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Introduction to Mathematical Thinking

- Study material –

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1 주차 (p10~p22)

2. Getting precise about language

One American dies of melanoma almost every hour

→ Almost every hour, an American dies of melanoma

We always share a common knowledge of the world, and that common knowledge can be drawn upon to determine the intended meaning. But, when mathematicians use language in their work, there often is limited or no shared, common understanding. Especially, the need for precision is paramount in mathematics,

∴ When mathematicians use language in doing mathematics, they rely on the literal meaning. → We have to learn to eliminate the sloppiness of expression that we are familiar with in everyday life, and master a highly constrained, precise way of writing and speaking.

2.1 Mathematical statements

Before we can prove whether a certain statement is true or false, we must be able to understand precisely what the statement says.

Mathematics is a very precise subject.

When we use language in an everyday setting, we often rely on context to determine what our words convey (Summer, Small)

In everyday life, we use context and our general knowledge of the world and of our lives to fill in missing information in what is written or said, and to eliminate the false interpretations that can result from ambiguities.

The man saw the woman with a telescope.

Sisters reunited after ten years in checkout line at Safeway.

Prostitutes appeal to the Pope.

Large hole appears in High Street. City authorities are looking into it.

Mayor says bus passengers should be belted.

In mathematics, precision is crucial, and it cannot be assumed that all parties have the same contextual and background knowledge in order to remove ambiguities.

The cost of miscommunication through an ambiguity can be high, possibly fatal.

To make the use of language in mathematics sufficiently precise is possible because of the special, highly restricted nature of mathematical statements.

Exercise 2.1.1

2.

→ 1) Using a telescope, the man saw the woman.

2) The man saw the woman who had got a telescope.

5. Elevator has two meanings : 1) elevator, 2) a grain depot

According to a solution, First is 'The elevator is to be used only in case there is a fire. Second is 'If the building catches on fire, the use of the elevator is prohibited.

1st : When there is a fire in the building, do not use elevator.

2nd : Do not use the elevator, because there might be a fire.

8.

In mathematics, precise is really important. The sentence has to have an objective description how hot it is. As a result, a concrete temperature should be included in the sentence.

2.2 The logical combinators and, or, and not

The Combinator and

Abbreviation for and → $\vee$, &

$\vee$ is called wedge.

$\phi \wedge \psi$ is usually read as ϕ and ψ

and ϕ, ψ are called the conjuncts of the combined statement.

ϕ and ψ are both true. → $\phi \wedge \psi$ will be true.

If one or both of ϕ, ψ is false, then $\phi \wedge \psi$ is also false.

and is independent of order in mathematics : $\phi \wedge \psi$ means the same thing as $\psi \wedge \phi$

John took the free kick and the ball went into the net.

The ball went into the net and John took the free kick.

Exercise 2.2.1

1. False. As an example above, 'and' is sometimes used differently between the mathematical concept and a daily expression concept.

4. I will explain why each case is not false.

7. Once 'and' is included in a sentence, it looks there is more detailed other features on Alice. So, (e) looks simple and better than others.

The Combinator or

Abbreviation for or → $\vee$

$\vee$ is called vee.

$\phi \vee \psi$ is usually read as ϕ or ψ

and ϕ, ψ are called the disjunction of the combined statement.

One of ϕ, ψ is true. → $\phi \vee \psi$ will be true.

$(3<5) \vee (1=0)$ is not only mathematically meaningful but actually true.

Exercise 2.2.2

2.

(a). π is bigger than 3.

(b). x is not equal to zero.

(c). x is greater than or equal to zero.

(d). x is greater than or equal to zero.

(e). x doesn't have the value from −3 to 3.

5. They are same because one of three variables are true, they are true.

Exercise 2.2.3

1.

(a). $\pi \leq 3.2$

(b). $x \geq 0$

(c). $x^2 \leq 0$

(d). $x \neq 1$

(e). ψ

4.

ϕ	$\neg\phi$
T	F
F	T

7. We can capture this kind of use of language from a context, a feeling, a sixth sense etc.

2 주차 (p23~p36)

2.3 Implication

Exercise 2.3.2

1. T, T

Exercise 2.3.3

2-(a) : $T \Rightarrow (D \wedge Y)$

(b) : $T \Rightarrow (Y \Rightarrow \neg D)$

(c) : $T \Rightarrow (\neg D \wedge Y)$

(d) : $D \Rightarrow \neg Y$

(e) : $T \Rightarrow (D \Leftrightarrow Y)$

5.

ϕ	ψ	$\neg \psi$	$\phi \Rightarrow \psi$	$\phi \not\Rightarrow \psi$	$\phi \wedge \neg \psi$
T	T	F	T	F	F
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	F	F

Exercise 2.3.4

2.

ϕ	ψ	$\neg \phi$	$\phi \Rightarrow \psi$	$\neg \phi \vee \psi$	$(\phi \Rightarrow \psi) \Leftrightarrow (\neg \phi \vee \psi)$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

5.

M	N	M X N	M + N
1	1	1	0
1	0	0	1
0	1	0	1
0	0	0	0

8. We should apply the concept of ‘implication’ to the rule. Using the notation ‘implication’, we can make the statement which is true “There is a vowel on one side $\Rightarrow$ There is an odd number on the other side.” We know that the contrapositive proposition of this state is also true. That is “ There is an even number on one side $\Rightarrow$ There is not a vowel on the other side.”. Thus, we need to turn over at least 2 cards E and 4 to verify the rule.

Exercise 2.3.5

3-(a) : Necessary : $\frac{n}{6}=\frac{1}{2}*\frac{n}{3}$

(b) : Not necessary : $\frac{n}{9}=\frac{1}{1.5}*\frac{n}{6}\rightarrow$ 1.5 is not natural number.

(c) : $\frac{n}{6}=2*\frac{n}{12}=\frac{1}{0.5}*\frac{n}{12}\rightarrow$ 0.5 is not natural number.

(d) :

(e) : $(\frac{n}{6})^2=\frac{n^2}{36}=\frac{1}{12}*\frac{n^2}{3}$

(f) : “n is even” means that n is divisible by 2. So, we can prove this using problem 3-(a).

6. m=(2a-1), n=(2b-1) and a and b is natural number bigger than zero. Then, we can make mn=(2a-1)(2b-1)=4ab-2a-2b+1. We can prove this formula have only odd number with any a,b value.

9.

ϕ	ψ	$\phi \Leftrightarrow \psi$	θ	$\psi \vee \theta$	$\phi \Rightarrow (\psi \vee \theta)$
T	T	T	T	T	T

T	F	F	T	T	T
F	T	F	F	T	T
F	F	T	F	F	T
T	T		F	T	T
T	F		F	F	F
F	T		T	T	T
F	F		T	T	T

12.

ϕ	ψ	$\neg\phi$	$\neg\psi$	$\phi \Rightarrow \psi$	$\neg\psi \Rightarrow \neg\phi$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

15-(a) : If two rectangles have the same area, they are congruent.

(b) : If in a triangle with sides a,b,c(c largest), $a^2+b^2=c^2$, then the triangle is right-angled.

(c) : If n is prime, then 2^n-1 is prime.

(d) : If the Dollar falls, the Yuan will rise.

18. A : antecedent, C : consequent

(a). A : The apples are red. C : The apples are ready to eat.

(b). A : f is differentiable. C : f is continuous.

(c). A : f is integrable. C : f is bounded.

(d). A : s is convergent. C : s is bounded.

(e). A : 2^n-1 is prime. C : n is prime.

(f). A : Karl is playing. C : The team wins.

(g). A : Karl plays. C : The team wins.

(h). A : Karl plays. C : The team wins.

21.

ϕ	ψ	$\neg\Phi$	$\neg\psi$	$(\phi \wedge \neg\psi)$	$(\psi \wedge \neg\phi)$	$(\phi \wedge \neg\psi) \vee (\psi \wedge \neg\phi)$
T	T	F	F	F	F	F

T	F	T	T	T	F	T
F	T	T	F	F	T	T
F	F	F	T	F	F	F

24.

1	2	3	4
Bear	Coke	Under drinking ages	Above drinking ages

- If 1 is above age to drink, then he can drink. = "1 is above age to drink. $\Longrightarrow$ He can drink."

- If 3 is drinking bear then it's illegal. = "3 is drinking bear. $\Longrightarrow$ It's illegal."

3주차 (p36~p50)

2.4 Quantifiers

Exercise 2.4.2

1. (a) : $(\exists x \in \mathbb{N})(x^3 = 27)$

   (b) : $(\exists m \in \mathbb{N})(m > 1{,}000{,}000)$

   (c) : $(\exists n \in \mathbb{N})(\forall a, b \in \mathbb{N})(n = a / b)$

4. (a) : $(\exists x \in \mathbb{R})(\forall a \in \mathbb{R})(x^2 + a = 0)$

   (b) : $(\exists x \in \mathbb{R})(\forall a \in \mathbb{R})[(a < 0) \wedge (x^2 + a = 0)]$

   (c) : $(\exists a, b \in \mathbb{N})(\forall x \in \mathbb{R})(x = \frac{a}{b})$

   (d) : $(\forall a, b \in \mathbb{R})(\exists x \in \mathbb{R})(x \neq \frac{a}{b})$

   (e) : $(\forall a, b \in \mathbb{R})(\exists x \in \mathbb{R})[(x \neq \frac{a}{b}) \wedge (x < \infty)]$

7. $$\begin{aligned}&(\exists time)(\forall people)[f(people, time)]\\ &\wedge(\forall time)(\exists people)[f(people, time)]\\ &\wedge(\forall time)(\forall people)[\neg f(people, time)]\end{aligned}$$

Exercise 2.4.3

2. A(x) : Researching AI

   x : Lab members

   $\neg[\exists x A(x)]$: It is not case that there is at least one member who researches AI

   $\forall x[\neg A(x)]$: All members do not research AI

Exercise 2.4.5

2. (a). T

(b). F : $x = \frac{2}{3}$

(c). T : $x = 1$

(d). F : $\pm\sqrt{2}\, are\, irrational$

(e). T : $\pm\sqrt{2}\, are\, real\, numbers.$

(f). F : $x = -100$

(g). T : $x = 1$

(h). T : $y = -x$

(i). F

(j). T

(k). T

(l). F : $z\, has\, to\, be\, y$

(m). F : $z\, has\, to\, be\, y$

(n). T

(o). F : $y^2\, has\, to\, be\, bigger\, than\, 0$

(p). F

5. (a). $(\exists x \in \mathbb{N})(\forall y \in \mathbb{N})(x + y \neq 1)$

(b). $(\exists x > 0)(\forall y < 0)(x + y \neq 0)$

(c). $\forall x(\exists \varepsilon > 0)[(x > -\varepsilon) \vee (x < \varepsilon)]$

(d). $(\exists x \in \mathbb{N})(\exists y \in \mathbb{N})(\forall z \in \mathbb{N})(x + y \neq z^2)$

4주차 (p51~p66)

3. Proofs

Exercise 3.2.1

1. Let's assume that $\sqrt{3}$ is rational. Using p and q, we can make this equation $\sqrt{3}=\frac{p}{q}$ where p and q are natural number and have no common factors. Squaring gives $3=\frac{p^2}{q^2}$ which rearranges to give $p^2=3q^2$.

