

**Exercises 2.2.1 ~ 4.4.1**  
**Solution Note**

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**Exercises 2.2.1.**

**2. Simplify the following symbolic statements as much as you can, leaving your answer in the standard symbolic form**

(a)  $(\pi > 0) \wedge (\pi < 10)$  [Answer:  $0 < \pi < 10$ ]

(b)  $(p \geq 7) \wedge (p < 12)$  [Answer:  $7 \leq p < 12$ ]

(c)  $(x > 5) \wedge (x < 7)$  [Answer:  $5 < x < 7$ ]

(d)  $(x < 4) \wedge (x < 6)$  [Answer:  $(x < 4)$ ]

(e)  $(y < 4) \wedge (y^2 < 9)$  [Answer:  $(y^2 < 9)$ ]

(f)  $(x \geq 0) \wedge (x \leq 0)$  [Answer:  $x = 0$ ]

**4. What strategy would you adopt to show that the conjunction  $\phi_1 \wedge \phi_2 \wedge \dots \wedge \phi_n$  is true?**

We need to show that  $\forall_i : (\phi_i)$  is true.

**5. What strategy would you adopt to show that the conjunction  $\phi_1 \wedge \phi_2 \wedge \dots \wedge \phi_n$  is false?**

We need to show that  $\exists_i : (\neg \phi_i)$  is true.

**6 Is it possible for one of  $(\phi \wedge \varphi) \wedge \theta$  and  $\phi \wedge (\varphi \wedge \theta)$  to be true and the other false, or does the associative property hold for conjunction? Prove your answer.**

Let us make the truth table of them.

$\phi$	$\varphi$	$\theta$	$(\phi \wedge \varphi) \wedge \theta$	$\phi \wedge (\varphi \wedge \theta)$
T	T	T	T	T
T	T	F	F	F
T	F	T	F	F
T	F	F	F	F
F	T	T	F	F
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

As we can see the above truth table, the associative property holds for conjunction.

**8. In the following table, T denotes ‘true’ and F denotes ‘false’. The first two columns list all the possible combinations of values of T and F that the two statements  $\phi$  and  $\varphi$  can have. The third column should**

give the truth value (T or F)  $\phi \wedge \varphi$  achieves according to each assignment of T or F to  $\phi$  and  $\varphi$ . Fill in the final column. The resulting table is an example of a “propositional truth table”.

$\phi$	$\varphi$	$\phi \wedge \varphi$
T	T	T
T	F	F
F	T	F
F	F	F

**Exercises 2.2.2**

**2. Simplify the following symbolic statements as much as you can, leaving your answer in the standard symbolic form**

(a)  $(\pi > 3) \vee (\pi > 10)$  [Answer:  $(\pi > 3)$ ]

(b)  $(x < 0) \vee (x > 0)$  [Answer:  $x \in \mathbb{R} \setminus \{0\}$ ]

(c)  $(x = 0) \vee (x > 0)$  [Answer:  $x \geq 0$ ]

(d)  $(x > 0) \vee (x \geq 0)$  [Answer:  $(x \geq 0)$ ]

(e)  $(x > 3) \vee (x^2 > 9)$  [Answer:  $(x^2 > 9)$ ]

**4. What strategy would you adopt to show that the conjunction  $\phi_1 \vee \phi_2 \vee \dots \vee \phi_n$  is true?**

We need to show that  $\exists_i : (\phi_i)$  is true.

**5. What strategy would you adopt to show that the conjunction  $\phi_1 \vee \phi_2 \vee \dots \vee \phi_n$  is false?**

We need to show that  $\forall_i : (\neg \phi_i)$  is true.

**6 Is it possible for one of  $(\phi \vee \varphi) \vee \theta$  and  $\phi \vee (\varphi \vee \theta)$  to be true and the other false, or does the associative property hold for conjunction? Prove your answer.**

Let us make the truth table of them.

$\phi$	$\varphi$	$\theta$	$(\phi \vee \varphi) \vee \theta$	$\phi \vee (\varphi \vee \theta)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	T	T
F	F	T	T	T
F	F	F	F	F

As we can see the above truth table, the associative property holds for conjunction.

**8. Fill in the entries in the final column of the following truth table:**

$\phi$	$\varphi$	$\phi \vee \varphi$
T	T	T
T	F	T
F	T	T
F	F	F

### Exercises 2.2.3

1. Simplify the following symbolic statements as much as you can, leaving your answer in a standard symbolic form (assuming you are familiar with the notation):

(a)  $\neg(\pi > 3.2)$  Answer:  $\pi \leq 3.2$

(b)  $\neg(x < 0)$  Answer:  $x \geq 0$

(c)  $\neg(x^2 > 0)$  Answer:  $x^2 \leq 0$

(d)  $\neg(x = 1)$  Answer:  $x \neq 1$

(e)  $\neg\neg\psi$  Answer:  $\psi$

3. Is showing that the negation  $\neg\phi$  is true the same as showing that  $\phi$  is false?

Yes. If  $\phi$  is false,  $\neg\phi$  is true.

4. Fill in the entries in the final column of the following truth table:

$F, T$ .

5. (a)  $D \wedge Y$

(b)  $\neg D \rightarrow \neg T$

(c)  $T \wedge \neg D \wedge Y$

(d)  $T \wedge \neg Y$

(e)  $T \wedge \neg Y \wedge D$

(f)  $(D \wedge \neg Y) \vee (\neg D \wedge Y)$

(g)  $T \wedge \neg(D \wedge Y)$

(h)  $T \wedge (\neg D \wedge \neg Y)$

(i)  $D \wedge Y \wedge \neg T$

(j)  $T \wedge ((\neg D \wedge S) \vee (D \wedge \neg S))$

6. Yes, the inverse of “guilty” is the same with “Not guilty”. Likewise, “Not proven” is the same with “ $\neg$  Proven”.

### Exercise 2.3.1.

1. F

2. As shown in the table, the subsequent statement  $\psi$  is false. From the true premise, we have no way to get truth in the implication.

### Exercise 2.3.2

1. T, T

2. The premise is already false, thus it does not matter if the subsequent statement is true and the premise implies both truth and false of subsequent statement.

