This note is to illustrate the example given in Cover and Thomas (2nd Edition) on **Sufficient Statistic**. It is (2.127) in p. 37.

Definition. *Sufficient Statistics*. A function $T(X)$ is said to be a *sufficient statistic* relative to the family $\{f_\theta(x)\}$ if X is *independent* of θ given $T(X)$ for any distribution on θ.

Let $f_\theta = uniform(\theta, \theta+1)$. Then, T is sufficient statistic for θ, i.e.,

$$T(X) := \left(\max\{X_1, X_2, \cdots, X_n\}, \min\{X_1, X_2, \cdots, X_n\}\right). \tag{1.1}$$

Proof. We need to show $\theta \to T \to X$ or

$$\Pr\{X = x \mid T = t, \theta = c\} = \Pr\{X = x \mid T = t\} \tag{1.2}$$

for any x, t and c. But it is difficult to evaluate since it is out of order. Let us consider to change the order between *T* and *X*. How to do that? Recall my probability primer. For any x, consider

$$\Pr\{X = (x_1, x_2, \cdots, x_n) \mid T = (x_{\max}, x_{\min}), \theta = c\}$$

$$\underset{(a)}{=} \frac{\Pr\{X = (x_1, x_2, \cdots, x_n), T = (x_{\max}, x_{\min}) \mid \theta = c\}}{\Pr\{T = (x_{\max}, x_{\min}) \mid \theta = c\}} \qquad \text{(Conditional probability)}$$

$$= \frac{\Pr\{T = (x_{\max}, x_{\min}) \mid X = (x_1, x_2, \cdots, x_n), \theta = c\} \Pr\{X = (x_1, x_2, \cdots, x_n) \mid \theta = c\}}{\Pr\{T = (x_{\max}, x_{\min}) \mid \theta = c\}}$$

(Conditional probability)

$$= \frac{\Pr\{T = (x_{\max}, x_{\min}) \mid X = (x_1, x_2, \cdots, x_n), \theta = c\} \Pr\{X = (x_1, x_2, \cdots, x_n) \mid \theta = c\}}{\sum_x \Pr\{T = (x_{\max}, x_{\min}) \mid X = (x_1, x_2, \cdots, x_n), \theta = c\} \Pr\{X = (x_1, x_2, \cdots, x_n) \mid \theta = c\}}$$

(Total Probability Theorem + Conditional Probability)

$$\underset{(b)}{=} \begin{cases} \dfrac{1.0}{\int_{(x_{\min}, \cdots, x_{\min})}^{(x_{\max}, \cdots, x_{\max})} dx} & \text{for } \forall\ c \le x_i \le c+1 \\ 0.0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \dfrac{1.0}{\int_{(x_{\min}, \cdots, x_{\min})}^{(x_{\max}, \cdots, x_{\max})} dx} & \text{for } \forall\ \min(x_1, x_2, \cdots, x_n) \le x_i \le \max(x_1, x_2, \cdots, x_n) \\ 0.0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \dfrac{1.0}{\int_{(x_{\min}, \cdots, x_{\min})}^{(x_{\max}, \cdots, x_{\max})} dx} & \text{for } \forall\ x_{\min} \le x_i \le x_{\max} \\ 0.0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \dfrac{1.0}{\left(\int_{x_{\min}}^{x_{\max}} dx\right)^n} = (x_{\max} - x_{\min})^{-n} & \text{for } \forall\ x_{\min} \le x_i \le x_{\max} \\ 0.0 & \text{otherwise} \end{cases} \qquad \text{(if } X_i \text{ i.i.d.)}$$

(1.3)

Q.E.D.

Q1. 교수님 질문 있습니다. 아래 사진에서 왜 분모가 integral 형태로 표현되는지 이해가 안됩니다. c<= xi <= c+1 이라면 1/n 이 되는게 아닌가요??

$$= \frac{\Pr\left\{T=(x_{\max},x_{\min})\mid X=\left(x_1,x_2,\cdots,x_n\right),\theta=c\right\}\Pr\left\{X=\left(x_1,x_2,\cdots,x_n\right)\mid\theta=c\right\}}{\sum_x \Pr\left\{T=(x_{\max},x_{\min})\mid X=\left(x_1,x_2,\cdots,x_n\right),\theta=c\right\}\Pr\left\{X=\left(x_1,x_2,\cdots,x_n\right)\mid\theta=c\right\}}$$

(Total Probability Theorem + Conditional Probability)

$$=\begin{cases} \dfrac{1.0}{\int_{(x_{\min},\cdots,x_{\min})}^{(x_{\max},\cdots,x_{\max})} dx} & \text{for } \forall\ c \le x_i \le c+1 \\ 0.0 & \text{otherwise} \end{cases}$$

Answer: The question is why there is an integration in the denominator. If $c \le x_i \le c+1$, shouldn't it be 1/n.

I do not know what it means to be "it" right there. Anyways, since the question is about the denominator part, I will consider the denominator part. But the whole is the application of the Bayes rule we talked about in my probability primer note. Recall the part that I explain the need for changing the order of conditioning to solve a difficulty problem.

Look at the first equality in (1.3). What we need to concentrate is the denominator part. It is

$$\Pr\left\{T=(x_{\max},x_{\min})\mid\theta=c\right\}. \tag{1.4}$$

Note that this probability is for a fixed numbers c and $T=(x_{\max},x_{\min})$.