Exercise 3.3.1

1. Necessarily irrational. Using contradiction, if $r+3=\frac{p}{q}$, then $r=\frac{p-3q}{q}$. This is rational.
2. Necessarily irrational. Using contradiction, if $5r=\frac{p}{q}$, then $r=\frac{p}{5q}$. This is rational.
3. Not necessarily irrational. Let $r=\sqrt{x}, s=1-\sqrt{x}$. Then, $r+s=1$
4. Not necessarily irrational. Let $r=\sqrt{x}, q=\sqrt{x}$. Then, $rq=x$
5. Necessarily irrational. Using contradiction, if $\sqrt{r}=\frac{p}{q}$, then $r=\frac{p^2}{q^2}$. This is rational.
6. Refer to p58

Exercise 3.3.2

3. Dividing a payout of $2 million with same amount, each investor can get $400,000. If one investor give others his own money - $400,000, then others' money will be more than $400,000 even though the investor who gave his own money to others will have less than $400,000 (Equivalence exchange). So, this statement is true.
6. (a). Let $r\in\mathbb{N}$. Then, m=2r, n=2r as they are even. That means m+n=4r, and it's even number.

(b). According to above proof, mn = $4r^2$. So, mn is divisible by 4.

(c). Let m=($2r_1$-1), n=($2r_2$-1) and $r_1, r_2 \in \mathbb{N}$. We can make m+n=$2r_1$+$2r_2$-2. m+n is divisible by 2. So, m+n is even.

(d). Let m=(2r-1), n=2r and $r \in \mathbb{N}$. m+n=4r-1. Substituting r to the equation, we can see m+n is odd.

(e). According to (d), mn=$4r^2$-2r. mn is divisible by 2. That means mn is even.

Exercise 3.4.1

1. There are some birds which can't fly such as chicken, ostrich, penguin etc.

4. (a). True $\rightarrow$ x=0

   (b). True $\rightarrow$ y=-x

   (c). False $\rightarrow$ $n = \frac{12-3m}{5}$. So, some n values are not natural number on m

   (d). True $\rightarrow$ ($\exists n$ is integer)($\frac{a}{bc} = n$)$\Rightarrow$ ($\exists n$ is integer)[$(\frac{a}{b} = nc) \vee (\frac{a}{c} = nb)$]

   (e). True $\rightarrow$ Let any five consequtive intergers are n, n+1, n+2, n+3, n+4 (n is integer). The sum of the integers is 5n+10. This is divisible by 5.

   (f). Ture $\rightarrow$ n^2+n must be odd for n^2+n to be even. However, It's impossible for n^2+n to make odd because we can't make the sum with odd and even or even and or on n^2 and n. As a result, the negation of (f) is false, and that means (f) is true

   (g). True$\rightarrow$Let $x = \frac{p}{q}$, $y = \frac{r}{s}$. There is $z = \frac{x+y}{2} = \frac{ps+rq}{2qs}$. It means x<z<y

   (h). True $\rightarrow$ $(\forall x, \forall y \in \mathbb{R})(x\, is\, rational \wedge y\, is\, irrational \rightarrow x+y\, is\, irrational)$. Using contradiction, we can make

   $$(\exists x, \exists y \in \mathbb{R})(x\, is\, rational \wedge y\, is\, irrational \rightarrow x+y\, is\, rational)$$

   If $x = \frac{p}{q}$ and $x+y = \frac{r}{s}$, then $y = \frac{rq-ps}{sq}$

   (i). True$\rightarrow$ $(\forall x, \forall y \in \mathbb{R})(x+y\, is\, irrational \rightarrow x\, is\, irrational \vee y\, is\, irrational)$

Let $(\exists x, \exists y \in \mathbb{R})(x+y\, is\, irrational \rightarrow x\, is\, rational \wedge y\, is\, rational)$

If $x=\dfrac{p}{q}$ and

(j).

7. Using the quadratic formula, we can express the equation to composite of a linear equation. According to the quadratic formula, $n=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$. That means $f(n)=n^2+bn+c=[n+(\dfrac{-b+\sqrt{b^2-4c}}{2})][n-(\dfrac{-b+\sqrt{b^2-4c}}{2})]$.

Exercise 3.5.1

1. $A(n)=1+2+3+...+n=\dfrac{1}{2}n(n+1)$

2. $A(1)=1$

3. $(\forall n\in\mathbb{N})[\dfrac{1}{2}n(n+1)\Rightarrow\dfrac{1}{2}(n+1)(n+2)]$

Exercise 3.5.2

3. (a). $\sum_{r=1}^{n+1} r^2=\dfrac{1}{6}n(n+1)(2n+1)+(n+1)^2=\dfrac{1}{6}(n+1)(n+2)(2n+3)$

(b). $\sum_{r=1}^{n+1} 2^r=2^{n+1}-2+2^{n+1}=2\cdot 2^{n+1}-2=2^{n+2}-2$

(c). $\sum_{r=1}^{n+1} r\cdot r!=(n+1)!-1+(n+1)\cdot(n+1)!=(n+2)(n+1)!-1=(n+2)!-1$

5, 6주차(p67~p90)

4. Proving results about numbers

Exercises 4.1.1

2. The rooms where there is a guest : n

   Tour members : N

   Except for n-1, ask the guest who accommodated in the room n to move n+N room. Then, the tour members enter the room from n to n+N-1

Exercises 4.1.3

1. (i). $$\begin{aligned}&a \mid 0 \to 0 = aq \; and \; q = 0\\&a \mid a \to a = aq \; and \; q = 1\\&\therefore True\end{aligned}$$

   (ii). $$\begin{aligned}&a \mid 1 \Leftrightarrow a = \pm 1\\&1 = aq \; and \; a \le 1, a \ne 0\\&\therefore a = \pm 1 \; because \; a, q \in \mathbb{Z}\end{aligned}$$

   (iii). $$\begin{aligned}&b = aq_1, d = cq_2\\&bd = (q_1 q_2) ac\\&\therefore a \mid b \wedge c \mid d \to ac \mid bd : True\end{aligned}$$

   (iv). $$\begin{aligned}&b = aq_1, c = bq_2 = (q_1 q_2) a\\&\therefore True\end{aligned}$$

   (v). $$\begin{aligned}&b = aq_1, a = bq_2 = aq_1 q_2\\&q_1 q_2 = 1\\&\therefore q_1, q_2 = \pm 1 \; because \; they \; are \; \mathbb{Z}\\&Let's \; square \; to \; b = aq_1 \; or \; a = bq_2. Then, b^2 = a^2 q_1^2 \; or \; a^2 = b^2\\&\therefore b = \pm a \; or \; a = \pm b\end{aligned}$$

   (vi). $$\begin{aligned}&b = aq\\&Thus, |b| = |a||q|. Since \; b \ne 0, we \; must \; have \; q \ne 0. So, |q| \ge 1\\&\therefore |a| \le |b|\end{aligned}$$

(vii). $$\begin{aligned}&b = aq_1, c = aq_2\\&bx = axq_1, cx = axq_2\\&bx + cx = a[x(q_1 + q_2)]\\&\therefore True\end{aligned}$$

4. $$\begin{aligned}&(\forall x \in \mathbb{N})(a = 2x - 1)\\&a(a^2 - 1) = 24q, q \in \mathbb{Z}\\&a^3 - a = 8x^3 - 12x^2 + 4x = 24q\\&\therefore q = \frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{6}x\\&q\, is\, integer\, on\, all\, x.\end{aligned}$$

Exercises 4.1.4

2. To show this note that among any n consequtive numbers there is one divisible by n. In particular, one of p, p+1, p+2, is divisible by 3. Thus, one of p, p+2, p+4 is divisible by 3. Hence, if (p, p+2, p+4) is a prime triplet, this forces p to be actually eaqual to 3. So, we conclude that (3, 5, 7) is the only prime triplet.

Exercises 4.1.5

2. Euclid's lemma : If a prime p divides the product ab of two integers α and β, p must divide at least one of those integers α and β.

   If c, a prime number measure ab, c measures either a or b. Suppose c does not measure a. Therefore c,a are prime to one another. Suppose ab=mc. Therefore, c:a=b:m. Hence b=nc, where n is some integer. Therefore c measures b. Similarly, if c does not measure b, c, measures a. Therefore c measures one or other of the two numbers a, b.

Exercises 4.2.1

2. $(\forall a \in \mathbb{Z})(\forall b \in \mathbb{Z})((Greatest\ common\ divisor\ of\ a, b = 1) \wedge (b \neq 0)) \to (\frac{a}{b} is\ rational\ number)$

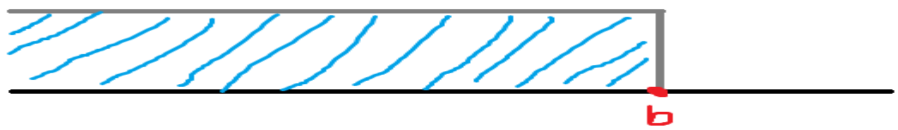
Exercises 4.3.2

1. $(\forall x \in A)(x \le a) \Rightarrow a\, has\, infinitely\, many\, different\, values.$

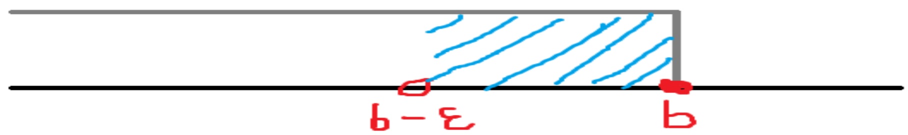
Because there is no maximum value in a set of integers, rationals, and reals, there are a lot of different values as upper bounds.

4. The lub of A means the least upper bound of a set A.

(a)



(b)



Let $b-\varepsilon = c$. Then, $a > b-\varepsilon$ become $a > c$ and also, $\forall \varepsilon > 0$ become $\forall b > c$. This is totally same with Question 3.