### Exercises 2.3.3

1. Which of the following are true and which are false?

(a)  $(\pi^2 > 2) \Rightarrow (\pi > 1.4) \Leftrightarrow T \Rightarrow T: T$

(b)  $(\pi^2 < 0) \Rightarrow (\pi = 3) \Leftrightarrow F \Rightarrow F: T$

(c)  $(\pi^2 > 0) \Rightarrow (1+2=4) \Leftrightarrow T \Rightarrow F: F$

(d)  $(\pi < \pi^2) \Rightarrow (\pi = 5) \Leftrightarrow T \Rightarrow F: F$

(e)  $(e^2 \geq 0) \Rightarrow (e < 0) \Leftrightarrow T \Rightarrow F: F$

(f)  $\neg(5 \text{ is an integer}) \Rightarrow (5^2 \geq 1) \Leftrightarrow F \Rightarrow T: T$

(g)  $(\text{The area of a circle of radius 1 is } \pi) \Rightarrow (3 \text{ is prime}) \Leftrightarrow T \Rightarrow T: T$

(h)  $(\text{Squares have three sides}) \Rightarrow (\text{Triangles have four sides}) \Leftrightarrow F \Rightarrow F: T$

(i)  $(\text{Elephants can climb trees}) \Rightarrow (3 \text{ is irrational}) \Leftrightarrow F \Rightarrow T: T$

(j)  $(\text{Euclid's birthday was July 4}) \Rightarrow (\text{Rectangles have four sides}) \Leftrightarrow F \Rightarrow T: T$

**Exercises 2.3.4.**

**5. Mod-2 arithmetic has just the two numbers 0 and 1 follows the usual rules of arithmetic together with the additional rule  $1+1=0$ . (It is the arithmetic that takes place in a single bit location in a digital computer.) Complete the following table:**

$M$	$N$	$M \times N$	$M + N$
1	1	?	?
1	0	?	?
0	1	?	?
0	0	?	?

Answer) the product rule does not change. But, the addition rule changes. Hence, the above table becomes

$M$	$N$	$M \times N$	$M + N$
1	1	1	0
1	0	0	1
0	1	0	1
0	0	0	0

**6. In the table you obtained in the above exercise, interpret 1 as T and 0 as F and view  $M$ ,  $N$  as statements.**

We have

$M$	$N$	$M \times N$	$M + N$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	F

(a) Which of the logical combinators  $\wedge$ ,  $\vee$  corresponds to  $\times$  ?

Answer)  $\wedge$  is corresponding to  $\times$ .

(b) Which of logical combinators corresponds to  $+$  ?

Answer)  $\otimes$  (exclusive or) is corresponding to  $+$ .

(c) Does  $\neg$  correspond to  $-$  (minus)?

Answer) Typically,  $-$  is one of binary operations while  $\neg$  is one of unitary operations. Thus,  $\neg$  does not correspond to  $-$ .

**7. Repeat the above exercise, but interpret 0 as T and 1 as F. What conclusions can you draw?**

We have

$M$	$N$	$M \times N$	$M + N$
F	F	F	T
F	T	T	F
T	F	T	F
T	T	T	T

(a) Which of the logical combinators  $\wedge$ ,  $\vee$  corresponds to  $\times$  ?

Answer)  $\vee$  is corresponding to  $\times$ .

(b) Which of logical combinators corresponds to  $+$  ?

Answer)  $\neg \otimes$  (not exclusive or) is corresponding to  $+$ .

(c) Does  $\neg$  correspond to  $-$  (minus)?

Answer) Typically,  $-$  is one of binary operations while  $\neg$  is one of unitary operations. Thus,  $\neg$  does not correspond to  $-$ .

**8. The following puzzle was introduced by the psychologist Peter Wason in 1966, and is one of the most famous subject tests in the psychology of reasoning. Most people get it wrong. (So you have been warned!)**

Four cards are placed on the table in front of you. You are told (truthfully) that each has a letter printed on one side and a digit on the other, but of course you can only see one face of each. What you see is:

B E 4 7



You are now told that the cards you are looking at were chosen to follow the rule “If there is a vowel on one side, then there is an odd number on the other side.” What is the least number of cards you have to turn over to verify this rule, and which cards do you in fact have to turn over?

Answer) The rule can be expressed by using the implication operator. Namely, we have

$$p \Rightarrow q$$

where  $p$  is “there is a vowel on one side” and  $q$  is there is an odd number on the other side.

The question is to find the least number of cards to make  $p \Rightarrow q$  true. By using the contrapositive proposition of  $p \Rightarrow q$  is “if there is an even number on the side, then there is not a vowel on the other side”. Thus, to verify the rule, we need to turn over two card such as E and 4.

**Exercises 2.3.5.**

**6. Let  $m, n$  denote any two natural numbers. Prove that  $mn$  is odd if and only if  $m$  and  $n$  are odd**

Answer) First, we need to prove that “if  $m$  and  $n$  are odd, then  $mn$  is odd”. Let us suppose that  $m = 2K + 1$  and  $n = 2L + 1$ , where  $K = 0, 1, \dots$  and  $L = 0, 1, \dots$ . Then, we have  $mn = 4KL + 2K + 2L + 1$ . It becomes  $mn = 2(2KL + K + L) + 1$ . Thus,  $mn$  is odd.

Second, we need to prove that “if  $mn$  is odd, then  $m$  and  $n$  are odd.” We aim to prove this by using a contrapositive proof. Thus, instead of proving “if  $mn$  is odd, then  $m$  and  $n$  are odd”, we need to prove “if  $m$  or  $n$  are even, then  $mn$  is even”. Let us suppose that  $m = 2K + 1$  and  $n = 2(L + 1)$ . Then, we have  $mn = 2(2K + 1)(L + 1)$ . Thus,  $mn$  is even. Also,  $mn$  is even when  $m = 2(L + 1)$  and  $n = 2K + 1$ . Last,  $mn$  is again even when  $m = 2(K + 1)$  and  $n = 2(L + 1)$ .

Thus, the proposition such that  $mn$  is odd if and only if  $m$  and  $n$  are odd is true.

**7. With reference to the previous questions, is it true that  $mn$  is even iff  $m$  and  $n$  are even?**

Answer) False, here is one example,  $mn = 6$  but  $m = 3$  and  $n = 2$ . Thus, the propositions such that if  $mn$  is even, then  $m$  and  $n$  are even is false.

**8. Show that  $\phi \Leftrightarrow \varphi$  is equivalent to  $(\neg\phi) \Leftrightarrow (\neg\varphi)$ . How does this related to your answers to Questions 6 and 7 above?**

Answer) By using a truth table, then we can easily prove that  $\phi \Leftrightarrow \varphi$  is equivalent to  $(\neg\phi) \Leftrightarrow (\neg\varphi)$ .

$\phi$	$\varphi$	$\phi \Leftrightarrow \varphi$	$(\neg\phi) \Leftrightarrow (\neg\varphi)$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	T	T

To prove the proposition such that “if  $mn$  is odd, then  $m$  and  $n$  are odd”, we used the contrapositive

proposition of it. Namely,  $\phi \Rightarrow \varphi$  is equivalent to  $\neg\varphi \Rightarrow \neg\phi$ .

