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| Support recovery with orthogonal matching pursuit in the presence of noise. |

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| Publication: | IEEE transactions on signal processing, vol. 63, no. 21, Nov. 2015. |
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**Short summary**:

OMP is one of greedy algorithms for finding a sparse solution in underdetermined linear simultaneous equations. It is popular by its simple principle.

There are many literatures for the performance analysis of OMP, especially for the sufficient conditions for successful working. Whereas, relatively few literatures deal with necessary conditions, and I haven’t found the literatures which dictate error rate of OMP.

In this paper, Wang provides the sufficient and necessary conditions of OMP. In addition, Wang’s article provides upper bound for recovery error characterized by SNR and magnitudes of the sparse vector. Sufficient conditions have been analyzed in many other papers, therefore, it is not a novel. The necessary condition resembles some empirical results in prior works as well. The sufficient and necessary conditions have a restriction on restricted isometry constant (RIC) of sensing (system) matrix. However, in my opinion, the bound for recovery error seems a completely fresh result since it doesn’t require any restriction on the RIC. The restriction can be an obstacle in adopting the conditions into our practical models, where it is hard to calculate RIC since it requires intractable computations.

Unfortunately, I’ve found defects in the proof of the bound for recovery error. Regardless of whether the author intentionally missed it or not, the corrected bound requires a strong restriction on the RIC even more than the sufficient and necessary conditions. I’ve tried to find a nontrivial bound without any restriction on the RIC based on the proof of Wang’s paper, but I failed. In conclusion, a nontrivial bound without the restriction on RIC cannot be found unless there is a novel alternative tricks in the proof.

In this presentation, I first list the results of Wang’s paper, and then I focus on the proof for the recovery rate and the defects.

# Preliminaries

## Notations

Let . For ,  is the set of all elements contained in  but not in .  represents a restriction of the vector  to the elements with indices in .  is a submatrix of  that only contains columns indexed by . If  is full column rank, then  is the pseudoinverse of .  represents the span of columns in .  is the projection onto the orthogonal complement of , where  denotes the identity matrix.  denotes the th column of .

## Sensing model and restricted isometry property

Consider following noisy CS model,



where, , , , and . Suppose that  is sparse with support  with cardinality .

Wang’s paper presents sufficient and necessary conditions for the recovery of unknown  for given  and . The bounds in conditions are function of RIC, , and , where  and  is the minimum-to-average ratio of the input signal. I therefore introduce the definition of restricted isometry property, which indicates orthornomality of when mapping sparse vectors.

*Definition(Restricted isometry property)*

Consider the model . If there exists a constant  such that, for every -sparse vector ,



Then,  is said to satisfy the RIP of order  with the RIC .

One of major goal in compressed sensing theory is finding a good sensing matrix  with smaller restricted isometry constant(RIC) . Note that computation of  is NP-hard problem, therefore, any analysis having restriction on  would not be adoptable in practical since no one do know exact value of RIC of their sensing matrix unless the intractable computation is done.

## Orthogonal matching pursuit

Now I introduce the pseudo code of OMP algorithm.

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| Table I, The OMP algorithm | |
| **Input** | , , and sparsity . |
| **Initialize** | iteration counter , |
|  | Estimated support , |
|  | and residual vector |
| **While** | **do** |
|  |  |
|  | **Identify** |
|  | **Enlarge** |
|  | **Estimate** |
|  | **Update** |
| **End** |  |
| **Output** | the estimated  and vector |

In short, at each iteration OMP compares correlations between  and , the maximum argument is chosen as one element of estimated support, and then it removes the estimated ingredient from .

# Main results

## Sufficient and necessary conditions

A sufficient condition for the exact recovery with OMP in the presence of noise is presented in Wang’s paper.

*Theorem 3.1 (Sufficient condition)* :

Suppose that the measurement matrix  satisfies the RIP with . Then OMP performs the exact support recovery of any -sparse signal  from its noisy measurements , provided that



Besides, a necessary condition is also presented, and Wang claims that other related or previous works haven’t provided the necessary condition.