7. lub(a,b) : b. because $b \le b$

lub[a,b] : b

max(a,b) : We can't define

max[a,b] : b

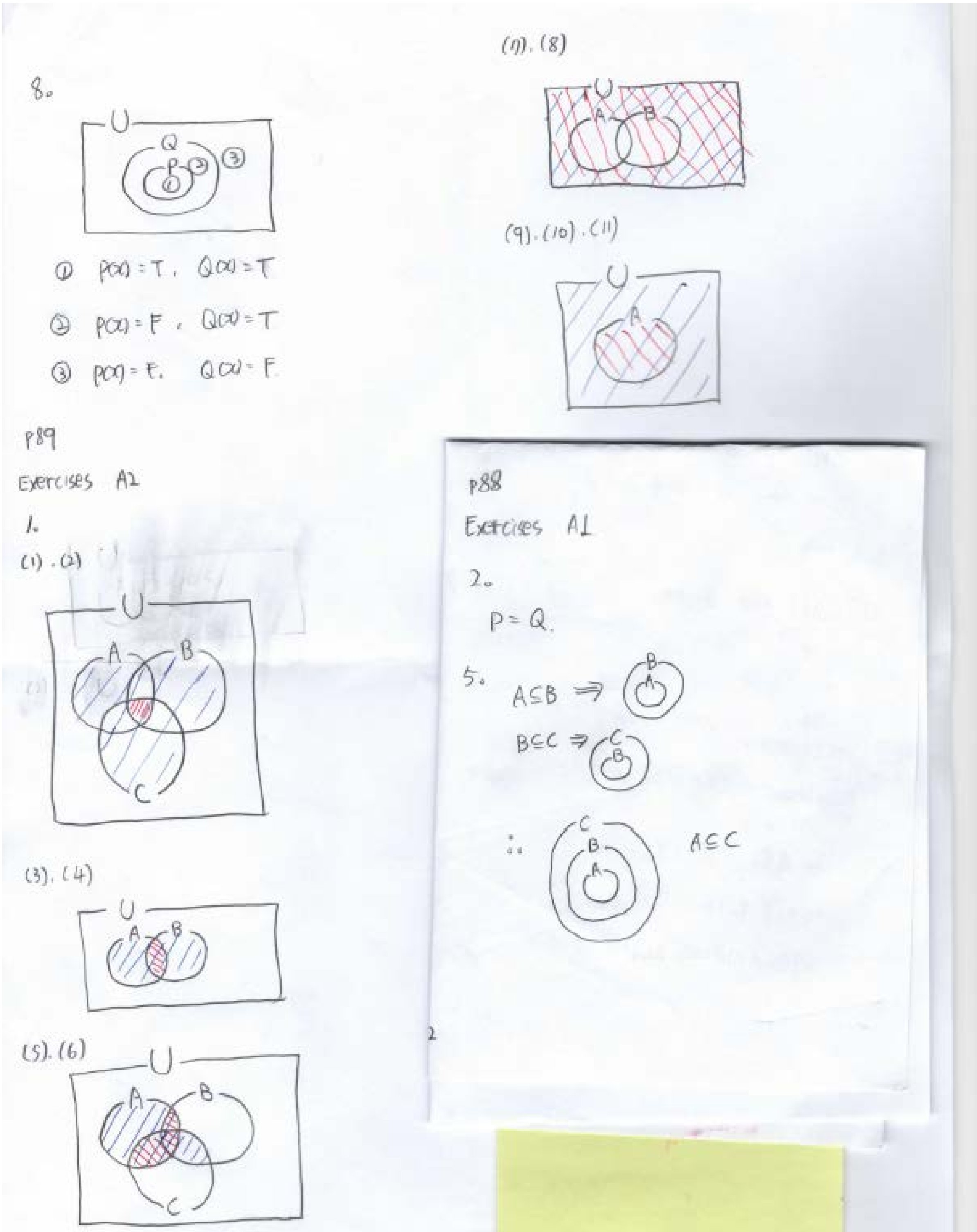
10. $(\forall a \in A)(a \ge b); and$
$(\forall \varepsilon > 0)(\exists a \in A)(a < b + \varepsilon)$

13. Let A is "a nonempty set of reals that has a lower bound." So, let $(\forall a \in A)(a \ge \min(A))$. min(A) is the b and this will be a lower bounds which is the greatest among values $\neg A$ ...

Exercises 4.3.3

2. r<s means r-s>0. Using the Archimedean property, we can make n(s-r)>1. We can see that ns>nr+1. This means nr<m (like nr+1)<ns. As a result, $r < \dfrac{m}{n} < s$ is proved from the Archimedean property.

Appendix problems : I draw the Venn diagram to solve the problems. Refer to the paper I wrote.



8.

U Q P ① ② ③

① $P(x) = T$, $Q(x) = T$

② $P(x) = F$, $Q(x) = T$

③ $P(x) = F$, $Q(x) = F$

P89

Exercises A2

1.

(1). (2)

U A B C

(3). (4)

U A B

(5). (6)

U A B C

(7). (8)

U A B

(9). (10). (11)

U A

P88

Exercises A1

2.

$P = Q$.

5. $A \subseteq B \Rightarrow$ B A

$B \subseteq C \Rightarrow$ C B

$\therefore$ C B A $A \subseteq C$

Done by Altynay Kadyrova

Chapter 2: Getting precise about language

2.1 Mathematical statements

Exercises 2.1.1

3. a). Sisters reunited after 10 years near checkout line at Safeway.
Explanation: I changed preposition from in to near. With “in”, we might think that sisters did appointment to meet inside Safeway. However, if we change it to “near”, it will seem like that they met accidentally.

Why I solve in this way is that it is less probable to put this information of appointment of sisters after 10 years into the headline of the newspaper while if they would meet unintentionally, then it is more probable to be in the headline of the magazine.

b). Prostitutes appeal for the Pope.
Explanation: Again here, I take into account prepositions. As a result, with “for”, this sentence will mean as a asking for a help or support. In contrast, with “to”, the meaning is confusing, (i.e. Pope likes prostitutes), which is not logical.

c). Large hole appears in High Street. City authorities are working on it.
Explanation: The problem was in wrong usage of phrasal verbs. Naturally, no one will look into hole in order to solve the sudden problem; authorities will start to work on it.

d). Mayor says bus passengers should use seat belts.
Explanation: Initially, ambiguous meaning was in improper use of word “belt” which made the sentence to hurt bus passengers. However, the meaning is to provide safety. Therefore, I changed old sentence to the above one.

6. This page intentionally left blank:

- This sentence does not make a true statement for mathematicians because it is already written in the paper which actually makes paper not completely blank.

- Purpose of making such statement is to prevent a confusion of readers that no error happened during the production process.

- Reformulation: The rest of the page intentionally left blank.

9. The words “nature and”, “and and”, and “and economy” are interconnected.

2.2 The logical combinators and, or and not

Exercises 2.2.1

2.

a). $0<\pi<10$

b). $7\leq p<12$

c). $5<x<7$

d). $x<4$

e). $-3<x<3$

f). $x=0$

5. I will prove that if one of the $\varphi(i)$ is false while others are true, then the conjunction will of all elements will be false.

8. $\varphi\wedge\theta$: T, F, F, F respectively

Exercises 2.2.2

3. I will prove that if either all $\varphi(i)$ or some of them is true while remaining is false in latter case, then overall disjunction is true.

6. e

Exercises 2.2.3

2. a). π is less or equal to 3.2

b). x is greater or equal to zero

c). Square of x is less or equal to zero

d). x is not equal to one

e). φ is exist

5.

a). $D \wedge Y$

b). $\neg T \wedge \neg D$

c). $T \wedge \neg D\wedge Y$

d). $D\wedge\neg Y$

e). $\neg Y\wedge D\wedge T$

f). $(D\wedge\neg Y) \vee (\neg D\wedge Y)$

g). $(T\wedge D) \vee (T\wedge Y)$

h). $T\wedge\neg Y\wedge\neg D$

i). $\neg T \wedge D \wedge Y$

j). $(T \wedge D) \vee \quad (T \wedge Y)$

2.3 Implication

Exercises 2.3.1

1. $\varphi \to \omega$ is False when φ is True and ω is false

Exercises 2.3.2

2. Third row: $\varphi \to \omega$ is true when φ is false and ω is true.

   Explanation: when φ is false, it means that φ won't imply ω, even if ω is true, this leads that implication is true.

3. Fourth row: $\varphi \to \omega$ is true when φ is false and ω is false.

   Explanation: if both of elements are false, then one implies another which is true. Hence, implication is also true.

Exercises 2.3.3

3.

φ	$\neg\varphi$	θ	$\varphi \to \theta$	$\neg\varphi \vee \theta$
T	F	T	T	T
T	F	F	F	F
F	T	T	T	T
F	T	F	T	T

6.

φ	θ	$\neg\theta$	$\varphi \to \theta$	$\varphi ! \to \theta$ or $\neg(\varphi \to \theta)$	$\varphi \wedge \neg\theta$
T	T	F	T	F	F
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	F	F

Conclusions from above table is that 5th and 6th columns values are same:

$\neg(\varphi \to \theta) \equiv \varphi \wedge \neg\theta$

Exercises 2.3.4

3.Refer to the 6th and 5th task in exercises 2.3.3

6. a). ^

b). XOR

c). No, minus one (-)1 = -1 whereas ¬1 = 0;

Exercises 2.3.5

1. I will use De Morgan's law.

$\neg(\varphi \vee \theta) = (\neg\varphi) \wedge (\neg\theta)$ which shows us that these two statements are equivalent.

4. C,d,f

7.No, counterexample 3*2=6

10. a). $\neg(\varphi\rightarrow\theta)\equiv \varphi\wedge\neg\theta$

φ	θ	$\neg\theta$	$\varphi\rightarrow\theta$	$\varphi!\rightarrow\theta$ or $\neg(\varphi\rightarrow\theta)$	$\varphi\wedge\neg\theta$
T	T	F	T	F	F
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	F	F

b). $\varphi\rightarrow(\omega\wedge\theta) \equiv(\varphi\rightarrow\omega) \wedge (\omega\rightarrow\theta)$

φ	ω	θ	$\omega\wedge\theta$	$\varphi\rightarrow(\omega\wedge\theta)$	$\varphi\rightarrow\omega$	$\omega\rightarrow\theta$	$(\varphi\rightarrow\omega) \wedge (\omega\rightarrow\theta)$
T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	F
T	F	T	F	F	F	T	F
F	T	T	T	T	T	T	T
F	F	T	F	T	T	T	T
T	F	F	F	F	F	T	F
F	T	F	F	F	T	F	F
F	F	F	F	F	T	T	T

c). $(\varphi \vee \omega) \to \theta \equiv (\varphi \to \theta) \wedge (\omega \to \theta)$

φ	ω	θ	$\varphi \vee \omega$	$(\varphi \vee \omega) \to \theta$	$\varphi \to \theta$	$\omega \to \theta$	$(\varphi \to \theta) \wedge (\omega \to \theta)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	T	T	T	T	T
F	T	T	T	T	T	T	T
F	F	T	F	T	T	T	T
T	F	F	T	F	F	T	F
F	T	F	T	F	T	F	F
F	F	F					

13. In all cases we can use negations of each statement and that one implies another

(a) If two rectangles do not have the same area , they are not congruent.

(b) If square of a + square of b is not equal to square of c, then a triangle with sides a,b,c , (c largest) is not right-angled

(c) If n is not prime, then 2^n–1 is not prime.

(d) If the Dollar will rise , the Yuan falls.

16. If φ is "It is sunny outside" and θ is "so they will go for hiking", φ implies θ - True. By contrast, its converse is false: "If we go for hiking it will be sunny outside" is an absurd.

