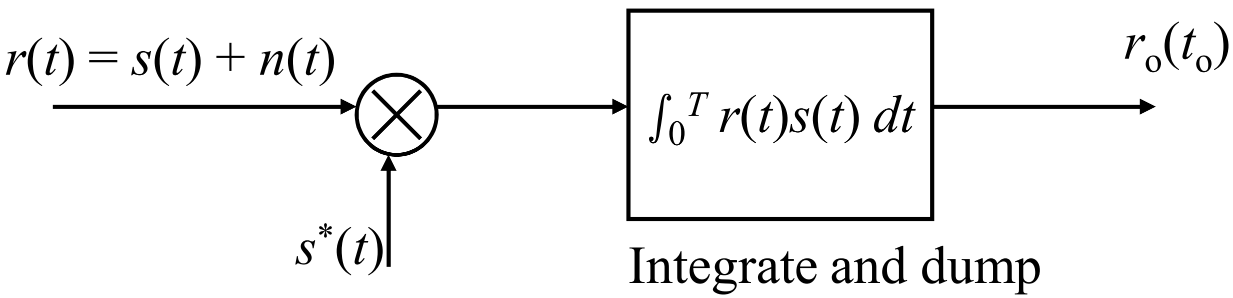
  $= (N_o/2) \int H^*(f)H(f)\, df$

  $= (N_o/2) \int |H(f)|^2\, df$

  $= (N_o/2) \int |S(f)|^2\, df$

  $= (N_o/2)\, E_s$
- Or simply we note that the PSD of the noise after the filter is $(N_o/2)\,|S(f)|^2$. Thus, the noise power at the output of the matched filter is $(N_o/2) \int |S(f)^2|\, df = E_s\,(N_o/2)$.
- The energy (power) of the signal $s_o(T)$ is E_s^2.
- Thus $SNR_o = E_s^2/[E_s(N_o/2)] = E_s/(N_o/2)$.

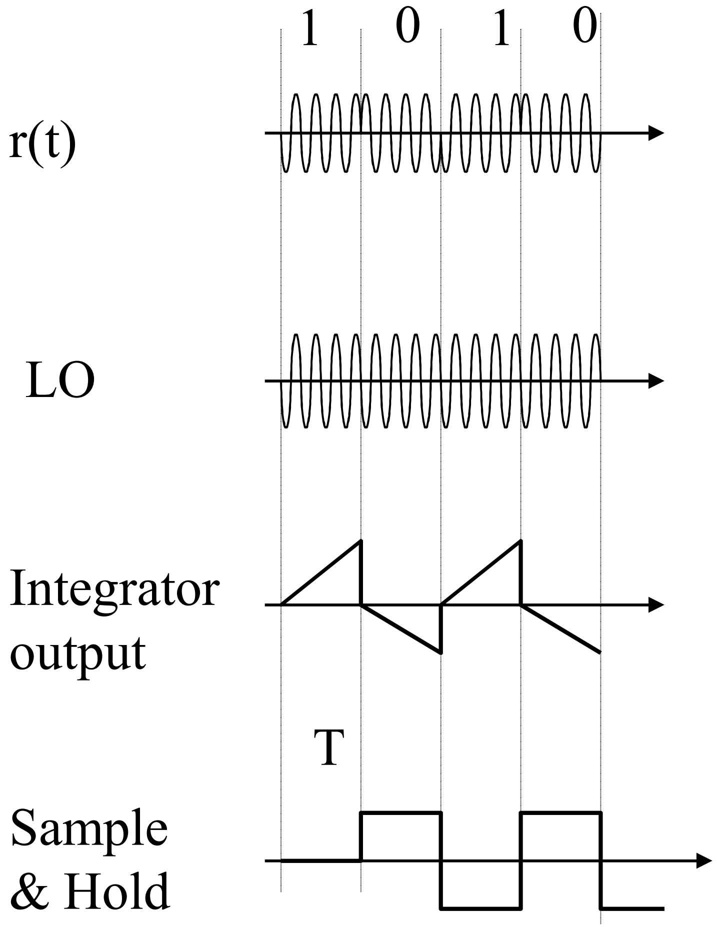
Correlator Realization of Matched Filtering (White noise only)



- For white noise, the matched filter result is the same as the correlator output—integrated and dumped.