**9. Construct truth tables to illustrate the following:**

(a)  $\phi \Leftrightarrow \varphi$

$\phi$	$\varphi$	$\phi \Leftrightarrow \varphi \equiv (\phi \Rightarrow \varphi) \wedge (\phi \Leftarrow \varphi)$
T	T	T
T	F	F
F	T	F
F	F	T

(b)  $\phi \Rightarrow (\varphi \vee \theta)$

$\phi$	$\varphi$	$\theta$	$\phi \Rightarrow (\varphi \vee \theta)$
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	T

**10. Use truth tables to prove that the following are equivalent**

(a)  $\neg(\phi \Rightarrow \varphi)$  and  $\phi \wedge (\neg\varphi)$

$\phi$	$\varphi$	$\neg(\phi \Rightarrow \varphi)$	$\phi \wedge (\neg\varphi)$
T	T	F	F
T	F	T	T

F	T	F	F
F	F	F	F

Thus,  $\neg(\phi \Rightarrow \varphi)$  is equivalent to  $\phi \wedge (\neg\varphi)$ .

(b)  $\phi \Rightarrow (\varphi \vee \theta)$  and  $(\phi \Rightarrow \varphi) \wedge (\phi \Rightarrow \theta)$

$\phi$	$\varphi$	$\theta$	$\phi \Rightarrow (\varphi \vee \theta)$	$(\phi \Rightarrow \varphi) \wedge (\phi \Rightarrow \theta)$
T	T	T	T	T
T	T	F	T	F
T	F	T	T	F
T	F	F	F	F
F	T	T	T	T
F	T	F	T	T
F	F	T	T	T
F	F	F	T	T

Thus,  $\phi \Rightarrow (\varphi \vee \theta)$  is not equivalent to  $(\phi \Rightarrow \varphi) \wedge (\phi \Rightarrow \theta)$ .

(c)  $(\phi \vee \varphi) \Rightarrow \theta$  and  $(\phi \Rightarrow \theta) \wedge (\varphi \Rightarrow \theta)$

$\phi$	$\varphi$	$\theta$	$(\phi \vee \varphi) \Rightarrow \theta$	$(\phi \Rightarrow \theta) \wedge (\varphi \Rightarrow \theta)$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	F	F
F	T	T	T	T
F	T	F	F	F
F	F	T	T	T

F	F	F	T	T
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Thus,  $(\phi \vee \varphi) \Rightarrow \theta$  is equivalent to  $(\phi \Rightarrow \theta) \wedge (\varphi \Rightarrow \theta)$ .

**11. Verify the equivalences in (b) and (c) in the previous question by means of a logical arguments.**

(b)  $\phi \Rightarrow (\varphi \wedge \theta)$  and  $(\phi \Rightarrow \varphi) \wedge (\phi \Rightarrow \theta)$

$\phi \Rightarrow (\varphi \wedge \theta)$  because  $(\phi \Rightarrow \varphi) \wedge (\phi \Rightarrow \theta)$  by following table:

$\phi$	$\varphi$	$\theta$	$(\phi \Rightarrow \varphi)$	$(\phi \Rightarrow \theta)$	$(\phi \Rightarrow \varphi) \wedge (\phi \Rightarrow \theta)$	$\phi \Rightarrow (\varphi \wedge \theta)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	T	F	F
T	F	F	F	F	F	F
F	T	T	T	T	T	T
F	T	F	T	T	T	T
F	F	T	T	F	F	F
F	F	F	T	T	T	T

(c)  $(\phi \vee \varphi) \Rightarrow \theta$  and  $(\phi \Rightarrow \theta) \wedge (\varphi \Rightarrow \theta)$

$(\phi \vee \varphi) \Rightarrow \theta$  because  $(\phi \Rightarrow \theta) \wedge (\varphi \Rightarrow \theta)$  by following table:

$\phi$	$\varphi$	$\theta$	$(\phi \Rightarrow \theta)$	$(\varphi \Rightarrow \theta)$	$(\phi \Rightarrow \theta) \wedge (\varphi \Rightarrow \theta)$	$(\phi \vee \varphi) \Rightarrow \theta$
T	T	T	T	T	T	T
T	T	F	F	F	F	F
T	F	T	T	F	F	F
T	F	F	F	F	F	F
F	T	T	T	T	T	T

F	T	F	T	T	T	T
F	F	T	T	F	T	T
F	F	F	T	T	T	T

12. Use truth tables to prove the equivalence of  $(\phi \Rightarrow \varphi)$  and  $(\neg\varphi) \Rightarrow (\neg\phi)$

$\phi$	$\varphi$	$\theta$	$(\phi \Rightarrow \varphi)$	$(\neg\varphi) \Rightarrow (\neg\phi)$
T	T	T	T	T
T	T	F	T	T
T	F	T	F	F
T	F	F	F	F
F	T	T	T	T
F	T	F	T	T
F	F	T	T	T
F	F	F	T	T

Thus,  $(\phi \Rightarrow \varphi)$  and  $(\neg\varphi) \Rightarrow (\neg\phi)$  are equivalence.

13. Write down the contrapositives of the following statements:

(a) If two rectangles are congruent, they have the same area.

: If two rectangles have different area, they are not congruent.

(b) If triangle with sides  $a, b, c$  ( $c$  largest) is right-angled, then  $a^2 + b^2 = c^2$ .

: If sides  $a, b, c$  ( $c$  largest) of triangle satisfy  $a^2 + b^2 \neq c^2$ , the triangle is not right-angled.

(c) If  $2^n - 1$  is prime, then  $n$  is prime.

: If  $n$  is not prime,  $2^n - 1$  is not prime.

(d) If the Yuan rises, the Dollar will fall.

: If the Dollar rises, the Yuan will fall.

14. It is important not to confuse the contrapositive of a conditional  $(\phi \Rightarrow \varphi)$  with its converse  $(\varphi \Rightarrow \phi)$ .

Use truth tables to show that the contrapositive and the converse of  $(\phi \Rightarrow \varphi)$  are not equivalent.

$\phi$	$\varphi$	$\theta$	$(\phi \Rightarrow \varphi)$	$(\varphi \Rightarrow \phi)$	$(\neg\varphi) \Rightarrow (\neg\phi)$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	F	T	F
F	T	T	T	F	T
F	T	F	T	F	T
F	F	T	T	T	T
F	F	F	T	T	T

Thus, by the table, the contrapositive and the converse of  $(\phi \Rightarrow \varphi)$  are not equivalent.