19. **Converse**

a). The apples are ready to eat, if they are red

b). F will be continuous, if the differentiability of a function f is sufficient.

c). f is integrable if a function is bounded.

d). Whenever s is convergent, a sequence s is bounded

e). 2^n–1 is will be prime when n is necessarily prime f). Only if Karl is playing then the team wins

g). The team wins if Karl plays

h). If Karl plays, then team wins.

Contrapositive:

a). The apples are not ready to eat, if they are not red

b). F will be discontinuous, if the differentiability of a function f is not sufficient.

c). f is not integrable if a function is not bounded.

d). Whenever s is not convergent, a sequence s is not bounded

e). 2^n–1 is will not be prime when n is unnecessarily prime

f). Only if Karl is not playing then the team loses

g). The team loses if Karl does not play

h). If Karl does not play, then team loses.

22. a). The ¬(P ∨ Q) and ¬P ∧¬Q are equivalent

P	Q	P ∨ Q	¬(P ∨ Q)	¬P	¬Q	¬P ∧¬Q
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

b). ¬P ∨ ¬Q and¬(P ∨¬ Q) are not equivalent

P	Q	¬P ∨ ¬Q	P ∨¬ Q	¬P	¬Q	¬(P ∨¬ Q)
T	T	F	T	F	F	F
T	F	T	T	F	T	F
F	T	T	F	T	F	T
F	F	T	T	T	T	F

c). ¬(P ∧ Q) and ¬P ∨¬Q are equivalent

P	Q	P ∧ Q	¬(P ∧ Q)	¬P	¬Q	¬P ∨¬Q
T	T	T	F	F	F	F
T	F	F	T	F	T	T

F	T	F	T	T	F	T
F	F	F	T	T	T	T

d). ¬P→(Q∧R) and¬(P→Q) ∨ ¬(P→R) are equivalent

P	Q	R	Q∧R	P→(Q∧R)	¬(P→(Q∧R))	P→Q	¬(P→Q)	P→R	¬(P→R)	¬(P→Q) ∨ ¬(P→R)
T	T	T	T	T	F	T	F	T	F	F
T	T	F	F	F	T	T	F	F	T	T
T	F	T	F	F	T	F	T	T	F	T
F	T	T	T	T	F	T	F	T	F	F
F	F	T	F	T	F	T	F	T	F	F
T	F	F	F	F	T	F	T	F	T	T
F	T	F	F	T	F	T	F	T	F	F
F	F	F	F	T	F	T	F	T	F	F

e). P→(Q→R) and (P∧Q)→R are equivalent

P	Q	R	Q→R	P→(Q→R)	P∧Q	(P∧Q)→R
T	T	T	T	T	T	T
T	T	F	F	F	T	F
T	F	T	T	T	F	T
F	T	T	T	T	F	T
F	F	T	T	T	F	T
T	F	F	T	T	F	T
F	T	F	F	T	F	T
F	F	F	T	T	F	T

25. 24^{th} exercise was easier than ex 2.3.4 (8) because in the former case I used simple logic while in the latter case I used implication. For example, in 24^{th}, we just need to check those IDs which are either underage or drinking alcohol. Because 1^{st} person is drinking beer, so we need to check his age, second person is drinking coke, so no need to check his age, his not breaking a rule not to drink alcohol if you are under age. Next one is at drinking age, so again no need to check. The last one is underage, need to check. Finally only two peoples' ID need to be checked.

In ex 2.3.4 (8), implication will be to solve the puzzle. We can at least turn 2 cards: E,4

"If vowel, then other side is odd number"

But if one side is odd, then other side is vowel – Is it true or false, we do not know

Therefore, in order to check we consider contrapositive, if one side is even, then other side is consonant.

Hence, to check we turn 4 and its other side should be consonant; And from initial statement we turn E and its other side should be odd

Answer: 2 cards to be turned: E,4

2.4 Quantifiers

Exercise 2.4.2

2.a). $(\forall x \in N)(x^3 \neq 28)$ For all natural number x, its cube is not equal to 28.

b). $(\forall x \in N)(0<x)$ For all natural number x, x is greater than zero.

c). $(\forall n \in N)(\text{prime } (n))$ For all natural number n, n is prime

5. a). $(\forall x \in C)(D(x) \Rightarrow M(x))$.

b). $(\forall x \in C)(\neg(D(x)) \Rightarrow M(x))$.

c). $(\forall x \in C)\ M(x) \Rightarrow (D(x))$.

d). $(\exists x \in C)(D(x) \wedge (\neg M(x)))$

e). $(\exists x \in C)((\neg D(x)) \wedge M(x))$

8. $\exists x \forall t A(x,t)$ - there is a driver in every six second involved in an accident.

Exercise 2.4.4

There is an even prime bigger than 2 - false

Proof: Assume, P(x) is the property that x is a prime; O(x) is property that x is odd

The statement: $(\exists x>2)(P(x) \wedge (\neg O(x)))$

In order to prove, I take negation of whole statement: $(\forall x>2)(P(x) \Rightarrow O(x))$ which is true.

Logically, by definition of prime number, it can be divisible to 1 and itself, so x greater than 2 is always divisible to two.

Exercise 2.4.5

3.a).All students like pizza.

Assume S(x) – x students; P – domain of all people; L(x) is a property that x (i.e students) like pizza

$(\forall x\in P)(S(x) \Rightarrow L(x))$ - all people if they are students, then they like pizza

Negation: $(\exists x\in P)\ (S(x) \wedge (\neg L(x)))$ - There are people who are students and do not like pizza

b). One of my friends does not have a car

Assume S(x) – one of my friends; P – domain of all people; C(x) is a property that x does not have a car

$(\exists x\in P)(S(x) \wedge C(x))$ - there is a person who is my friend and he doesn't have a car

Negation: $(\forall x\in P)(S(x) \Rightarrow \neg C(x))$ All people if they are my friend, they do have a car

c). Some elephants do not like muffins

Assume S(x) – some elephants; A – domain of all animals; M(x) is a property that x does not like muffins.

$(\exists x\in A)(S(x) \wedge M(x))$ - there are some animals which are elephants and doesn't like muffins

Negation: $(\forall x\in A)(S(x) \Rightarrow \neg M(x))$ - For every animal, if it is an elephant, then it does like muffins

d). Every triangle is isosceles

Assume S(x) – triangle; A – all geometric figures; I(x) is a property that triangle is isosceles

$(\forall x\in A)(S(x) \Rightarrow I(x))$ - every geometric figure if it is triangle, then it is an isosceles

Negation: $(\exists x\in A)(S(x) \wedge \neg I(x))$ there is a geometric figure which is a triangle and not isosceles

e). Some of the students in the class are not here today

Assume $S(x)$ – students; P – domain of all people; $N(x)$ is a property that x (i.e students) absent today

$(\exists x \in P)(S(x) \wedge N(x))$ - there are people who are students and not attending class today

Negation: $(\forall x \in P)(S(x) \Rightarrow \neg N(x))$ - every people if they are students, then they are present in a class today

f). Everyone loves somebody

Assume x,y – people; P – domain of all people; $L(x,y)$ is a property that x loves y

$(\forall x \in P)(\exists x)(L(x,y))$ - for all people there is a person who loves another one

Negation: $(\exists x \in P)(\forall x)(\neg L(x,y))$ There are people for every person who doesn't love another one

g). Nobody loves everybody

Assume x,y – people; P – domain of all people; $L(x,y)$ is a property that x doesn't love y

$(\exists x \in P)\forall y(L(x,y))$ - there are people x, for every y people in the case that former does not love latter

Negation: $(\forall x \in P)\exists y(\neg L(x,y))$ For every people there is a person who loves another one

h). If a man comes, all the women will leave

Assume $M(x)$ - man; C – man comes; $W(x)$- women; P – domain of all people; $L(x)$ is a property that x will leave

$[\exists x\{M(x) \wedge C\}] \Rightarrow \forall x\ (W(x) \Rightarrow L(x))$ - IF there is a person who is a man and he comes, then for all people if they are women, then they will leave.

Negation: $[\forall x\{M(x) \wedge C\}] \wedge \exists x\ (W(x) \wedge (\neg L(x)))$ For every man who comes, there are women who will not leave

i).All people are tall or short

Assume $T(x)$ – tall people; $S(x)$ – short people; P – domain of all people;

$(\forall x \in P)\ (T(x) \vee S(x))$ - all people are tall or short

Negation: $(\exists x \in P)\ (T(x) \wedge S(x))$ There are people who are tall and short //absurd

j). All people are tall or all people are short

Assume $T(x)$ – tall people; $S(x)$ – short people; P – domain of all people;

$[(\forall x \in P)\ (T(x)) \vee\ [(\forall x \in P)\ S(x))]$ All people are tall or all people are short

Negation: $[(\exists x \in P)\ (T(x)) \wedge\ [(\exists x \in P)\ S(x))]$ There are tall people and there are short people k).Not all precious stones are beautiful

Assume T(x) –all precious stones; S(x) –beautiful; P – domain of all stone;

$(\exists x\in P)(\forall T(x) \wedge S(x))$ there are stones which are not all precious and not beautiful

Negation: $(\forall x\in P)(T(x) \Rightarrow S(x))$ For all stone, there is precious stone or not beautiful

l). Nobody loves me

Assume x – people; y – me; P – domain of all people; L(x,y) is a property that x doesn't love y

$(\exists x\in P)\forall x(L(x,y))$ There is a person for whom everyone doesn't love me

Negation: $(\forall x\in P)\exists x(\neg L(x,y))$ For all people, there is a person who loves me

m). At least one American snake is poisonous

Assume S(x) – American snake; B(x)- snake is poisonous; P – domain of all snakes;

$(\exists x\in P)(S(x) \wedge B(x))$ There are snakes which are both American and poisonous

Negation: $(\forall x\in P)(S(x) \Rightarrow (\neg B(x)))$ For every snake if it is American then it is not poisonous

n). At least one American snake is poisonous

Assume S(x) – American; B(x)- poisonous; C(x) – snake; P – domain of all animals;

$(\exists x\in P)(S(x) \wedge C(x) \wedge B(x))$ There are snakes which are both American and poisonous

Negation: $(\forall x\in P)(S(x) \Rightarrow C(x) \Rightarrow (\neg B(x)))$ For every animal which is snake if it is American then it is not poisonous

6."You may fool all the people some of the time, you can even fool some of the people all of the time, but you cannot fool all the people all the time"

P – domain of all people; F(p,t) – property that person p can fool at t – time

$\exists t(\forall p\in P)F(p,t) \wedge \exists p\in P(\forall t)F(p,t) \wedge \neg\forall p\in P(\forall t)F(p,t)$ and

$\exists t(\forall p\in P)F(p,t) \wedge (\forall t)(\exists p\in P)F(p,t) \wedge \neg(\forall t)(\forall p\in P)F(p,t)$

Negation: $\forall t(\exists p\in P)(\neg F(p,t)) \vee \forall p\in P(\exists t)(\neg F(p,t)) \vee \forall p\in P(\forall t)F(p,t)$ and

$\forall t(\exists p\in P)(\neg F(p,t)) \vee (\exists t)(\forall p\in P)(\neg F(p,t)) \vee (\forall t)(\forall p\in P)F(p,t)$

You may not fool some people all time or at some time, you may not fool all people or you may fool all people all time.