Correlator for Switched Sinusoids

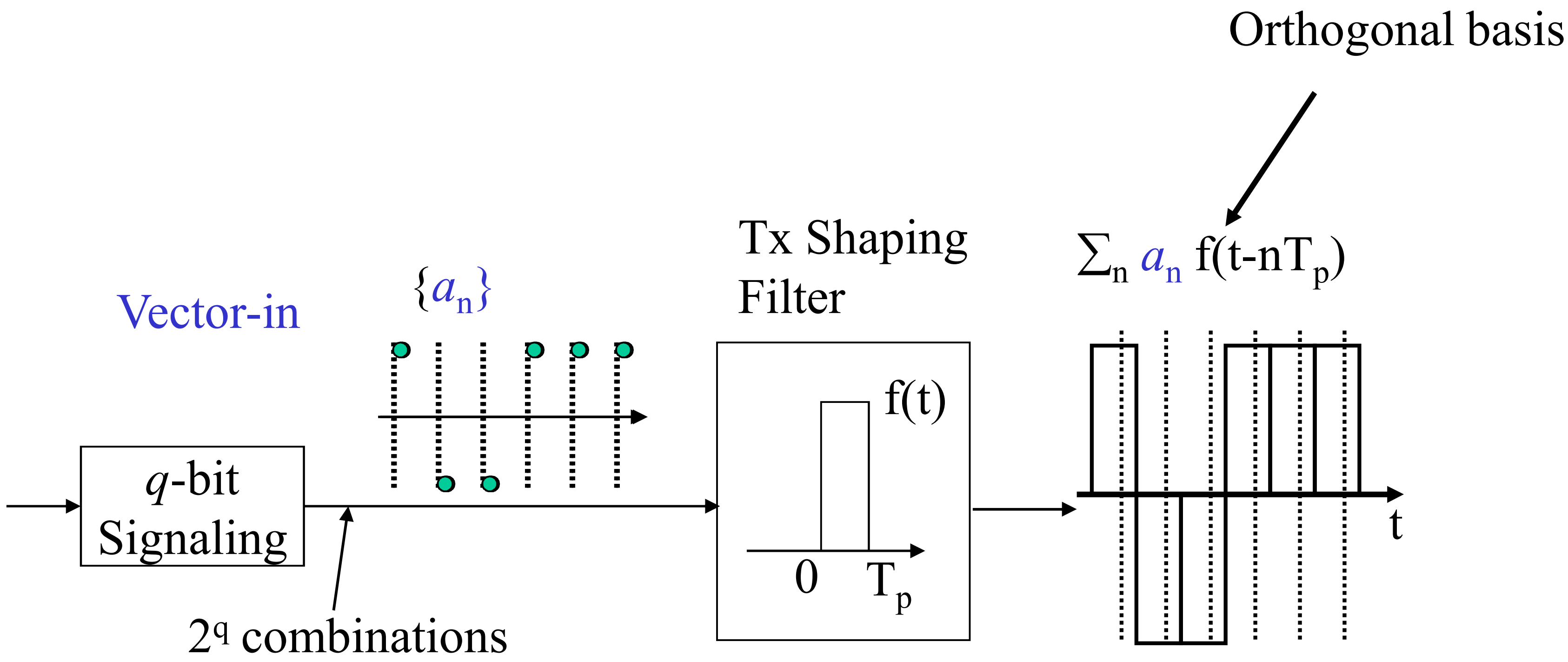
- Integrator resets every $t = T$
- The sampler takes sample every $t = T$



- Now let's consider a very simple transmitter and receiver pair.
- The purpose is to illustrate the overall idea of what we have learned so far.
  - What are they???