**15. Write down the converses of the four statements in Question 13.**

(a) If two rectangles are congruent, they have the same area.

: If two rectangles have the same area, they are congruent.

(b) If triangle with sides  $a, b, c$  ( $c$  largest) is right-angled, then  $a^2 + b^2 = c^2$ .

: If sides  $a, b, c$  ( $c$  largest) of triangle satisfy  $a^2 + b^2 = c^2$ , the triangle is right-angled.

(c) If  $2^n - 1$  is prime, then  $n$  is prime.

: If  $n$  is prime,  $2^n - 1$  is prime.

(d) If the Yuan rises, the Dollar will fall.

: If the Dollar falls, the Yuan will rise.

**16. Show that for any two statements  $\phi$  and  $\psi$  either  $\phi \Rightarrow \psi$  or its converse is true (or both). This is another reminder that the conditional is not the same as implication.**

**17. Express the combinator “ $\phi$  unless  $\psi$ ” in terms of the standard logical combinators.**

**18. Identify the antecedent and the consequent**

(a)

Antecedent: the apples are red.

Consequent: they are ready to eat.

(b)

Antecedent:  $f$  is differentiable.

Consequent:  $f$  is continuous

(c)

Antecedent:  $f$  is integral

Consequent:  $f$  is bounded

(d)

Antecedent:  $S$  is convergent

Consequent:  $S$  is bounded

(e)

Antecedent:  $2^n - 1$  is prime

Consequent:  $n$  is prime

(f)

Antecedent: Karl is playing

Consequent: The team wins

(g)

Antecedent: Karl plays

Consequent: Team wins

(h)

Antecedent: Karl is playing

Consequent: Team wins

**21. Express  $a \dot{\vee} b$  in terms of  $\wedge, \vee, \neg$**

a	b	$a \dot{\vee} b$
T	T	F
T	F	T
F	T	T
F	F	F

$$a \dot{\vee} b = (a \vee b) \wedge \neg(a \wedge b)$$

**22. Which are equivalent?**

a)  $\neg(P \vee Q), \neg P \wedge \neg Q$ . True

b)  $\neg P \vee \neg Q, \neg(P \vee \neg Q)$  False

c)  $\neg(P \wedge Q), \neg P \vee \neg Q$  True

d)  $\neg(P \Rightarrow (Q \wedge R)), \neg(P \Rightarrow Q) \vee \neg(P \Rightarrow R)$  True

$$\begin{aligned} \neg(P \Rightarrow (Q \wedge R)) &= \neg((P \Rightarrow Q) \wedge (P \Rightarrow R)) \\ &= \neg(P \Rightarrow Q) \vee \neg(P \Rightarrow R) \end{aligned}$$

e)  $P \Rightarrow (Q \Rightarrow R), (P \wedge Q) \Rightarrow R$  True



P	Q	R	$Q \Rightarrow R$	$P \Rightarrow (Q \Rightarrow R)$	$(P \wedge Q)$	$(P \wedge Q) \Rightarrow R$
T	T	T	T	T	T	T
T	T	F	F	F	T	F
T	F	T	T	T	F	T
F	T	T	T	T	F	T
F	T	F	F	T	F	T
F	F	T	T	T	F	T
F	F	F	T	T	F	T

**23.**

a)  $P \Rightarrow Q$  If you catch a cold then you have fever.

$Q \Rightarrow P$  (True) If you have fever then you catch a cold

b)  $P \Rightarrow Q$  If you under drinking age then you can't drink alcohol.

$Q \Rightarrow P$  (False) If you can't drink alcohol you are under drinking age.

c)  $P \Rightarrow Q$  If you catch a cold then you have fever.

$\tilde{Q} \Rightarrow \tilde{P}$  (True) If you did not catch a cold then you do not have fever

d) -

**24. in charge of a party**

Let's use short notation for: Alcohol=A, Legal drinkers = L

First column filed from statement "One person has beer, another has coke". Second column "One is under the drinking age, the other is above"

Old enough to drink alcohol means  $A \Rightarrow L$  it false when you are under the age and drinks alcohol and true otherwise (it false when A is true and L is false). So we need to check row 1 and 4 and make sure that  $A \Rightarrow L$  is not false.

	A	L	$A \Rightarrow L$
1	T		
2	F		
3		T	
4		F	

**25. Compare with Wason's problem.**

In Wason's problem we need to check 3 cards (E,7). Because rule "If there vowel on one side, then there is an odd number on the other side." The rule could be false if card E has even number or card 7 has a vowel. We need to make sure both cards follows rule. We do not care about B and 4 cards. Rule does not include any information about consonants (card B). Card 4 can be vowel or consonant if it vowel then it follows our rule if it is consonant then rule does not cover consonant case.

I personally spent more time thinking on Wason's problem. Basically two problems are same but previous problem has more familiar situation.

## Exercises 2.4.2

### 1. Express the following as existence assertions.

(a) The equation  $x^3 = 27$  has a natural number solution.

$$: (\exists x \in \mathcal{N})(x^3 = 27)$$

(b) 1,000,000 is not the largest natural number.

$$: (\exists x \in \mathcal{N})(x > 1,000,000)$$

(c) The natural number  $n$  is not a prime.

$$: (\exists n \in \mathcal{N})(n \text{ is not a prime})$$

### 2. Express the following as 'for all' assertions

(a) The equation  $x^3 = 28$  does not have a natural number solution.

$$: (\forall x \in \mathcal{N})(x^3 \neq 28)$$

(b) 0 is less than every natural number.

$$: (\forall x \in \mathcal{N})(x > 0)$$

(c) the natural number  $n$  is a prime.

$$: (\forall \text{prime } n)(n \text{ is a natural number})$$

### 3. Express the following in symbolic form, using quantifiers for people:

(a) Everybody loves somebody.

$$: (\forall x \in \text{Peoples})(\exists y \in \text{Peoples}, x \neq y)(x \text{ loves } y)$$

(b) Everyone is tall or short

$$: (\forall x \in \text{Peoples})(x \text{ is tall or short})$$

(c) Everyone is tall or everyone is short

$$(\forall x \in \text{Peoples})(x \text{ is tall}) \vee (\forall x \in \text{Peoples})(x \text{ is short})$$

(d) Nobody is at home

$$: (\forall x \in \text{Peoples})(x \text{ is not at home})$$

(e) If John comes, all the women will leave.

: (John comes)  $\Rightarrow$  ( $\forall w \in \text{woman}$ ) ( $w$  will leave)

(f) If a man comes, all the women will leave.