Ch3 Proofs

Exercise 3.2.1

2. No, not true and e.g. counter example:

$\sqrt{4} = 2$ where 2 is rational

Exercise 3.3.2

1. Because by definition of biconditional: $\varphi \Leftrightarrow \phi$ = $(\varphi \Rightarrow \phi) \wedge (\phi \Rightarrow \varphi)$ therefore proving both $(\varphi \Rightarrow \phi) and (\phi \Rightarrow \varphi)$ will give us the truth of $\varphi \Leftrightarrow \phi$

4.a). If the Yan will rise, then the Dollar falls

b). If –y<-x then x<y. (For x,y real numbers)

c). If triangles have the same area, then they are congruent

d). Whenever $b^2 \geq 4a$, then the quadratic equation $ax^2 + bx + c = 0$ has a solution. (where a,b,c,x denote real numbers and $a \neq 0$).

e). Let ABCD be a quadrilateral. If opposite angles of ABCD are pairwise equal, then the opposide sides are pairwise equal.

f). Let ABCD be a quadrilateral. If all four angles of ABCD are equal, then all four sides are equal.

g). If $n^2 + 5$ is divisible by 3,, then n is not divisible by 3. (For n natural number).

7."An integer n is divisible by 12 iff n^3 is divisible by 12". Prove or disprove

This is biconditional. So first part: By Contradiction method. Assume that above statement is false. Then there is an integer n which is not divisible by 12. However, we know that e.g. when n = 12 or 144 etc, n is divisible by 12 and n^3 is divisible by 12. So, contradiction and statement is true.

Only if part: false, counter example when n= 6. If n^3 is divisible by 12 (which is true), but n is not divisible to 12: 6/12.

Conclusion: The above claim is false. Disproved.

Exercise 3.4.1

2. Prove or disprove the claim $(\forall x, y \in \Re)[(x-y)^2 > 0]$.

By contradiction, suppose that ($\neg (\forall x, y \in \Re)[(x-y)^2 > 0]$) = $(\exists x, y \in \Re)[(x-y)^2 \leq 0]$

is true. However, we know that square of any real number is never negative. But when x=y it is true. Thus, negation of the claim is true. Hence, above claim is false.

5. Prove or disprove claim that there are integers m.n s.t. $m^2 + mn + n^2$ is a perfect square.

Perfect square: when m=n=3 and mn = 9; So 9,25,16 are perfect squares.

Assume that $m^2 + mn + n^2$ is not a perfect square. For example if m=0 or n=0, then it will be a perfect square. Contradiction. So initial claim is true.

8. By contradiction, if we assume that there are points inside/ in interior of triangle, then from that point, we can make 3 more triangles. So contradiction, Instead of one triangle we got 3.

Exercise 3.5.2

1. A(n) = the sum of first n odd numbers is equal to n^2.

Method of induction:

1.1A(1) = 1 = 1^2

1.2 A(n+1):

$$
\begin{aligned}
&1+3+5+...+(2n-1) = n^2 \\
&1+3+5+...+(2n-1)+(2n+1) = (n+1)^2 \\
&1+3+5+...+(2n-1)+(2n+1) = n^2 + 2n + 1
\end{aligned}
$$

4.
$$
\begin{aligned}
&1+2+...+n = N \\
&n+(n-1)+...+1 = N \\
&(n+1)+(n+1)+...(n+1) = 2N \\
&N = (n(n+1))/2
\end{aligned}
$$

Ch 4: Proving results about numbers

Exercises 4.1.2

1. Relationship between b|a and a/b

b|a means that a is divisible to b and $b \neq 0$, domain is integers whereas a/b means that a divided to b, domain can be all real numbers.

Exercises 4.1.3

2. Assume that m is odd number so

   $m = 2n+1$ where n is int;

$$m = 2n+1 = 2(\frac{2n}{2})+1 = 4(\frac{n}{2})+1 = 4n+1$$
$$m = 2n+1 = 2(\frac{2n-2}{2}+1)+1 = 4(\frac{n-1}{2})+3 = 4n+3$$

   Conclusion: every odd number can be one of the forms 4n+1 or 4n+3 . Proved.

6. Answer comes from Generalized Division Theorem: Let a,b = int; b≠0. Then there are unique integers q,r such that a = qb+r and $0 \leq r < |b|$

Exercises 4.1.4

3. Original: If p is prime, then whenever p divides product ab, p divides at least one of a,b.

   Converse: If p|ab implies that p|a or p|b, then p is prime

Exercises 4.3.1

1.Let assume that we have two separate intervals:

$$2 < r < 8$$
$$6 < s < 16$$
$$6 < q < 8$$

Then, intersection of these two intervals gives us third new interval:

However, in case of unions, it is not always true. Union of two intervals will give new interval iff their intersection is non-empty or one side of interval is open and another is closed.

Exercises 4.3.2

2. Let's say that a = lub of A. If there exist another b, which is b<a, but it contradicts that a is lub. Hence, least upper bound is unique.

5. S = (a∈Z|a>0), this means that set S has infinite upper bound. Hence, no upper bound.

8. x,y∈(a,b) = (a<x<b) and (a<y<b)

a-b<x-y<b-a and 0≤$\left|x-y\right|$<b-a

$\left|x-y\right|$'s upper bound is b-a

11. a). (∀∈A) [(a≥b) or (a>b)] - means that b is lower bound

b). whenever c>b there is a∈A s.t a<c - means that b is greatest lower bound iff no c>b is lower bound.

-- Iff, c is not lower bound for any c>b

-- Iff, there is a∈A for any c>b s.t a<c

-- Iff there is a∈A for any c>b s.t no the case of a≥c

14. Every non-empty set of real numbers that is bounded above has a lub that is real
Every non-empty set of real numbers that is bounded below has a glb that is also real

Ch4.4 Sequences

Exercise 4.4.1

1. In symbols, a_n not getting closer to a means: lim $a_n = a$ (as $n \to \infty$) iff $(\exists n \in N)(\forall m \geq n)(|a_m - a| < \varepsilon)$

In words, there is n such that for all m greater than or equal to n, the distance from a_m to a is less than ε.

2. Need to prove that $(\forall \varepsilon > 0)(\exists n \in N)(\forall m \geq n)(|(\frac{m}{m+1})^2 - 1| < \varepsilon)$

Let ε>0 be arbitrarily and we need to find n∈N s.t m≥n:

$$|(\frac{m}{m+1})^2 - 1| < \varepsilon$$

Pick n so large that n>1/ε. Then for m≥n

$$|(\frac{m}{m+1})^2 - 1| = |\frac{m^2}{(m+1)^2} - 1| = |\frac{-2m-1}{(m+1)^2}| = \frac{2m+1}{(m+1)^2} < \frac{1}{m} \leq \frac{1}{n} < \varepsilon$$ as required.

3. Need to prove that $(\forall \varepsilon > 0)(\exists n \in N)(\forall m \geq n)(|\frac{1}{m^2}| < \varepsilon)$

Let ε>0 be arbitrarily and we need to find n∈N s.t m≥n:

$$|\frac{1}{m^2}| < \varepsilon$$

Pick n so large that n>1/ε. Then for m≥n

$$\frac{1}{m^2} \le \frac{1}{n} < \varepsilon$$

In other words, $(\forall \varepsilon > 0)(\exists n \in N)(\forall m \ge n)(|\frac{1}{m^2}| < \varepsilon)$ as required.

4. $(\frac{1}{2^n})_{n=1}^{\infty} = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots, \frac{1}{2^n}, \dots$

Need to show that $(\forall \varepsilon > 0)(\exists n \in N)(\forall m \ge n)(|\frac{1}{2^m}| < \varepsilon)$

Let's choose n large enough so that $(|\frac{1}{2^n}| < \varepsilon)$ for arbitrarily ε>0.

So if m≥n, $\frac{1}{2^m} \le \frac{1}{2^n} < \varepsilon$. Thus, $(\forall m \ge n)(\frac{1}{2^m} < \varepsilon)$. Therefore, $(\forall \varepsilon > 0)(\exists n \in N)(\forall m \ge n)(|\frac{1}{2^m}| < \varepsilon)$

5. $|a_n|_{n=1}^{\infty} \lim a_n = \infty$

Need to prove that $(\forall \varepsilon > 0)(\exists n \in N)(\forall m \ge n)(|a_m - \infty| < \varepsilon)$

Let ε>0 be arbitrarily and we need to find n∈N s.t m≥n:

$|a_m - \infty| < \varepsilon$

Pick n so large that n>1/ε. Then for m≥n

If m≥n $|a_m - \infty| = a_m + \infty < a_m \le a_n < \varepsilon$ as required.

For $(n)_{n=1}^{\infty}$ Let m>0 be arbitrarily and set N=M, $a_n = n > M$ whenever n>M. Thus, $(n)_{n=1}^{\infty}$ goes to infinity.

For $(2^n)_{n=1}^{\infty}$ Let m>0 be arbitrarily and set N=M, $a_n = 2^n > M$ whenever n>M. Thus, $(2^n)_{n=1}^{\infty}$ goes to infinity.

6. $(\forall \varepsilon > 0)(\exists n \in N)(\forall m \ge n)(|a_m - a| < \varepsilon)$

Lub condition: $(\forall a \in A)(a \le b) \& (\forall \varepsilon > 0)(\exists a \in A)(a > b - \varepsilon)$

Let a be element of set and b- value of lub.

$asn \to \infty \lim a_n = b$

$(\forall\varepsilon > 0)(\exists n \in N)(\forall m \geq n)(|a_m - b| < \varepsilon)$

Lub condition: $(\forall a \in A)(a \leq b) \,\&\, (\forall\varepsilon > 0)(\exists a \in A)(a > b - \varepsilon)$

With rearrangement of $|a_m - b| < \varepsilon \Rightarrow b - \varepsilon < a_m < b + \varepsilon$

Let $A = (a_n)_{n=1}^{\infty}$, then we can get $(\forall a \in A)(a \leq b) \,\&\, (\forall\varepsilon > 0)(\exists a \in A)(a > b - \varepsilon)$

Hence, as $n \to \infty \lim a_n = a = \mathrm{lub}(a_{1,} a_2, \ldots)_{n=1}^{\infty}$

7. $(a_n)_{n=1}^{\infty}$ is increasing: $a_n < a_{n+1}$ for all n;

Is bounded: $(\forall a \in A)(a \leq b)$ all elements of A is smaller or equal to b. Then we can pick n so large, $|a_m - b| < \varepsilon$ because A is increasing for all m such that m≥n $|a_m - a| < \varepsilon$

Hence, we can make statement: $(\forall\varepsilon > 0)(\exists n \in N)(\forall m \geq n)(|a_m - a| < \varepsilon)$ and say that $n \to \infty \lim a_n = b$ and it goes to a limit.