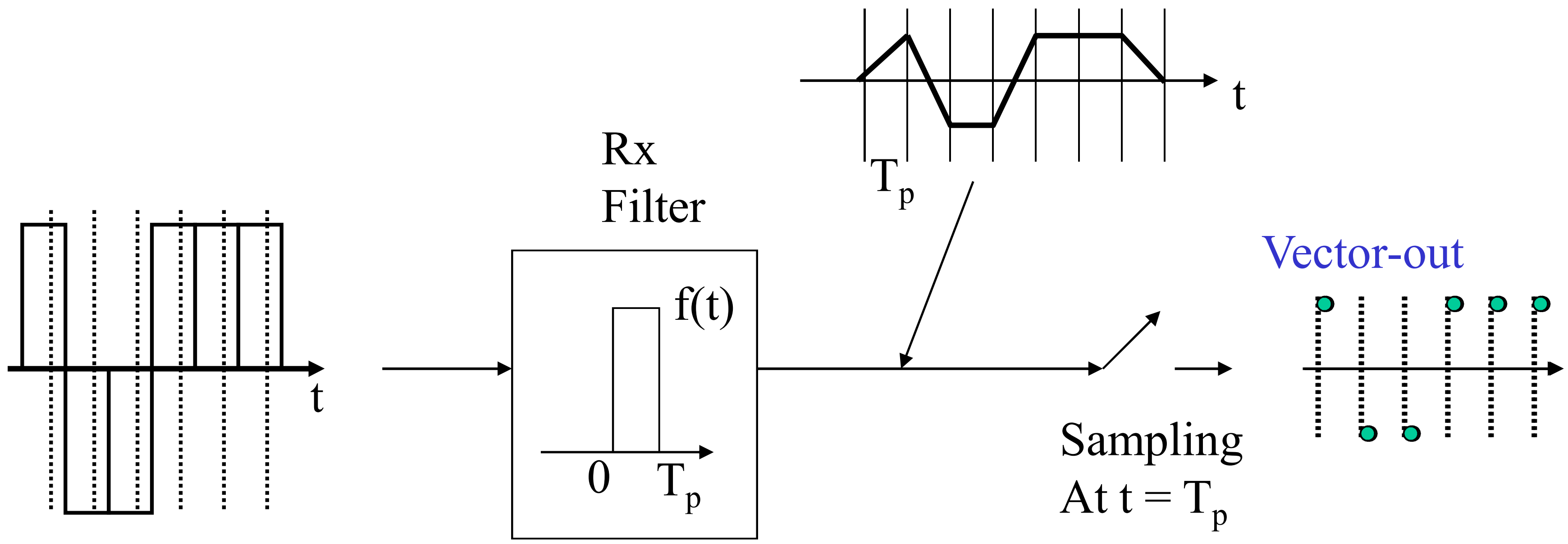
Transmitter

- Consider the following example



Receiver

- The Receiver is a mirrored system of the transmitter



Binary Signals: Pass Band

- On-Off Keying:
  - $A_c \cos(w_c t)$ for "1"
  - Nothing for "0"
- Binary Phase-Shift Keying (BPSK)
  - $s(t) = A_c \cos[w_c\, t + \pi/2\;\; m(t)]$
  - m(t) is the polar baseband signal.
  - $A_c \sin(w_c t)$ for "1" and $-A_c \sin(w_c t)$ for "0."
- Binary Frequency-Shift Keying (FSK)
  - Two FM signals for "1" and "0."
  - $s(t) = A_c \cos(w_1\, t + \theta_1)$ or $A_c \cos(w_2\, t + \theta_2)$

Binary Bandpass Signals

- Shown left are two kinds of signals.
  - Baseband signals
  - Bandpass signals
- Baseband signals
  - Unipolar
  - Polar
- Bandpass signals
  - On-Off keying
  - Phase Shift Keying
  - Frequency Shift Keying

The pictures scanned from Crouch, 6th Edition.

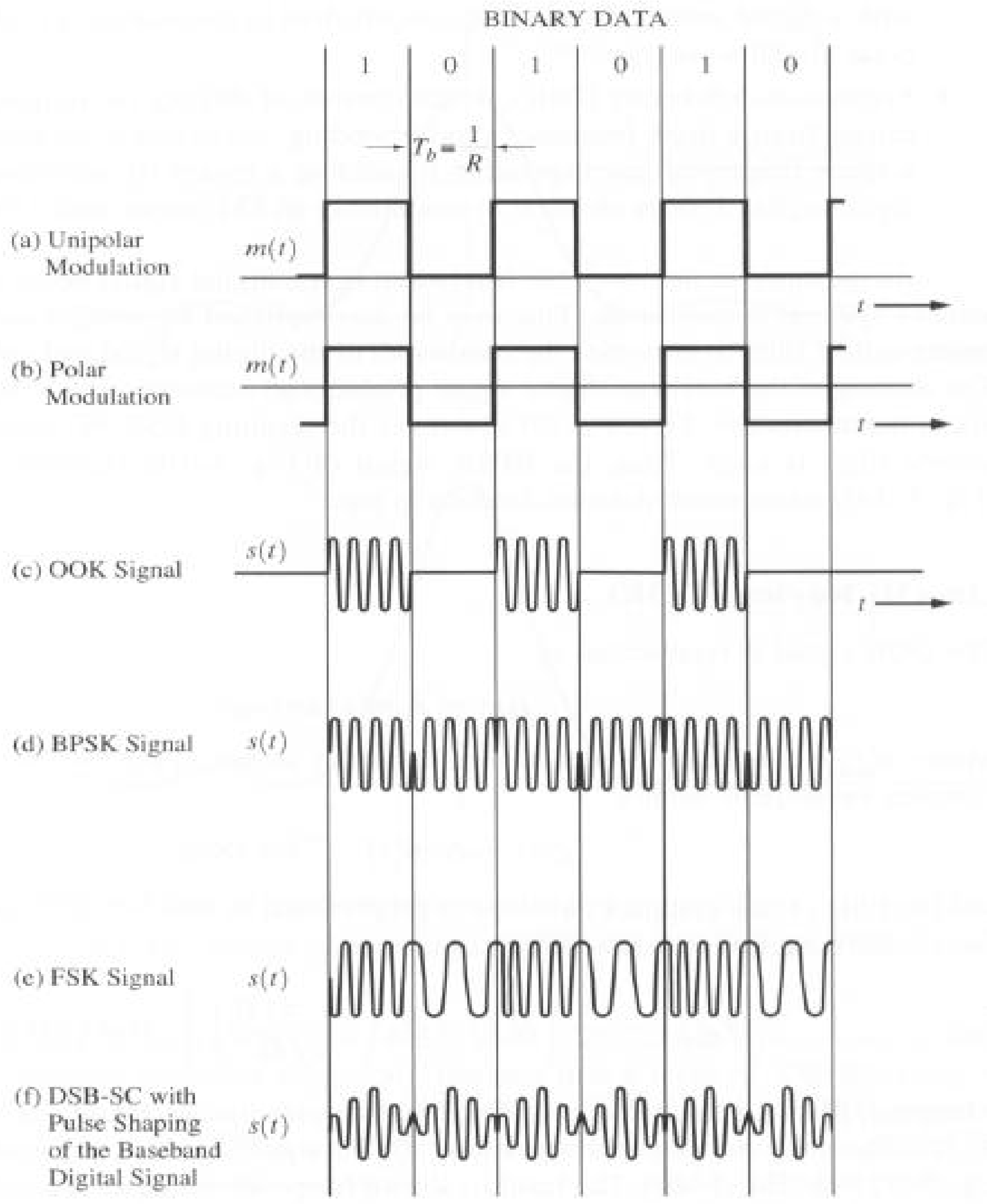
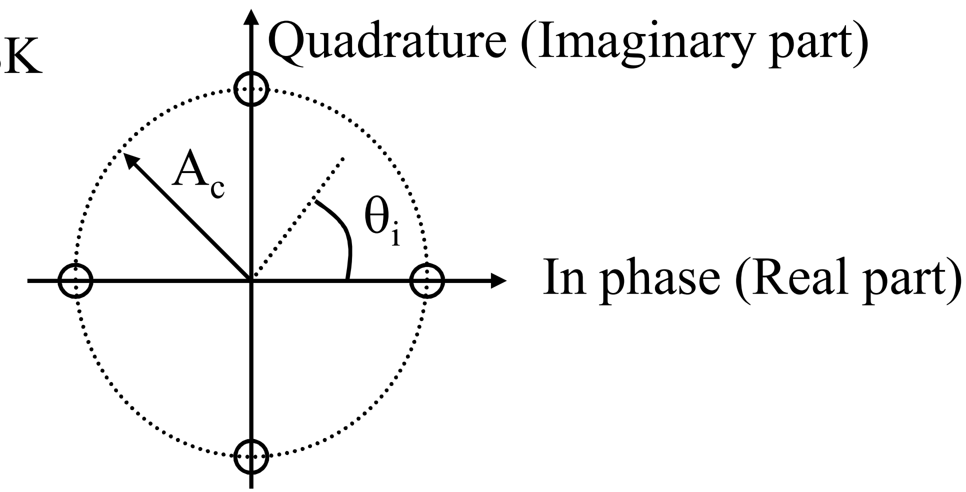