*Theorem 3.2 (Necessary condition)* :

If one wishes to accurately recover the support of any -sparse signal  from its noisy measurements  with OMP, then the  should satisfy



In Theorem 3.2, note that the denominator in the right hand side of should be positive to be a nontrivial bound since  is nonnegative. In other words, it should be assumed that . The paper doesn’t refer about this.

Both sufficient and necessary conditions provided in Wang’s paper have restrictions on , therefore, it is hard to exploit the conditions into our practical problems since the computation of RIC is intractable.

## Upper bound for recovery error rate

*Theorem 4.1* :

Let . Then if , OMP recovers the support of -sparse signal  from its noisy measurements  with error rate



where  is a constant.

Theorem 4.1 claims that recovery error rate would be bounded to a proportion of RIC if the bound is nontrivial. Remarks in Wang’s paper claim that  is reasonably small and the bound is reasonable (nontrivial) by explaining based on some examples. In specific, the author claims that the small  is obtained when , , and  is sufficiently large. In addition, the remarks insist the bound is reasonable from the fact that  when  and  is an orthonormal matrix, i. e., . However, I think the remarks are not sufficient to say the bound is a good or nontrivial. Indeed, I numerically evaluate the right hand side of based on the proof, where  is provided in a implicit form, for varying  when , , and . Following Fig. 1 shows the result.



Fig. Numerical evaluations for the upperbound of recovery error rate

This evaluation represents that the bound is nontrivial only approximately for . Otherwise, the bound doesn’t provide any insight since we already know .

# Defects in provided proof

In this section, I intoduce sketch of proof for Theorem 4.1 not to make you understand, but to point out the defects, which result in constaint on  even bitter than the constraint in Theorem 3.1 and Theorem 3.2.

Before proceeding, I introduce notations and definitions used in proof. For notational simplicity, let . At the th iteration of OMP (), let  denote the set of missed detection of support indices. For given constant , let  denote the subset of  corresponding to the  largest elements (in magnitude) of . Also, let  denote the th largest element (in magnitude) in . The author fixes . If , then set  and . Since OMP totally runs  iterations before stopping, the error rate of the support recovery can be given by



The recovery error rate is obtained from lower and upper bounds for energy of the residue vector at last iteration, i. e., bounds for energy of . The lower bound can be simply obtained by



where  uses the fact that  for any , and ,  follows from the RIP,  is because  and , and  is due to the fact  is supported on  and hence .

Here, defect is that  should be nonnegative. The author assumes that  but doesn’t refer. This assumption leads to a restriction on RIC of .

In consequent, lower bound of is trivial. It is not for worth as a meaningful bound, but just for deriving the term . An upper bound for  derived from now on therefore should be meaningful. For this purpose, the author observes the energy behavior in residue vector for every iteration as presented by following proposition.

*Proposition 4.3:*

For any , the residual of OMP satisfies



Proof is omitted here. Proof of Proposition 4.3 provided in Wang’s paper also has same kind of defects in proof of Theorem 4.1. By similar reasons for defect in ,  is assumed.

Now I turn to obtaining the upper bound for . Without loss of generality, we assume that  and that the elements of  are in descending order of their magnitudes. Then from the definition of  we have that for any , ,



By applying Proposition 4.3, the upper bound of energy is obtained by



where  uses the facts that  and that the energy of residual of the OMP algorithm is always non-increasing with the number of iterations (i. e. ), and  is from .

In common with the case in , it should be assumed that , i.e., , which is missed in the statement of Theorem 4.1.

In conclusion, Theorem 4.1 also has a restriction on the RIC, even stricter than the restrictions in sufficient and necessary conditions. I tried to modify the proof to remove the restriction while following the idea of Wang’s proof, but I failed. The key in Wang’s proof for a nontrivial bound is Proposition 4.3. Without employing Proposition 4.3, we can remove the restriction, but the derived bound would be trivial. Meanwhile, Wang fixed , but it is not mandatory since we can tighten the bound and loosen the restriction on the RIC by letting  be a variable of . In specific, when we choose , then it follows that if  and , then



where



and



Evaluation of the right hand side of is presented by following figure,



Fig. Numerical evaluation of the modified bound for error rate