: ( $\forall m \in \text{man}$ ) ( $m$  comes)  $\Rightarrow$  ( $\forall w \in \text{woman}$ ) ( $w$  will leave)

**4. Express the following using quantifiers that refer (only) to the sets  $R$  and  $N$ .**

(a) The equation  $x^2 + a^2 = 0$  has a real root for any real number  $a$ .

: ( $\exists a \in R$ ) ( $\exists x \in R$ ) ( $x^2 + a^2 = 0$ )

(b) The equation  $x^2 + a^2 = 0$  has a real root for any negative real number  $a$ .

: ( $\exists a \in R, a < 0$ ) ( $\exists x \in R$ ) ( $x^2 + a^2 = 0$ )

(c) Every real number is rational.

: ( $\forall x \in R$ ) ( $x$  is rational)

(d) There is an irrational number.

: ( $\exists x \in R$ ) ( $x$  is irrational)

(e) There is no largest irrational number.

: ( $\forall \text{irrational } a$ ) ( $\exists \text{irrational } x$ ) ( $x > a$ )

**5.  $C =$  set of all cars,  $D(x) = x$  is domestic,  $M(x) = x$  is badly made**

(a) All the domestic cars are badly made

( $\forall x \in C$ ) [ $D(x) \Rightarrow M(x)$ ]

(b) All foreign cars are badly made

( $\forall x \in C$ ) [ $M(x) \in C$ ]

(c) All badly made cars are domestic

( $\forall x \in C$ ) [ $M(x) \Rightarrow D(x)$ ]

(d) There is a domestic car that is not badly made

( $\forall x \in C$ ) [ $\exists x \notin C$ ]

(e) There is a foreign car that is badly made

( $\forall x \in C$ ) [ $\exists x \in C$ ]

**6. There is a rational number between any two unequal real numbers.**

$$(\exists x \in \mathbb{R})(\exists y \in \mathbb{R})x < y \wedge (\forall Q \in \mathbb{Q})Q \leq y$$

7. “You may fool all the people some of the time, you can even fool some of the people all of the time, but you cannot fool all of the people all the time.”

Let’s break this into three

Let  $P$  be the set of all people, let  $T$  be the set of all times, and let  $F(p, t)$  be true if it is possible to fool person  $p$  at time  $t$ .

You may fool all the people some of the time

$$(\exists t \in T)(\forall p \in P)F(p, t)$$

You can even fool some of the people all of the time

$$(\exists p \in P)(\forall t \in T)F(p, t)$$

You cannot fool all of the people all the time

$$(\forall p \in P)(\forall t \in T)\neg F(p, t)$$

Now combining whole statement “*You may fool all the people some of the time, you can even fool some of the people all of the time, but you cannot fool all of the people all the time.*”

$$((\exists t \in T)(\forall p \in P)F(p, t)) \wedge ((\exists p \in P)(\forall t \in T)F(p, t)) \wedge ((\forall p \in P)(\forall t \in T)\neg F(p, t))$$

8. “A driver is involved in an accident every six seconds.” Let  $x$  be a variable to denote a drive,  $t$  a variable for a six-second interval, and  $A(x, t)$  the property that  $x$  is in an accident during interval  $t$ . Express the headline in logical notation.

$$\exists x[A(x, t)]$$

### Exercise 2.4.5

1.

- a)  $(\forall x \in P)[S(x) \Rightarrow \text{Like}P(x)]$
- b)  $(\exists x \in P)[F(x) \wedge \neg C(x)]$
- c)  $(\exists x \in A)[E(x) \wedge \neg M(x)]$
- d)  $(\forall x \in F)[T(x) \Rightarrow I(x)]$
- e)  $(\exists x \in P)[S(x) \wedge \neg H(x)]$
- f)  $(\forall x \in P)(\exists y \in P)[L(x, y)]$
- g)  $(\forall x \in P)(\exists y \in P)[\neg L(x, y)]$
- h)  $(\exists x \in P)(\forall x \in P)[MC(x) \Rightarrow WC(y)]$
- i)  $(\forall x \in P)[T(x) \dot{\vee} S(x)]$
- j)  $(\forall x \in P)[T(x)] \dot{\vee} (\forall x \in P)[S(x)]$
- k)  $(\exists x \in P)[P(x) \wedge \neg B(x)]$
- l)  $(\forall x \in P)(\exists y \in P)[\neg L(x, y)]$
- m)  $(\exists x \in P)[A(x) \wedge P(x)]$
- n)  $(\exists x \in P)[A(x) \wedge S(x) \wedge P(x)]$

3.

- a)  $\neg((\forall x \in P)[S(x) \Rightarrow \text{Like}P(x)]) = (\exists x \in P)[S(x) \wedge \neg \text{Like}P(x)]$
- b)  $\neg((\exists x \in P)[F(x) \wedge \neg C(x)]) = (\forall x \in P)[F(x) \Rightarrow C(x)]$
- c)  $\neg((\exists x \in A)[E(x) \wedge \neg M(x)]) = (\forall x \in A)[E(x) \Rightarrow M(x)]$
- d)  $\neg((\forall x \in F)[T(x) \Rightarrow I(x)]) = (\exists x \in F)[T(x) \wedge \neg I(x)]$
- e)  $\neg((\exists x \in P)[S(x) \wedge \neg H(x)]) = (\forall x \in P)[S(x) \wedge \neg H(x)]$
- f)  $\neg((\forall x \in P)(\exists y \in P)[L(x, y)]) = (\exists x \in P)(\forall y \in P)[\neg L(x, y)]$
- g)  $\neg((\forall x \in P)(\exists y \in P)[\neg L(x, y)]) = (\exists x \in P)(\forall y \in P)[L(x, y)]$
- h)  $\neg((\exists x \in P)(\forall x \in P)[MC(x) \Rightarrow WC(y)]) = (\forall x \in P)(\exists x \in P)[MC(x) \wedge \neg WC(y)]$
- i)  $\neg((\forall x \in P)[T(x) \dot{\vee} S(x)]) = (\exists x \in P)[(T(x) \vee S(x)) \wedge \neg(T(x) \vee S(x))]$
- j)  $\neg((\forall x \in P)[T(x)] \dot{\vee} (\forall x \in P)[S(x)]) = (\forall x \in P)[T(x)] \wedge \neg(\forall x \in P)[S(x)] = (\forall x \in P)[T(x)] \wedge (\exists x \in P)[\neg S(x)]$
- k)  $\neg((\exists x \in P)[P(x) \wedge \neg B(x)]) = (\forall x \in P)[P(x) \Rightarrow B(x)]$
- l)  $\neg((\forall x \in P)(\exists y \in P)[\neg L(x, y)]) = (\exists x \in P)(y \in P)[L(x, y)]$
- m)  $\neg((\exists x \in P)[A(x) \wedge P(x)]) = (\forall x \in P)[A(x) \vee \neg P(x)]$
- n)  $\neg((\exists x \in P)[A(x) \wedge S(x) \wedge P(x)]) = (\forall x \in P)[A(x) \vee S(x) \vee \neg P(x)]$