Figure 5–19 Bandpass digitally modulated signals.

Multi-level Modulated Bandpass Signals

- M-ary Phase Shift Keying
  - $s(t) = A_c \cos(w_c t + \theta_i)$, for $i = 0, 1, 2, \ldots, M - 1$.
  - The phase carries information
  - We use two-dimensional signal space plot in order to represent the signal.

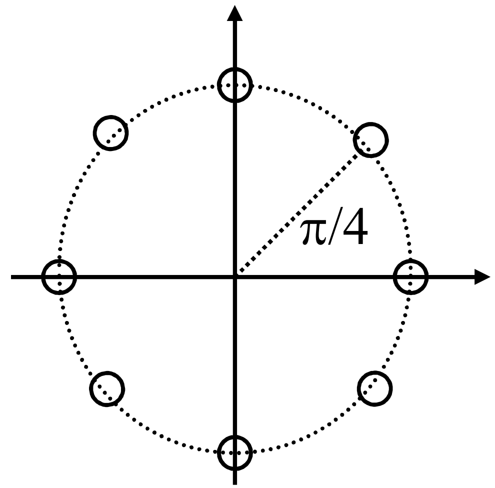
4-PSK



8 PSK

- 8 PSK example

  i=0, 1, 2, …, 7
- $\pi/4$ is the angle between any two adjacent points in the signal space plot.
- We call the plot shown right *digital signal constellation.*



Baseband vs. Pass band

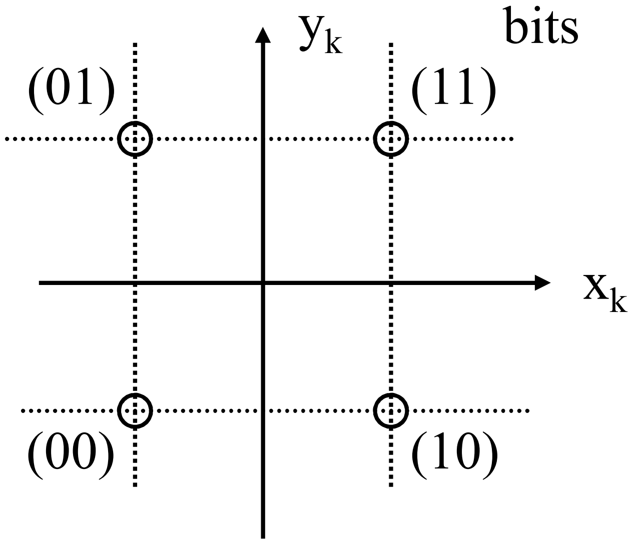
- $s(t) = A_c \cos(w_c t + \theta_i)$, for $i = 0, 1, 2, \ldots, M - 1$.
- Get rid of the carrier
- $s(t) = \mathrm{Re}\{A_c \exp(\mathrm{j}\,(w_c t + \theta_i))\}$