APPENDIX:

Exercise A1:

3. X is a positive real number which is square is equal to 3.

6.set {1,2,3,4} its subset: {1}, {2}, {3}, {4}, {1,2}, {1,3}, {1,4}, {2,3}, {2,4}, {3,4}, {1,2,3}, {1,2,4},{2,3,4}, {1,2,3,4}, {1,3,4}, {Ø}.

9.n elements has 2^n subsets

Proof by induction:

a). n=1: set A = {1} has one element and two subsets: {1},{Ø}

b).n+1: $2^n = 2^{n+1} = 2^n * 2 = 2^{n+1}$

Exercise A2:

2.Venn diagrams represent set theory in diagram form.

Done by Hyunwoo, Jung

Exercises 2.1.1

1. **How would you show that not every number of the form $N=(p_1 \cdot p_2 \cdot p_3 \cdot \cdots \cdot p_n)+1$ is prime, where $p_1, p_2, p_3, \cdots, p_n, \cdots$ is the list of all prime numbers?**

I can show the counter-examples.

$N=15(2 \cdot 7+1)N$ is not prime in spite of the fact that 2 and 7 is prime.

4. The following notice was posted on the wall of a hospital emergency room:

NO HEAD INJURY IS TOO TRIVIAL TO IGNORE.

Reformulate to avoid the unintended second reading. (The context for this sentence is so strong that many people have difficulty seeing there is an alternative meaning.)

A TRIVIAL HEAD INJURY should not be ignored.

7. Find (and provide citations for) three examples of published sentences whose literal meaning is (clearly) not what the writer intended.

1. Stop the red light

: without context situation, we have to stop when we see the every red light.

2. A killed someone with a gun.

: the first, A has a gun, the second, someone has a gun

3. British left waffles on Falkland Islands.

: the statement “on Falkland Islands” can be connected with 1) left, and 2) waffles.

10. Provide a context and a sentence within that context, where the words and, or, and, or, and occur in that order, with no other word between them.

There are ‘AND. OR.’ and ‘OR. AND’ gate

Exercises 2.2.1

3. Express each of your simplified statements from Question 2 in natural English.

(a) π is greater than 0 and less than 10.

(b) p is greater than or equal to 7 and p is less than 12.

(c) x is greater than 5 and x is less than 7.

(d) x is less than 4

(e) y is greater than -3 and y is less than 3.

(f) x is equal to 0.

6. Is it possible for one of $(\phi \wedge \psi) \wedge \theta$ and $\phi \wedge (\psi \wedge \theta)$ to be true and the other false, or does the associative property hold for conjunction? Prove your answer.

<u>First,</u>

In math, associative property hold for conjunction.

If the first proposition is true, $(\phi \wedge \psi)$and $\theta$is true.

Also, Because $(\phi \wedge \psi)$is true, $\phi$and $\psi$is true.

Consequently, ϕ, $\psi$and $\theta$are true. So, the second proposition is true.

-> associative property holds for the conjunction.

<u>Second</u>

We can check the truth table for the associative property of the conjunction.

ϕ	ψ	θ	$(\phi \wedge \psi) \wedge \theta$	$\phi \wedge (\psi \wedge \theta)$
T	T	T	T	T
T	T	F	F	F
T	F	T	F	F
T	F	F	F	F
F	T	T	F	F
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

Exercises 2.2.2

1. Simplify the following symbolic statements as much as you can, leaving your answer in a standard symbolic form (assuming you are familiar with the notation) :

(a) $(\pi > 3) \vee (\pi > 10)$: $\pi > 3$

(b) $(\pi < 0) \vee (\pi > 0)$: $x \neq 0$

(C) $(x = 0) \vee (x > 0)$: $x \geq 0$

(d) $(x > 0) \vee (x \geq 0)$: $x \geq 0$

(e) $(x > 3) \vee (x^2 > 9)$: $x^2 > 9$

4. What strategy would you adopt to show that the disjunction $\phi_1 \vee \phi_2 \vee \cdots \vee \phi_n$ is false?

All the propositions (ϕ_1, ϕ_2, $\cdots$, ϕ_n) prove to be false.

7. Fill in the entries in the final column of the following truth table :

ϕ	ψ	$\phi \vee \psi$
T	T	T
T	F	T
F	T	T
F	F	F

Exercises 2.2.3

3. Is showing that the negation $\neg\phi$ is true the same as showing that ϕ is false?

In math, This two showing is the same.

The definition of negation is to replace True with false or replace false with true.

6. In US law, a trial verdict of "Not guilty" is given when the prosecution fails to prove guilt. This, of course, does not mean the defendant is, as a matter of actual fact, innocent. Is this state of affairs captured accurately when we use "not" in the mathematical sense? (i.e., Do "Not guilty" and "¬ guilty" mean the same?) What if we change the question to ask if "Not proven" and "¬ proven" mean the same thing?

1. No.

in real life, not guilty means we fail to prove guilt. It did not mean innocent. However, in mathematics, not guilty means innocent

2. Different from (1), the statement about proven are only two : proven or not

proven.

Therefore, we can say that not proven is same meaning with $\neg$proven.

Exercises 2.3.1

2. Provide a justification of your entry.

: The truth of ∅ does not results in the truth of ψ. Therefore, second row is False.

Exercises 2.3.3

1. Which of the following are true and which are false?

(a) $(\pi^2 > 2) \Rightarrow (\pi > 1.4)$: T, (ϕ : T, ψ :T)

(b) $(\pi^2 < 0) \Rightarrow (\pi = 3)$: T, (ϕ : F, ψ :F)

(c) $(\pi^2 > 0) \Rightarrow (1+2=4)$: F, (ϕ : T, ψ :F)

(d) $(\pi < \pi^2) \Rightarrow (\pi = 5)$: F, (ϕ : T, ψ :F)

(e) $(e^2 \geq 0) \Rightarrow (e < 0)$: F, (ϕ : T, ψ :F)

(f) $\neg$(5 is an integer) $\Rightarrow (5^2 \geq 1)$: T, (ϕ : F, ψ :T)

(g) (The area of a circle of radius 1 is π) $\Rightarrow$ (3 is prime) : T, (ϕ : T, ψ :T)

(h) (Squares have three sides) $\Rightarrow$ (Triangles have four sides) : T, (ϕ : F, ψ :F)

(i) (Elephants can climb trees) $\Rightarrow$ (3 is irrational) : T, (ϕ : F, ψ :F)

(j) (Euclid's birthday was July 4) $\Rightarrow$ (Rectangles have four sides) : T, (ϕ : ?, ψ :T)

4. What conclusions can you draw from the above table?

: $(\phi \Rightarrow \psi) \equiv (\neg\phi \vee \psi)$

Exercises 2.3.4

1. Build a truth table to prove the claim I made earlier that $\phi \Leftrightarrow \psi$ is true if ϕ and ψ are both true or both false, and $\phi \Leftrightarrow \psi$ is false if exactly one of ϕ, ψ is true and the other false. (To constitute a proof, your table should have columns that show how the entries for $\phi \Leftrightarrow \psi$ are derived, one operator at a time, as in the previous exercises.)

: $\phi \Leftrightarrow \psi$ means $(\phi \Rightarrow \psi) \wedge (\psi \Rightarrow \phi)$

Also, $(\phi \Rightarrow \psi)$ means $(\phi \wedge \psi) \vee (\neg \phi)$ by definition of implication.

ϕ	ψ	$(\phi \wedge \psi)$	$(\phi \wedge \psi) \vee (\neg \phi)$ $(\phi \Rightarrow \psi)$	$(\phi \wedge \psi) \vee (\neg \psi)$ $(\psi \Rightarrow \phi)$	$(\phi \Rightarrow \psi) \wedge (\psi \Rightarrow \phi)$ $(\phi \Leftrightarrow \psi)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	T	T	F	F
F	F	F	T	T	T

4. The ancient Greeks formulated a basic rule of reasoning for proving mathematical statements. Called MODUS PONENS, it says that if you know ϕ and you know $(\phi \Rightarrow \psi)$, then you can conclude ψ.

(a) Construct a truth table for the logical statement

$$[\phi \wedge (\phi \Rightarrow \psi)] \Rightarrow \psi$$

ϕ	ψ	$(\phi \Rightarrow \psi)$	$\phi \wedge (\phi \Rightarrow \psi)$	$[\phi \wedge (\phi \Rightarrow \psi)] \Rightarrow \psi$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

(b) Explain how the truth table you obtain demonstrates that modus ponens is a valid rule of inference.

: In truth table, If statement A is TRUE, B must also be true for A$\Rightarrow$B to be true.

I find "MODUS PONENS" is a method that If A and A$\Rightarrow$B is true, then B is true.

7. Repeat the above exercise, but interpret 0 as T and 1 as F. What conclusions can you draw?

M	N	$M \times N$	$M + N$

F	F	F	T
F	T	F	F
T	F	F	F
T	T	T	F

Statement of the $M \times N$ equals statement of $M \wedge N$

Statement of the $M + N$ equals statement of $\neg(M \vee N)$

Exercises 2.3.5

2. By a denial of a statement ϕ we mean any statement equivalent to $\neg\phi$. Give a useful denial of each of the following statements.

(a) 34,159 is a prime number.

: 34,159 is not a prime number.

(b) Roses are red and violets are blue.

: Roses are not red or violets are not blue.

(c) If there are no hamburgers, I'll have a hot dog.

: There is no hamburger and I will not have a hot dog.

(A $\Rightarrow$ B) $\Leftrightarrow$ ($\neg$ A $\vee$ B). So, $\neg$ (A $\Rightarrow$ B) $\Leftrightarrow$ (A $\wedge$ $\neg$ B)

(d) Fred will go but he will not play.

: Fred will not go or he will play.

(e) The number x is either negative or greater than 10.

: The number x is not negative and less than or equal to 10.

A denial of "$(x<0) \vee (x>10)$" is $(x \geq 0) \wedge (x \leq 10)$

(f) We will win the first game or the second.

: We will not win the first game and the second.

5. In exercise 3, which conditions are necessary and sufficient for n to be divisible by 6?

: Necessary and sufficient = (Necessary $\wedge$ sufficient) = ({a, e, f} ∩ {c, d, f}) = f

8. Show that $\phi \Leftrightarrow \psi$ is equivalent to $(\neg\phi) \Leftrightarrow (\neg\psi)$. How does this relate to your answers to Questions 6 and 7 above?