2.

a)  $\forall x(x+1 \geq x)$  (Real numbers)

True

b)  $\exists x(2x+3 = 5x+1)$  (Natural numbers)

False  $x=2/3$

c)  $\exists x(x^2 + 1 = 2^x)$  (Real numbers)

True  $x=0$

d)  $\exists x(x^2 = 2)$  (Rational numbers)

True since all integer are rational  $x=1$

e)  $\exists x(x^2 = 2)$  (Real numbers) True

f)  $\forall x(x^3 + 17x^2 + 6x + 100 \geq 0)$  (Real numbers)

False because it is negative for large negative x

g)  $\exists x(x^3 + x^2 + x + 1 \geq 0)$  (Real numbers)

True  $x=0$

h)  $\forall x \exists y(x + y = 0)$  (Real numbers)

True. For all x there is exist y such that  $x=-y$

i)  $\exists x \forall y(x + y = 0)$  (Real numbers)

False. There is no one x for all y

j)  $\forall x \exists! y(y = x^2)$  (Real numbers)

True. For all x there is unique (only one) y. because  $x^2$  is positive

k)  $\forall x \exists! y(y = x^2)$  (Natural numbers)

True. For all x there is unique (only one) y

l)  $\forall x \exists y \forall z(xy = xz)$  (Real numbers)

False.

m)  $\forall x \exists y \forall z(xy = xz)$  (Prime numbers)

False.

n)  $\forall x \exists y(x \geq 0 \Rightarrow y^2 = x)$  (Real numbers)

True.  $y=\sqrt{x}$

o)  $\forall x(x < 0 \Rightarrow \exists y(y^2 = x))$  (Real numbers)

False. ( $x < 0 \Rightarrow \exists y(y^2 = x)$ ) If  $x=-2$  then y should be  $\sqrt{-2}$  but it is not a real number therefore  $T \Rightarrow F=F$

p)  $\forall x(x < 0 \Rightarrow \exists y(y^2 = x))$  (Positive real numbers)

True. because for positive real numbers there no  $x < 0$

so antecedant equal is false therefore conditional is always true.

4.

a)  $\forall x(x+1 \geq x)$  (Real numbers)

$\exists x(x+1 \leq x)$

b)  $\exists x(2x+3 = 5x+1)$  (Natural numbers)

$\forall x(2x+3 \neq 5x+1)$

c)  $\exists x(x^2+1 = 2^x)$  (Real numbers)

$\forall x(x^2+1 \neq 2^x)$

d)  $\exists x(x^2 = 2)$  (Rational numbers)

$\forall x(x^2 \neq 2)$

e)  $\exists x(x^2 = 2)$  (Real numbers)

$\forall x(x^2 \neq 2)$

f)  $\forall x(x^3 + 17x^2 + 6x + 100 \geq 0)$  (Real numbers)

$\exists x(x^3 + 17x^2 + 6x + 100 \leq 0)$

g)  $\exists x(x^3 + x^2 + x + 1 \geq 0)$  (Real numbers)

$\forall x(x^3 + x^2 + x + 1 \leq 0)$

h)  $\forall x \exists y(x + y = 0)$  (Real numbers)

$\exists x \forall y(x + y \neq 0)$

i)  $\exists x \forall y(x + y = 0)$  (Real numbers)

$\forall x \exists y(x + y \neq 0)$

j)  $\forall x \exists! y(y = x^2)$  (Real numbers)

$\exists x \forall y(y \neq x^2)$

k)  $\forall x \exists! y(y = x^2)$  (Natural numbers)

$\exists x \forall y(y \neq x^2)$

l)  $\forall x \exists y \forall z(xy = xz)$  (Real numbers)

$\exists x \forall y \exists z(xy \neq xz)$

m)  $\forall x \exists y \forall z(xy = xz)$  (Prime numbers)

$\exists x \forall y \exists z(xy \neq xz)$

n)  $\forall x \exists y(x \geq 0 \Rightarrow y^2 = x)$  (Real numbers)

$\exists x \forall y(x \geq 0 \wedge y^2 \neq x)$

o)  $\forall x(x < 0 \Rightarrow \exists y(y^2 = x))$  (Real numbers)

$\forall x(x < 0 \wedge \forall y(y^2 \neq x))$

p)  $\forall x(x < 0 \Rightarrow \exists y(y^2 = x))$  (Positive real numbers)

$\forall x(x < 0 \wedge \forall y(y^2 \neq x))$

Exercises 3.3.2

1. Explain why proving  $\phi \Rightarrow \varphi$  and  $\varphi \Rightarrow \phi$  establishes the truth of  $\phi \Leftrightarrow \varphi$ .

$\Rightarrow$  Obviously, if both  $\phi$  and  $\varphi$  are true, then  $\phi \Rightarrow \varphi$  and  $\varphi \Rightarrow \phi$  is true.

$\Rightarrow$  Also, if both  $\phi$  and  $\varphi$  are false, then  $\phi \Rightarrow \varphi$  and  $\varphi \Rightarrow \phi$  is true.

$\Rightarrow$  Thus, proving  $\phi \Rightarrow \varphi$  and  $\varphi \Rightarrow \phi$  establish the truth of  $\phi \Leftrightarrow \varphi$ .

2. Explain why proving  $\phi \Rightarrow \varphi$  and  $(\neg\varphi) \Rightarrow (\neg\phi)$  establishes the truth of  $\phi \Leftrightarrow \varphi$ .

$\Rightarrow$  Obviously, if both  $\phi$  and  $\varphi$  are true, then  $\phi \Rightarrow \varphi$  and  $(\neg\varphi) \Rightarrow (\neg\phi)$  is true.

$\Rightarrow$  Also, if both  $\phi$  and  $\varphi$  are false, then  $\phi \Rightarrow \varphi$  and  $(\neg\varphi) \Rightarrow (\neg\phi)$  is true.