  $= \mathrm{Re}\{\mathrm{A_c} \exp(\mathrm{j}\ \theta_i) \exp(\mathrm{j}\ w_c t\ )\}$

  $= \mathrm{Re}\{g(t) \exp(\mathrm{j} w_c\, t)\}$
- Baseband signal: $g(t) = A_c \exp(\mathrm{j}\ \theta_i))$
- Complex number → real and imaginary part

4-QAM signals

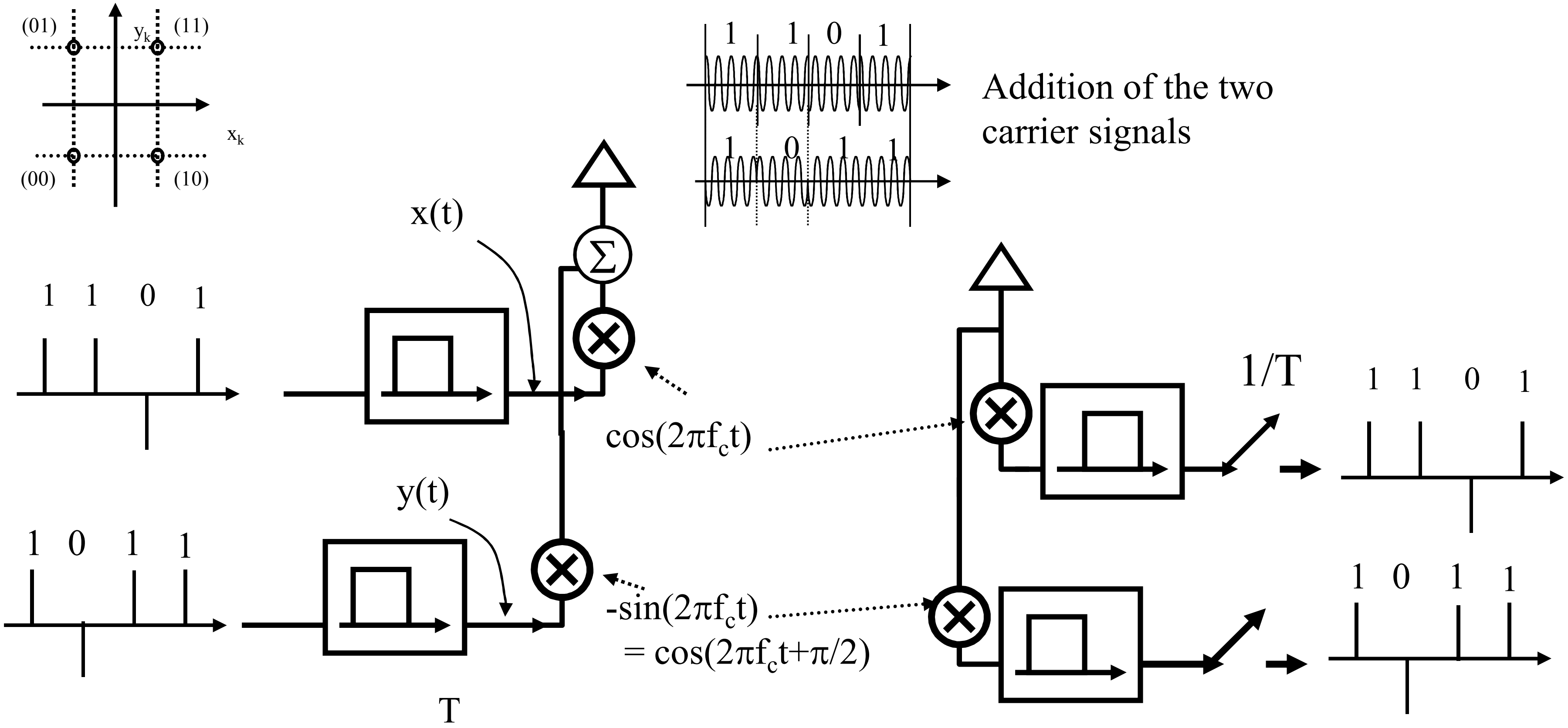
- Baseband signal: $g(t) = x(t) + j\, y(t) = R(t) \exp(j\,\theta(t))$
  - $R(t)$ is called the envelop signal (the magnitude part). It's A_c here.
- Let $f(t) = \mathrm{rect}((t - T/2)/T)$ for now.
- Suppose $g(t)$ in the k-th signaling interval takes a value from the four predefined constellation points.
  - $A_c\,(1 + j)$, $A_c(-1 + j)$, $A_c(-1 - j)$, $A_c(+1 - j)$
  - $A_c\,(x_k + j\, y_k)$, where x_k , y_k = +1 for bit "1" and -1 for bit "0"



$$
\begin{aligned}
g(t) &= \textstyle\sum_k A_c(x_k + j\, y_k)\, f(t - kT) \\
&= A_c \textstyle\sum_k x_k\, f(t-kT) + j\, A_c \textstyle\sum_k y_k\, f(t - kT) \\
&= x(t) + j\, y(t)
\end{aligned}
$$

