**Problem 2.23. Cover and Thomas. 2nd Edition.**

I noticed that many students have made mistakes or provided incomplete solutions on this problem. Thus, I made this discussion note.

P2.23 Conditional mutual information. Consider a sequence of  binary random variables , , …, . Each sequence with an even number of 1’s has probability , and each sequence with an odd number of 1’s has probability 0.

Find the mutual informations

 , , …, .

**Solution**:

Note that it is given that each sequence with even 1’s has the joint probability

 .

Note that there are a total of  such sequences. Each sequence is thus equally probable.

And each sequence with an odd number of 1’s has probability zero, i.e.,

 .

For a fixed *n*, what’s given is the joint distribution of . We need to derive the marginal distributions from this joint distribution.

Let *n* = 2. Consider . All possible outcomes for are (0, 0), (0, 1), (1, 0), (1, 1). Then, from , the joint distribution is, ,  and . The marginal distribution  is  and . The marginal distribution  is  and . Thus, the product distribution  is uniform 1/4 for each. Note that the two random variables  and  are not independent.

Once the joint distribution and the product distribution are given, it is just a matter of applying the definition to calculate the mutual information. Using , we have

 

We made a big progress.

Let *n* = 3 and consider . Again, we start with the joint distribution and then find out marginal distributions.

All possible outcomes for are

|  |  |  |  |
| --- | --- | --- | --- |
|  |  |  |  |
| (0,0,0) |  | 1/2 |  |
| (0,0,1) | 0 | 0 |  |
| (0,1,0) | 0 | 0 |  |
| (0,1,1) |  | 1/2 |  |
| (1,0,0) | 0 | 0 |  |
| (1,0,1) |  | 1/2 |  |
| (1,1,0) |  | 1/2 |  |
| (1,1,1) | 0 | 0 |  |

For the conditional marginal, we have

|  |  |
| --- | --- |
|  |  |
| (0, 0) |  |
| (0, 1) |  |
| (1, 0) |  |
| (1, 1) |  |

and

|  |  |
| --- | --- |
|  |  |
| (0, 0) |  |
| (0, 1) |  |
| (1, 0) |  |
| (1, 1) |  |

Furthermore, the marginal  is given by  and .

Using these results so far, we can obtain the product conditionals  in the table above. Then, we can calculate the conditional mutual information:



We note that the foregoing result can be generalized to



Thus, it can be generalized to

 

for all *n*.

**(Short version)**

Intuitively, one can say the following:

 

Now note that the conditional joint distribution , , , . Then, the marginal is uniform distributed, i.e.,  and . Thus, the final equality holds.

End.