$\Rightarrow$  In addition, if  $\phi$  is false and  $\varphi$  is true, then  $\phi \Rightarrow \varphi$  and  $(\neg\varphi) \Rightarrow (\neg\phi)$  is true

$\Rightarrow$  Thus, proving  $\phi \Rightarrow \varphi$  and  $(\neg\varphi) \Rightarrow (\neg\phi)$  cannot establish the truth of  $\phi \Leftrightarrow \varphi$ .

3. Prove that if five investors split a payout of \$2 million, at least one investor receives at least \$400,000.

$\Rightarrow$  Let us assume that the five investors fairly divide \$2 million. Then, each investor receives \$400,000. Now, let us assume that the five investors unfairly divide \$2 million. Namely, at least one investor receives at most \$400,000. Then, one investor always receives at least \$400,000. Thus, the proposition is true.

4. Write down the converse of the following condition statements:

(a) If the Dollar falls, the Yuan will rise

$\Rightarrow$  If the Yuan will rise, the Dollar falls

(b) If  $x < y$  then  $-y < -x$ . (For  $x, y$  real numbers.)

$\Rightarrow$  If  $-y < -x$ , then  $x < y$ .

(c) If two triangles are congruent, they have the same area.

$\Rightarrow$  If two triangles have the same area, then they are congruent.

(d) The quadratic equation  $ax^2 + bx + c = 0$  has a solution whenever  $b^2 \geq 4a$ . (Where  $a, b, c, x$  denote real number and  $a \neq 0$ .)

$\Rightarrow$   $b^2 \geq 4a$  whenever the quadratic equation  $ax^2 + bx + c = 0$  has a solution.

(e) Let ABCD be a quadrilateral. If the opposite sides of ABCD are pairwise equal, then the



**opposite angles are pairwise equal.**

⇒ If the opposite angles of ABCD are pairwise equal, then the opposite sides are pairwise equal.

**(f) Let ABCD be a quadrilateral. If all four sides of ABCD are pairwise equal, then all four angles are equal.**

⇒ If all four angles of ABCD are equal, then all four sides of ABCD are pairwise equal.

**(g) If  $n$  is not divisible by 3, then  $n^2 + 5$  is divisible by 3. (For  $n$  a natural number.)**

⇒ If  $n^2 + 5$  is divisible by 3, then  $n$  is not divisible by 3.

Exercises 3.4.1

1. Prove or disprove the statement “All birds can fly.”

All birds can fly

$B(x)$  :  $x$  is a bird

$F(x)$  :  $x$  can fly

$\exists x B(x) \rightarrow F(x)$

Not all birds can fly

$\exists x B(x) \rightarrow \neg F(x)$

2. Prove that between any two unequal rationales there is a third rational.

Let  $a$  and  $b$  be any two rational numbers with  $a < b$ .

To show that  $a < c < b$  proceeds as follows:

$a < b$  (given)

$a + a < b + a$

$2a < a + b$

$a < (a + b) / 2$

Similarly,

$a < b$  (given)

$a + b < b + b = 2b$

$(a + b) / 2 < b$

Therefore,

$c = (a + b) / 2$  is a rational number satisfying

$a < c < b$

3. Assume that there exist two rational numbers,  $n$  and  $m$  that do not have any rational numbers between them, where  $m > n$ .

Then  $m + n > n + n$

$$\frac{(m+n)}{2} > \frac{(n+n)}{2}$$

$$\frac{(m+n)}{2} > n$$

4. Say whether each of the following is true or false, and support your decision by a proof:

(a) There exist real numbers  $x$  and  $y$  such that  $x + y = y$

There is an  $x$  and  $y$  such that  $x + y = y$ . These mean the same thing. Order is not important with multiple “there is” phrases. This is Not true for mixtures of “there is” and “for all” phrases.

**(b) The sum of any five consecutive integers is divisible by 5 (without remainder).**

Let the five consecutive integers be  $x, x+1, x+2, x+3, x+4$ , where  $x$  can be any integers

So, sum of five consecutive integers is:

$$(x) + (x+1) + (x+2) + (x+3) + (x+4)$$

$$= 5x + 10$$

$$= 5(x+2)$$

Since the sum of any integers is  $5(x+2)$  which is a multiple of 5. So, the sum is always divisible by 5.

**(c) For any integer  $n$ , the number  $n^2 + n + 1$  is odd.**

Let  $n$  be an odd integer.

Then  $n = 2k + 1$  for some integer  $k$ .

Then  $n^2 = (2k + 1)^2$

And  $n^2 + n + 1 = (2k + 1)^2 + (2k + 1) + 1$

$$= 4k^2 + 4k + 2k + 3$$

$$= 4k^2 + 6k + 3$$

$$= 2(2k^2 + 3k) + 3$$

Therefore,  $n^2 + n + 1$  is odd.

**(d) Between any two distinct rational numbers there is a third rational number.**

**Given:** Any two rational numbers  $r$  and  $s$ , with  $r < s$ .

Prove: There exists a rational  $r < x < s$ .

From definition of rational, let  $r = a/b$ , and  $s = c/d$  for integers  $a, b, c, d$ , with  $b \neq 0$  and  $d \neq 0$ .

Re-express the fractions  $r$  and  $s$  over the same denominator  $2bd$ :

$$r = 2da/2bd \text{ and } s = 2bc/2bd.$$

Since  $r < s$ ,  $2da < 2bc$ .

Then  $(2da + 2bc)/2 = da + bc$  is the odd integer that lies halfway between the integers  $2da$  and  $2bc$ .

Hence  $(da + bc)/2bd$  is a rational number that lies between  $r$  and  $s$ .

**(e) For any real numbers  $x, y$ , if  $x$  is irrational and  $y$  is rational, then  $x + y$  is irrational.**

If  $x + y$  were rational, then it could always be written as  $\frac{(a+b)}{c}$  with  $a, b$  and  $c$  as integers when  $a+b$  is relatively prime to  $c$ .

$\frac{(a+b)}{c}$  (can be written as  $\frac{a}{c} + \frac{a}{b}$ , but this would be the sum of two rational numbers since both are written as

the ratio of two integers, thus  $x + y$  where  $x$  is irrational and  $y$  is rational is an irrational number.

**6. Let  $m$  and  $n$  be integers. Prove that:**

**(a) If  $m$  and  $n$  are even, then  $m+n$  is even.**

We start by assuming  $m=2u$ ,  $n=2v$  for  $u$  and  $v$  are positive integers. Then,  $m+n = 2u+2v = 2(u+v)$ . Here, since  $u+v$  is also a positive integer,  $m+n$  is multiple number of 2 and thus that is even.