Quadrature Amplitude Modulation

- Two independent channels can be obtained over the same RF spectrum (Orthogornality: $\int_0^T \cos(2\pi f_c t)\sin(2\pi f_c t)\,dt = 0$)
- $s(t) = x(t)\cos(w_c t) - y(t)\sin(w_c t)$



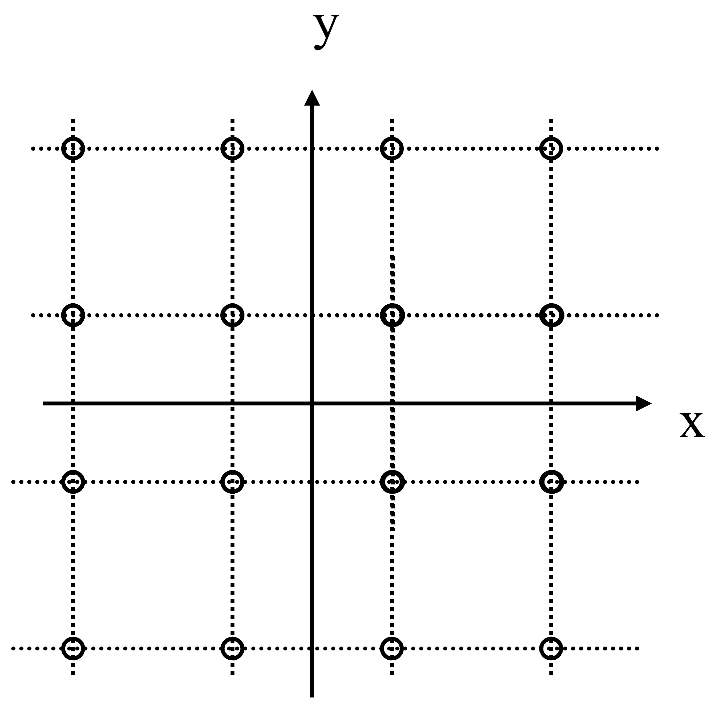
16 QAM Constellation

- Each symbol carries four information bits.
- In general the in-phase $x(t)$ and quadrature part $y(t)$ can be written as

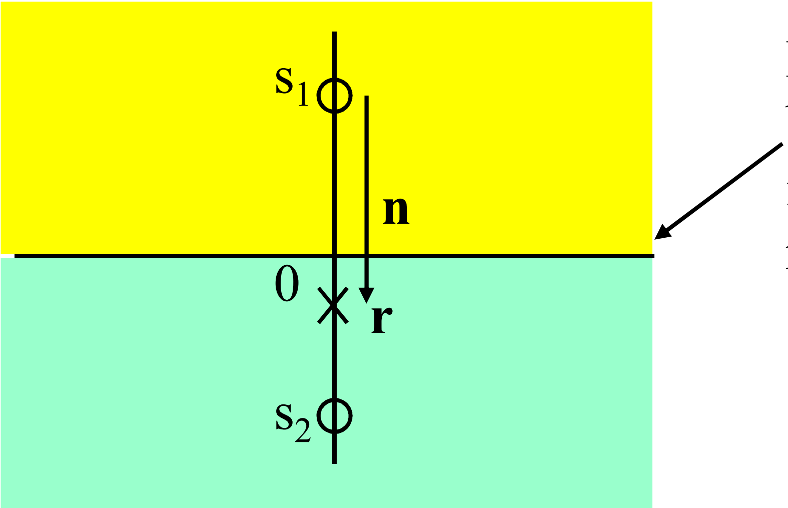
$$x(t) = \sum_k x_k f(t - kT_p)$$

$$y(t) = \sum_k y_k f(t - kT_p)$$

- For example of 16-QAM, y_k, $x_k \in \{A_c, 3A_c, -A_c, -3A_c\}$



Decision Regions for BPSK



Minimum distance rule gives a decision boundary which divides the entire space into two mutually-exclusive decision regions.