: We can prove it through truth table as follow. $\phi \Leftrightarrow \psi$ is equivalent to $(\phi \Rightarrow \psi) \wedge (\phi \Leftarrow \psi)$

And, $(\neg\phi) \Leftrightarrow (\neg\psi)$ is equivalent to $(\neg\phi \Rightarrow \neg\psi) \wedge (\neg\phi \Leftarrow \neg\psi)$

ϕ	ψ	$\phi \Rightarrow \psi$	$\phi \Leftarrow \psi$	$\phi \Leftrightarrow \psi$	$\neg\phi \Rightarrow \neg\psi$	$\neg\phi \Leftarrow \neg\psi$	$(\neg\phi) \Leftrightarrow (\neg\psi)$
T	T	T	T	T	T	T	T
T	F	F	T	F	T	F	F
F	T	T	F	F	F	T	F
F	F	T	T	T	T	T	T

Question 6 proves that "mn is even iff m or n are even."

Question 7 proves that "mn is odd iff m or n are odd."

So, Answer to Questions 6 and 7 relate the proposition of Question 8.

11. Verify the equivalence in (b) and (c) in the previous question by means of a logical argument. (So, in the case of (b), for example, you must show that assuming ϕ and deducing $\psi \wedge \theta$ is the same as both deducing ψ from ϕ and θ from ϕ.)

14. It is important not to confuse the contrapositive of a conditional $\phi \Rightarrow \psi$ with its converse $\psi \Rightarrow \phi$. Use truth tables to show that the contrapositive and the converse of $\phi \Rightarrow \psi$ are not equivalent.

ϕ	ψ	$\psi \Rightarrow \phi$	$(\neg\psi) \Rightarrow (\neg\phi)$
T	**T**	**T**	**T**
T	**F**	**T**	**F**
F	**T**	**F**	**T**
F	**F**	**T**	**T**

17. Express the combinatory ϕ unless ψ in terms of the standard logical combinators.

: $(\neg\psi) \Rightarrow \phi$

20. Let $\forall$ denote the `exclusive or' that corresponds to the English expression “either one or the other but not both". Construct a truth table for this connective.

ϕ	ψ	$\phi \forall \psi$
T	**T**	**F**
T	**F**	**T**
F	**T**	**T**
F	**F**	**F**

23. Give, if possible, an example of a true conditional sentence for which

(a) the converse is true.

: if 2n-1 is prime then n is prime. (True)

converse : if n is prime then 2n-1 is also prime. (True)

(b) the converse is false.

: if $a > 0$, $a^2 > 0$ (True)

converse : if $a^2 > 0$, $a > 0$ (False)

(c) the contrapositive is true.

: Every sentence is valid for this condition.

(d) the contrapositive is false.

: Every sentence is not valid for this condition.

Exercises 2.4.1

1. The same kind of argument I just outlined to show that the cubic equation $y = x^3 + 3x + 1$ has a real root, can be used to prove the “Wobbly Table Theorem.” Suppose you are sitting in a restaurant at a perfectly square table, with four identical legs, one at each corner. Because the floor is uneven, the table wobbles. One solution is to fold a small piece of paper and insert it under one leg until the table is stable. But there is another solution. Simply by rotating the table you will be able to position it so it does not wobble. Prove this. [WARNING : This is a thinking-outside-the-box question. The solution is simple, but it

can take a lot of effort before you find it. This would be an unfair question on a timed exam but is a great puzzle to keep thinking about until you hit upon the right idea.]

Exercises 2.4.2

3. Express the following in symbolic form, using quantifiers for people:

(a) Everybody loves somebody.

: $(\forall x)(\exists y): L(x, y)$, where $L(x, y)$ denotes that x loves y

(b) Everyone is tall or short.

: $(\forall x):\{Tall(x) \vee Short(x)\}$

(c) Everyone is tall or everyone is short.

: $(\forall x): Tall(x) \vee (\forall x): Short(x)$

(d) Nobody is at home.

: $(\forall x): H(x)$, where $H(x)$ denotes $'x'$ isn't at home.

(e) If John comes, all the women will leave.

: $comes(John) \Rightarrow (\forall x)\{women(x) \Rightarrow Leave(x)\}$

(f) If a man comes, all the women will leave.

: $(\exists x)\{Man(x) \wedge Comes(x)\} \Rightarrow (\forall y)\{women(y) \Rightarrow Leave(y)\}$

6. Express the following sentence symbolically, using only quantifiers for real numbers, logical connectives, the order relation <, and the symbol $Q(x)$ having the meaning 'x is rational' :

There is a rational number between any two unequal real numbers.

: $(\forall a \in \mathbb{R})(\forall b \in \mathbb{R}):\{(a < b) \Rightarrow \exists x:\{Q(x) \wedge a < x < b\}\}$

Exercises 2.4.3

1. Show that $\neg[\exists x\, A(x)]$ is equivalent to $\forall x[\neg A(x)]$.

$\neg[\exists x\, A(x)]$: I assume it is not the case that $\exists x\, A(x)$ is true. It is not the case that at least one x satisfies $A(x)$. In other words, All the x must fail to satisfy $A(x)$.

So, $\forall x[\neg A(x)]$ is equivalent to $\neg[\exists x\, A(x)]$.

Exercises 2.4.5

1. Translate the following sentences into symbolic form using quantifiers. In each case the assumed domain is given in parentheses.

(a) All students like pizza. (All people)

: $(\forall x \in People)(Student(x) \Rightarrow Like\ \ pizza(x))$

(b) One of my friends does not have a car. (All people)

: $(\exists x \in People)(One\ of\ \ my\ \ friends(x) \wedge \neg Have\ a\ car(x))$

(c) Some elephants do not like muffins. (All animals)

: $(\exists x \in Animals)(Elephants(x) \wedge \neg Like\ muffins(x))$

(d) Every triangle is isosceles. (All geometric figures)

: $(\forall x \in Geometric\ \ figures)(Triangle(x) \Rightarrow Isosceles(x))$

(e) Some of the students in the class are not here today. (All people)

: $(\exists x \in People)(Students\, in\ the\ class(x) \wedge \neg Today(x))$

(f) Everyone loves somebody. (All people)

: $(\forall x)(\exists y)L(x,y)$ Where $L(x,y)$ denotes " x loves y "

(g) Nobody loves everybody. (All people)

: $\neg(\exists x)(\forall y)L(x,y)$

(h) If a man comes, all the women will leave. (All people)

: $(\exists x)[Man(x) \wedge Comes(x)] \Rightarrow (\forall x)[Woman(x) \Rightarrow Leaves(x)]$

(i) All people are tall or short. (All people)

: $(\forall x):\{Tall(x) \vee Short(x)\}$

(j) All people are tall or all people are short. (All people)

: $(\forall x):Tall(x) \vee (\forall x):Short(x)$

(k) Not all precious stones are beautiful. (All stones)

: $(\exists x):\{Precious(x) \wedge \neg Beautiful(x)\}$

(l) Nobody loves me.

: $(\forall x)\neg Love\ me(x)$

(m) At least one American snake is poisonous. (All snakes)

: $(\exists x):\{American(x) \wedge Poisonous(x)\}$

(n) At least one American snake is poisonous. (All animals)

: $(\exists x):\{American(x) \wedge Snake(x) \wedge Poisonous(x)\}$

4. Negate each of the statements in Question 2, putting your answers in positive form.

(a) $\exists x(x+1<x)$ (Real numbers)

(b) $\forall x(2x+3\neq 5x+1)$ (Natural numbers)

(c) $\forall x(x^2+1\neq 2^x)$ (Real numbers)

(d) $\forall x(x^2\neq 2)$ (Rational numbers)

(e) $\forall x(x^2\neq 2)$ (Real numbers)

(f) $\exists x(x^3+17x^2+6x+100<0)$ (Real numbers)

(g) $\forall x(x^3+x^2+x+1<0)$ (Real numbers)

(h) $\exists x\forall y(x+y\neq 0)$ (Real numbers)

(i) $\forall x\exists y(x+y\neq 0)$ (Real numbers)

(j) $\exists x\forall !y(y\neq x^2)$ (Real numbers)

(k) $\exists x\forall !y(y\neq x^2)$ (Natural numbers)

(l) $\exists x\forall y\exists z(xy\neq xz)$ (Real numbers)

(m) $\exists x\forall y\exists z(xy\neq xz)$ (Prime numbers)

(n) $\exists x\forall y(x\geq 0\wedge y^2\neq x)$ (Real numbers)

(o) $\exists x[(x<0)\wedge\forall y(y^2\neq x)]$ (Real numbers)

(p) $\exists x[(x<0)\wedge\forall y(y^2\neq x)]$ (Positive real numbers)

7. The standard definition of a real function f **being** *continuous at a point* $x=a$ **is**

$$(\forall\varepsilon>0)(\exists\delta>0)(\forall x)[|x-a|<\delta\Rightarrow|f(x)-f(a)|<\varepsilon]$$

Write down a formal definition for f **being** *discontinuous at* a. **Your definition should be in positive form.**

$$(\exists\varepsilon>0)(\forall\delta>0)(\exists x)[|x-a|<\delta\wedge\neg\{|f(x)-f(a)|<\varepsilon\}]$$

Exercises 3.2.1

1. **If not, then for what N is $\sqrt{N}$ irrational? Formulate and prove a result of the form " $\sqrt{N}$ irrational if and only if N..."**

If $\sqrt{N}$ is a rational number, then N is perfect square.

Taking the contrapositive : If N isn't perfect square, $\sqrt{N}$ is a irrational.

Exercises 3.3.2

2. Explain why proving $\phi \Longrightarrow \psi$ and $(\neg\phi) \Rightarrow (\neg\psi)$ establishes the truth of $\phi \Leftrightarrow \psi$.

Proving $(\neg\phi) \Rightarrow (\neg\psi)$ is the same as proving $\psi \Longrightarrow \phi$

By Exercise 3.3.2-1, proving $\phi \Longrightarrow \psi$ and $(\neg\phi) \Rightarrow (\neg\psi)$ establishes the truth of $\phi \Leftrightarrow \psi$.

5. Discounting the first example, which of the statements in the previous exercise are true, for which is the converse ture, and which are equivalent? Prove your answers.

8. If you have not yet solved Exercise 3.3.1(6), have another attempt, using the hint to try $s = \sqrt{2}$.

Exercise 3.3.1(6) : $r^s = r^{\sqrt{2}}$

1) If $r^{\sqrt{2}}$ is rational, we can take r and $\sqrt{2}$ (irrational) and the theorem is proved.

2) If $r^{\sqrt{2}}$ is irrational, we can take $r' = r^{\sqrt{2}}$ and $s = \sqrt{2}$, and then

$$(r')^s = r^{\sqrt{2}^{\sqrt{2}}} = r^2 (r^2 \text{ can be a rational})$$

So, The theorem is proved.

Exercises 3.4.1

3. Prove that between any two unequal rationals there is a third rational.

Any two unequal rationals : $\frac{a}{b}, \frac{c}{d}$

We can take $x = \frac{\left(\frac{a}{b} + \frac{c}{d}\right)}{2}$. x is rationals and $\frac{a}{b} < x < \frac{c}{d}$

6. Prove that for any positive m there is a positive integer n such that $mn+1$ is a perfect square.

$mn + 1 = r^2 (r : positive)$

$mn = r^2 - 1$

$mn = (r+1)(r-1)$

1) $m = r+1$, $n = r-1$; $n = m-2$

2) $m = r-1$, $n = r+1$; $n = m+2$

9. Prove that if every even natural number greater than 2 is a sum of two primes (the Goldbach Conjecture), then every odd natural number greater than 5 is a sum of three primes.