In order to evaluate this, we need to introduce the total probability theorem.

$$\Pr\left\{T=(x_{\max},x_{\min})\mid\theta=c\right\}=\sum_x \Pr\left\{T=(x_{\max},x_{\min}),X=\left(x_1,x_2,\cdots,x_n\right)\mid\theta=c\right\}. \tag{1.5}$$

The summation here is not rigorous since the random variable x_i are continuous. In order to write this correctly, we have to write the RHS to be the following one:

$$\sum_x \Pr\left\{T=(x_{\max},x_{\min}),X=\left(x_1,x_2,\cdots,x_n\right)\mid\theta=c\right\}$$
$$=\int\int\cdots\int \Pr\left\{T=(x_{\max},x_{\min}),X=\left(X_1\in\left(x_1,x_1+dx_1\right],\cdots,X_n\in\left(x_n,x_n+dx_n\right]\right)\mid\theta=c\right\}dx_1dx_2\cdots dx_n \tag{1.6}$$

This is tedious. I cannot even hold them into a single line of expression.

That is why I use the short hand notation given in (1.5). That is, just remember that they are continuous random variables. Once we figure out what to do, then, we will reflect this fact to the final answer.

Now going back to the Equation (1.5), we can write the joint distribution as the product of the conditional and the prior as we have done in the numerator part.

$$\begin{aligned}&\sum_x \Pr\left\{T=(x_{\max},x_{\min}),X=\left(x_1,x_2,\cdots,x_n\right)\mid\theta=c\right\}\\&=\sum_x \Pr\left\{T=(x_{\max},x_{\min})\mid X=\left(x_1,x_2,\cdots,x_n\right),\theta=c\right\}\Pr\left\{X=\left(x_1,x_2,\cdots,x_n\right)\mid\theta=c\right\}\end{aligned} \tag{1.7}$$

At this point, let us consider $\Pr\left\{X=\left(x_1,x_2,\cdots,x_n\right)\mid\theta=c\right\}$ in Eq. (1.7) and note that

$$\Pr\left\{X=\left(x_1,x_2,\cdots,x_n\right)\mid\theta=c\right\}=\begin{cases}1.0 & \text{for } \forall\ c\le x_i\le c+1\\ 0.0 & \text{otherwise}\end{cases} \tag{1.8}$$

For example, suppose $c=0.5$, then for each $i=1,2,\cdots,n$, x_i ~ uniform on $\left(0.5,1.5\right]$. We may also let n = 3. Then, consider the following probabilities:

1. $\Pr\left\{X=(x_1=1.3,x_2=0.9,x_3=0.7)\mid\theta=0.5\right\}$
2. $\Pr\left\{X=(x_1=1.7,x_2=0.9,x_3=0.7)\mid\theta=0.5\right\}$
3. $\Pr\left\{X=(x_1=0.8,x_2=0.4,x_3=0.7)\mid\theta=0.5\right\}$

Again, using the rigorous notation, each of the above means the following kind of events:

4. $$\begin{aligned}&\Pr\left\{X=(x_1=1.3,x_2=0.9,x_3=0.7)\mid\theta=0.5\right\}\\&=\Pr\left\{X=\left(x_1=\left(1.3,1.3+dx_1\right],x_2=\left(0.9,0.9+dx_2\right],x_2=\left(0.7,0.7+dx_3\right]\right)\mid\theta=0.5\right\}\end{aligned}$$

Now we understand the result (1.8).

Next we apply this result (1.8) to (1.7). Then, we note (1.7) becomes the following:

$$\begin{aligned}&\sum_x \Pr\left\{T=(x_{\max},x_{\min}),X=\left(x_1,x_2,\cdots,x_n\right)\mid\theta=c\right\}\\&=\int_c^{c+1}\int_c^{c+1}\cdots\int_c^{c+1}\Pr\left\{T=(x_{\max},x_{\min})\mid X=\left(x_1,x_2,\cdots,x_n\right),\theta=c\right\}dx_1dx_2\cdots dx_n\end{aligned} \tag{1.9}$$

Note the lower limit and upper limit of each integration is the interval $\left(c,c+1\right]$.
Now we are left with the evaluation of the integrand

$$\Pr\left\{T=(x_{\max},x_{\min})\mid X=\left(x_1,x_2,\cdots,x_n\right),\theta=c\right\} \tag{1.10}$$

But by the definition of $T:=(x_{\max},x_{\min})$, this conditional probability shall satisfies the following:

$$\begin{aligned}&\Pr\left\{T=(x_{\max},x_{\min})\mid X=\left(x_1,x_2,\cdots,x_n\right),\theta=c\right\}\\&=\begin{cases}1.0 & \text{for } \forall\ x_{\min}\le x_i\le x_{\max}\\ 0.0 & \text{otherwise}\end{cases}\end{aligned} \tag{1.11}$$

Now applying (1.11) into (1.9), we have

$$
\begin{aligned}
&\sum_{x} \Pr\left\{T = (x_{\max}, x_{\min}), X = \left(x_1, x_2, \cdots, x_n\right) \mid \theta = c\right\} \\
&= \underbrace{\int_{c}^{c+1} \int_{c}^{c+1} \cdots \int_{c}^{c+1}}_{\substack{x_{\max} = \max(x_1, x_2, \cdots, x_n) \\ x_{\min} = \min(x_1, x_2, \cdots, x_n)}} dx_1 dx_2 \cdots dx_n \\
&= \int_{x_{\min}}^{x_{\max}} \int_{x_{\min}}^{x_{\max}} \cdots \int_{x_{\min}}^{x_{\max}} dx_1 dx_2 \cdots dx_n
\end{aligned}
\tag{1.12}
$$

Finally applying (1.12) into (1.9), we get the Equality (b) in Equation (1.3).