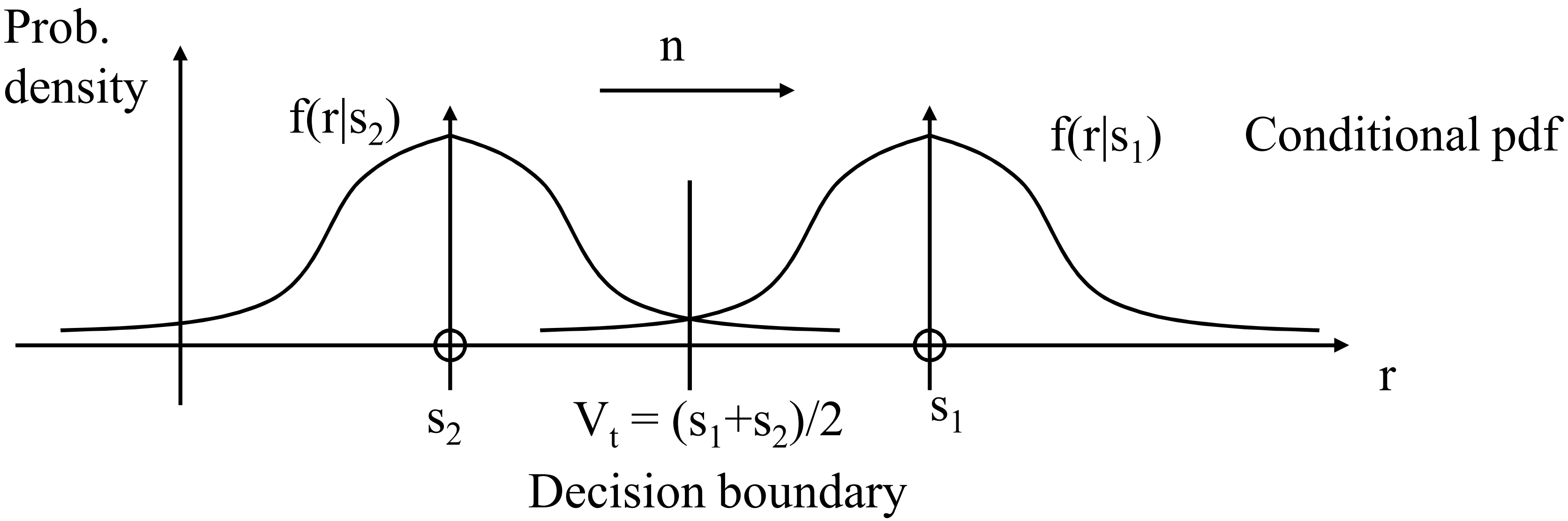
Let's assume s_1 was sent, we note that error occurs when $n \leq |\mathbf{s}_1 - \mathbf{s}_2|/2$

- $s_1(t) = A_c \sin(w_c t)$, $0 \leq t \leq T$, for binary digit "1 ."
- $s_2(t) = -A_c \sin(w_c t)$, $0 \leq t \leq T$, for binary digit "0."
- The dimensionality of the signal set is 1 (use $\psi(t) = \mathrm{sqrt}(2/T) \sin(w_c t)$).

  $s_1(t) \leftrightarrow s_1 = A_c\, \mathrm{sqrt}(T/2)$

  $s_2(t) \leftrightarrow s_2 = -A_c\, \mathrm{sqrt}(T/2)$
- Thus, the difference energy $E_d \leq d_E^2 = 2A_c^2\, T$.
  - "=" achieved when the matched filter is used.

Error Probability for Binary Signaling



- $\Pr(\text{error}) = P(\text{error}, s_1 \text{ sent}) + P(\text{error}, s_2 \text{ sent})$

$$= P(\text{error} \mid s_1 \text{ sent})\, P(s_1 \text{ sent}) + P(\text{error} \mid s_2 \text{ sent})\, P(s_2 \text{ sent})$$

$$= P(r \le V_t \mid s_1 \text{ sent})\, P(s_1 \text{ sent}) + P(r \ge V_t \mid s_2 \text{ sent})\, P(s_2 \text{ sent})$$

$$= P(n \le V_t - s_1)\, P(s_1 \text{ sent}) + P(n \ge V_t - s_2)\, P(s_2 \text{ sent})$$

--- Let pdf of n is P(n) = f(n)