Goldbach Conjecture : $2n = p + q (n > 1)$

$2n+3 = (p+q)+3 (n > 1 \ : 2n+3 > 5)$

Exercises 3.5.2

3. Prove the following by induction:

(a) $4^n - 1$ is divisible by 3.

First, $n = 1$

3 is divisible by 3

Second, Assume $4^n - 1$ is divisible by 3. ($4^n - 1 = 3k$)

$4^{n+1} - 1 \ = 4 \cdot (4^n) - 1 = 4 \cdot (3k+1) - 1 = 3 \cdot (4k+1)$

(b) $(n+1)! > 2^{n+3}$ for all $n \geq 5$.

First, $n = 5$

$6! = 720 > 2^8 = 256$

Second, Assume $(n+1)! > 2^{n+3}$

$\{(n+1)+1\}! \ = (n+2)(n+1)!$

$> (n+2) \cdot 2^{n+3}$ $\qquad \because (n+1)! > 2^{n+3}$

$> 2 \cdot 2^{n+3} = 2^{n+4}$ $\because\ n+2 > 2$

$\therefore\ (n+2)! > 2^{n+4}$

Exercises 4.1.1

1. **The Hilbert Hotel scenario is as before, but this time, two guests arrive at the already full hotel. How can they be accommodated (in separate rooms) without anyone having to be ejected?**

The clerk moves everyone into the next room. So the occupant of Room 1 moves into Room 2, the occupant of Room 2 moves into Room 3, and so on throughout the hotel. And then repeat this one more again. When the clerk has done that, Room 1 and Room 2 are empty. The clerk puts two guests in those rooms. (69p. the last sentences)

Exercises 4.1.2

2. Determine whether each of the following is true or false. Prove your answers.

(a) 0|7

We can't consider the visibility by a zero number.

(b) 9|0

$q = 0$ (True)

(c) 0|0

Likewise (a), this one also doesn't work.

(d) 1|1

$q=1$ (True)

(e) 7|44

7 simply does not divide into 44. (False)

(f) 7|(-42)

$q=-6$ (True)

(g) (-7)|49

$q=-7$ (True)

(h) (-7)|(-56)

$q=8$ (True)

(i) 2708|569401

q does not exist. (False)

(j) $(\forall n \in N)[2n|n^2]$

$q=\frac{n^2}{2n}=\frac{n}{2}$, If n is odd, q does not exist. (False)

(k) $(\forall n \in Z)[2n|n^2]$

Same (j)

(l) $(\forall n \in Z)[1|n]$

$q=n$ (True)

(m) $(\forall n \in N)[n|0]$

$q=0$ (True)

(n) $(\forall n \in Z)[n|0]$

If $n=0$, Likewise (a) (False)

(o) $(\forall n \in N)[n|n]$

$q=1$ (True)

(p) $(\forall n \in Z)\,[n|n]$

If $n=0$, Likewise (a) (False)

Exercises 4.1.3

3. Prove that for any integer n, at least one of the integers $n, n+2, n+4$ is divisible by 3.

1) Let $n=3k$

$n=3k$ is divisible by 3.

2) Let $n=3k+1$

$n+2=3k+3=3(k+1)$ is divisible by 3.

3) Let $n=3k-1$

$n+4=3k+3=3(k+1)$ is divisible by 3.

Exercises 4.1.4

1. Does the following statement accurately define prime numbers? Explain your answer. If the statement does not define the primes, modify it so it does.

$$p \text{ is prime iff } (\forall n \in N)\,[(n|p) \Rightarrow (n=1 \vee n=p)]$$

If $n|p$ is True and $(n=1 \vee n=p)$ is True, $(\forall n \in N)\,[(n|p) \Rightarrow (n=1 \vee n=p)]$ is True

and p is prime.

If $n|p$ is True and $(n=1 \vee n=p)$ is False, $(\forall n \in N)\,[(n|p) \Rightarrow (n=1 \vee n=p)]$ is

False. and p is not prime.

If $n|p$ is False, p is not prime. but, Regardless of the true and lie of

$(n=1 \vee n=p)$, $(\forall n \in N)\,[(n|p) \Rightarrow (n=1 \vee n=p)]$ is True.

So, That statement does not define the primes.

p is prime iff $(\forall n \in N)\,[(n|p) \wedge (n=1 \vee n=p)]$

1	2	3	4	5
T	T	T	T	T
T	F	F	F	F
F	T	T	F	F
F	T	T	F	F

1. $n|p$ 2. $(n=1 \vee n=p)$ 3. $(\forall n \in N)\,[(n|p) \Rightarrow (n=1 \vee n=p)]$

4. p is prime? 5. $(\forall n \in N)\,[(n|p) \wedge (n=1 \vee n=p)]$

Exercises 4.1.5

1. Try to prove Euclid's Lemma. If you do not succeed, move on to the following exercise.

Euclid's Lemma : If p is prime and $p|ab$, then $p|a$ or $p|b$.

1) If $ab=0$, then $a=0$ or $b=0$ So, the theorem is true in this case.

2) $ab \neq 0$, Let $ab=pk=p \cdot (k_1 \cdot k_2 \cdot \cdots \cdot k_n)$ ($k_1 \sim k_n$ are prime)

Because p is prime, a or b have a prime factor p

So, $p|a$ or $p|b$ is true.

Exercises 4.2.1

1. Take the integers, Z as a given system of numbers. You want to define a larger system, Q that extends Z by having a quotient a/b for every pair $a,\ b$ of integers, $b \neq 0$. How would you go about defining such a system? In particular, how would you respond to the question, "What is the quotient a/b?"

(You cannot answer in therms of actual quotients, since until Q has been defined, you don't have quotients.)

Definiton : $(\forall a \in Z)(\forall b \in Z)\left\{(\gcd(a,\ b)=1) \wedge (b \neq 0) \Rightarrow \left(\frac{a}{b}\ \ is\ \ a\ \ rational\ \ number\right)\right\}$

Exercises 4.3.1

2. Taking $\mathbb{R}$ as the universal set, express the following as simply as possible in terms of intervals and unions of intervals. (Note that A' denotes the complement of the set A relative to the given universal set, which in this case is $\mathbb{R}$. See the appendix.)

Appendix : p.89

(a) $[1,\ 3]' = (-\infty,\ 1) \cup (3,\ \infty)$

(b) $(1,\ 7]' = (-\infty,\ 1] \cup (7,\ \infty)$

(c) $(5,\ 8]' = (-\infty,\ 5] \cup (8,\ \infty)$

(d) $(3,\ 7)\cup[6,\ 8]=(3,\ 8]$

(e) $(-\infty,\ 3)'\cup(6,\ \infty)=[3,\ \infty)\cup(6,\ \infty)=[3,\ \infty)$

(f) $\{\pi\}'=(-\infty,\ \pi)\cup(\pi,\ \infty)$

(g) $(1,\ 4]\cap[4,\ 10]=\{4\}$

(h) $(1,\ 2)\cap[2,\ 3)=(1,\ 3)$

(i) A', where $A=(6,\ 8)\cap(7,\ 9]=(7,\ 8)\ \therefore A'=(-\infty,\ 7]\cup[8,\ \infty)$

(j) A', where $A=(-\infty,\ 5]\cup(7,\ \infty)A'=(5,\ 7]$

Exercises 4.3.2

3. Let A be a set of integers, rationals, or reals. Prove that b is the least upper bound of A iff:

(a) $(\forall a\in A)(a\leq b)$; and

(b) Whenever $c<b$ there is an $a\in A$ such that $a>c$.

(a) : b is an upper bound

(b) : for any $c(<b)$, c is not an upper bound.

So, b is the least upper bound of A

6. Show that any finite set of integers/rationals/reals has a least upper bound.

Any finite set of integers/rationals/reals has a maximum value.

So, We can write like this : $(\forall a \in A)(a \leq \max(A))$

$\max(A) = b$ is the least upper bound of set.

9. Define the notion of a $\boldsymbol{lower\ bound}$ **of a set of intergers/rationals/reals.**

$(\forall a \in A)(a \geq b)$: b is lower bound of A

12. State and prove the analog of Question 4 **for greatest lower bounds.**

Let $b + \varepsilon = c$. then, $(\forall \varepsilon > 0) \rightarrow (\forall c > b)$, and $(a < b + \varepsilon) \rightarrow (a < c)$

Then, This problem is equal to problem 11.

Exercises 4.3.3

1. Let $A = \{r \in Q \mid r > 0 \wedge r^2 > 3\}$**. Show that** A **has a lower bound in** Q **but no greatest lower bound in** Q**. Give all details of the proof along the lines of Theorem 4.3.1.**

Let $x = \frac{p}{q} (\in Q)$ be any upper bound of A, where $p, q \in N$

$x^2 < 3$; $p^2 < 3q^2$ Now, as n gets large, the expression

Exercises A1

1. What well-known set is this :

$$\{n \in N \mid (n > 1) \wedge (\forall x, y \in N)[(xy = n) \Rightarrow (x = 1 \vee y = 1)]\}$$

This is Prime number set.

4. Prove that for any set A:

$$\varnothing \subseteq A \text{ and } A \subseteq A$$

1. $\varnothing \subseteq A \Leftrightarrow (x \in \varnothing \Rightarrow x \in A)$

$x \in \varnothing$ is False, So $x \in \varnothing \Rightarrow x \in A$ is True.

2. $A \subseteq A \Leftrightarrow (x \in A \Rightarrow x \in A)$

If $x \in A$ is true, $(x \in A \Rightarrow x \in A)$ is true.

If $x \in A$ is False, $(x \in A \Rightarrow x \in A)$ is true.

7. List all subsets of the set $\{1,\ 2,\ 3,\ \{1,\ 2\}\}$

$\varnothing$,

$\{1\}, \{2\}, \{3\}, \{\{1,\ 2\}\}$,

$\{1,\ 2\}, \{1,\ 3\}, \{1,\ \{1,\ 2\}\}, \{2,\ 3\}, \{2,\ \{1,\ 2\}\}, \{3,\ \{1,\ 2\}\}$,

$\{1,\ 2,\ 3\}, \{1,\ 2,\ \{1,\ 2\}\}, \{1,\ 3,\ \{1,\ 2\}\}, \{2,\ 3,\ \{1,\ 2\}\}$,

$\{1,\ 2,\ 3,\ \{1,\ 2\}\}$

10. Let

$$A = \{o,\ t,\ f,\ s,\ e,\ n\}$$

Give an alternative definition of the set A.

(Hint: this is connected with Λ but is not entirely mathematical.)

Set A include the First letter from 1 to 9.