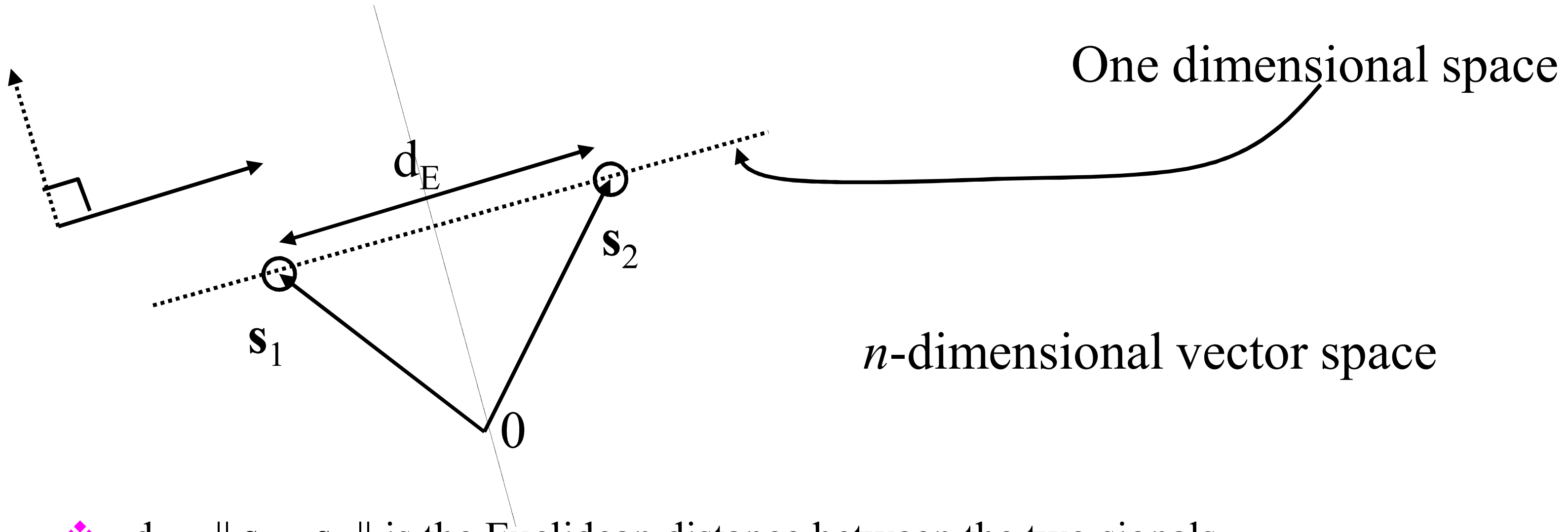
$$= P(s_1 \text{ sent}) \int_{-\infty}^{V_t} f(r|s_1)\, dr + P(s_2 \text{ sent}) \int_{V_t}^{\infty} f(r|s_2)\, dr \quad \text{--- (1)}$$

P_e for binary signaling

- Assume s_1 and s_2 equally likely $P(s_1 \text{ sent})=P(s_2 \text{ sent})=0.5$
- $f(n)$ is Gaussian with zero mean and variance σ^2
  - $f(r|s_1) = (\pi N_o)^{-1/2} \exp(-(r - s_1)^2/ N_o)$
  - $f(r|s_2) = (\pi N_o)^{-1/2} \exp(-(r - s_2)^2/ N_o)$
- We note that because of symmetry, the optimal choice of V_t is the half point $(s_1 + s_2)/2$.
- Making use of symmetry of the Gaussian pdf's, we can write
  - $P_e = \int_{(s_1-s_2)/2}^{\infty} \mathrm{sqrt}(\pi N_o)^{-1} \exp(-n^2/ N_o)\, dn$

$$= Q\left(\frac{|s_1-s_2|}{\sqrt{2N_o}}\right) = Q\left(\sqrt{\frac{(s_1-s_2)^2}{2N_o}}\right) = Q\left(\sqrt{\frac{E_d}{2N_o}}\right)$$

Probability of Making Errors for Binary Signaling



- $d_E = \| \mathbf{s}_1 - \mathbf{s}_2 \|$ is the Euclidean distance between the two signals.
- d_E^2 is the difference in signal energy.

- Decision error depends only upon the one dimensional noise acting along the line shown above (the noise components perpendicular to the line do not contribute toward generating the decision error event.).

- Assume that any one signal is sent. Then the error event is the set of all outcomes in which the received signal lands on the decision regions of the other signals. This event occurs when the one dimensional noise is directed toward the other codeword with its magnitude large enough to move across the decision boundary.

Appendix --
Often Used Identities

- $\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$
- $\cos(x \pm y) = \cos(x)\cos(y) \pm \sin(x)\sin(y)$
- $\cos(x)\cos(y) = ½ (\cos(x+y) + \cos(x-y))$
- $\sin(x)\sin(y) = -½ (\cos(x+y) + \cos(x-y))$
- $\cos(x)\sin(y) = ½ (\sin(x+y) – \sin(x-y))$
- $\sin(x) = (1/2j)(e^{jx} – e^{-jx})$
- $\cos(x) = (1/2)(e^{jx} + e^{-jx})$

Problems

1. (the Q(x) function) Assume *X* is a Gaussian distributed random variable with mean 1 and variance of σ^2. Find the probability Pr{X > 5}. Express the probability with the Gaussian Q function (see definition in Section 2.3).
2. Find out if sin and cos waveforms are orthogonal to each other. If yes under a certain condition, provide them.

3. P2.3 (KL Decomp),
4. P2.11 (Rep. of Signals),
5. P2.25 (Bounds on Q(x) function),
6. P2.51 (Sampling theorem)

7. P3.2 (Signal Representation)
8. P3.6 (Power Efficient Constellation)

9. P4.5 (Signal Representation/Constellation)

Problems

- Consider a communications system with given conditions:
  - There are eight users and one access point.
  - All eight users make accesses to the access point simultaneously.
  - They use the same frequency band as well. The bandwidth is 1MHz.
  - Each user sends 1Mbps with arbitrarily small errors.
- Is it possible to design a set of waveforms for such a multiple access system which support all the statements above ? If yes, please provide one design. For full credits, justification to the level of this lecture note should be given.