**(b) If  $m$  and  $n$  are even, then  $mn$  is divisible by 4.**

We start by assuming  $m=2u$ ,  $n=2v$  for  $u$  and  $v$  are positive integers. Then,  $mn = (2u)(2v) = 4(uv)$ . Here, since  $uv$  is also a positive integer,  $mn$  is multiple number of 4 and thus that is divisible by 4.

**(c) If  $m$  and  $n$  are odd, then  $m+n$  is even.**

We start by assuming  $m=2u-1$ ,  $n=2v-1$  for  $u$  and  $v$  are positive integers. Then,  $m+n = (2u-1)+(2v-1) = 2(u+v-1)$ . Here, since  $(u+v-1)$  is also a positive integer,  $m+n$  is multiple number of 2 and thus that is even.

**(d) If one of  $m, n$  is even and the other is odd, then  $m+n$  is odd.**

We start by assuming  $m=2u$ ,  $n=2v-1$  for  $u$  and  $v$  are positive integers. Then,  $m+n = (2u)+(2v-1) = 2(u+v)-1$ . Here, since  $(u+v)$  is also a positive integer, the remainder of  $m+n$  by dividing with 2 is 1 and thus that is odd.

**(e) If one of  $m, n$  is even and the other is odd, then  $mn$  is even.**

We start by assuming  $m=2u$ ,  $n=2v-1$  for  $u$  and  $v$  are positive integers. Then,  $mn = (2u)(2v-1) = 2(2uv-u)$ . Here, since  $(2uv-u)$  is also a positive integer,  $mn$  is multiple number of 2 and thus that is even.

**7. Prove or disprove the statement “An integer  $n$  is divisible by 12 if and only if  $n^3$  is divisible by 12.”**

Disprove) By counter example, for  $n^3 = 12 \cdot 18 = 216$ ,  $n = 6$  and 6 cannot be divided by 12.

**8. If you have not yet solved Exercise 3.3.1(6), have another attempt, using the hint to try  $s = \sqrt{2}$ .**

$r^{\sqrt{2}}$ .

For  $s = \sqrt{2}$ ,  $s = \underbrace{1}_{\text{Rational}} + \underbrace{(\sqrt{2}-1)}_{\text{Irrational}}$  and  $r^s = r^{1+(\sqrt{2}-1)} = r \cdot r^{(\sqrt{2}-1)}$ . Here, we note  $r$  is irrational and the product of

two numbers whose one of them is irrational is irrational. Thus,  $r^s$  is irrational.

**Exercises 4.1.3**

**1. Prove all the parts of Theorem 4.1.3.**

Let  $a, b, c, d$  be integers,  $a \neq 0$ . Then,

(i)  $a|0, a|a$

$\Rightarrow a|0 \Leftrightarrow 0 = a \times 0 + 0$  and  $a|a \Leftrightarrow a = a \times 1 + 0$ .

(ii)  $a|1$  if and only if  $a = \pm 1$ ;

$\Rightarrow a|1 \Leftrightarrow 1 = a \times b + 0$ . Hence,  $ab = 1$  that implies  $a = \frac{1}{b}$ . Since  $a, b$  are integers,  $a = 1, b = 1$  or

$a = -1, b = -1$ . Hence,  $a = \pm 1$ .

$\Rightarrow a = \pm 1$ . Then,  $1 = 1 \times 1 + 0$  and  $1 = (-1) \times (-1) + 0$ .

(iii) if  $a|b$  and  $c|d$ , then  $ac|bd$  for  $c \neq 0$

$\Rightarrow a|b \Leftrightarrow b = a \times q_1$  and  $c|d \Leftrightarrow d = c \times q_2$ . Then,  $bd = ac \times (q_1 q_2) = ac \times q_3 \Leftrightarrow ac|bd$

(iv) if  $a|b$  and  $b|c$ , then  $a|c$  for  $b \neq 0$

$\Rightarrow a|b \Leftrightarrow b = a \times q_1$  and  $b|c \Leftrightarrow c = b \times q_2$ . Then,  $c = b \times q_2 = a \times q_1 \times q_2 = a \times q_3 \Leftrightarrow a|c$

(v)  $[a|b$  and  $b|a]$  if and only if  $a = \pm b$

$\Rightarrow$  First, we aim to show if  $[a|b$  and  $b|a]$ , then  $a = \pm b$ .  $a|b \Leftrightarrow b = a \times q_1$  and  $b|a \Leftrightarrow a = b \times q_2$ . Then, we have  $b = a \times q_1 = b \times q_2 \times q_1$  that leads to  $q_2 q_1 = 1$ . Namely,  $q_1 = q_2 = 1$  or  $q_1 = q_2 = -1$  that lead to  $a = \pm b$ .

Second, we aim to show if  $a = \pm b$ , then  $[a|b$  and  $b|a]$ . It is true.

(vi) if  $a|b$  and  $b \neq 0$ , then  $|a| \leq |b|$

$\Rightarrow a|b \Leftrightarrow b = a \times q_1$ , where  $q_1$  is integer with  $|q_1| \geq 1$ . It leads to  $|a| \leq |b|$

(vii) if  $a|b$  and  $a|c$ , then  $a|(bx + cy)$  for any integers  $x, y$

$\Rightarrow a|b \Leftrightarrow b = a \times q_1$  and  $a|c \Leftrightarrow c = a \times q_2$ . Then, we have  $bx + cy = a \times q_1 x + a \times q_2 y = a(q_1 x + q_2 y)$  that leads to  $a|(bx + cy)$

**2. Prove that every odd number is of one of the forms  $4n+1$  or  $4n+3$ .**

Let  $p$  be an odd number. Then,  $p = 2K+1$ , where  $K = 0, 1, \dots, L, \dots$ . Obviously, we have

$$p = 2K+1 = 2\left(\frac{2K}{2}\right)+1 = 4\left(\frac{K}{2}\right)+1 = 4K_1+1$$

or

$$p = 2K+1 = 2\left(\frac{2K-2}{2}+1\right)+1 = 4\left(\frac{K-1}{2}\right)+3 = 4K_2+3.$$

Thus, the proposition “every odd number is of one of the forms  $4n+1$  or  $4n+3$ ” is true.

**3. Prove that for any integer  $n$ , at least one of the integers  $n$ ,  $n+2$ ,  $n+4$  is divisible by 3.**

$\Rightarrow$  Let  $n = 3K$ , then  $n$  is divisible by 3. Let  $n = 3K+1$ , then  $n+2 = 3(K+1)$  is divisible by 3. Let  $n = 3K+2$ , then  $n+4 = 3K+6 = 3(K+2)$  is divisible by 3. Therefore, the propositions “for any integer  $n$ , at least one of the integers  $n$ ,  $n+2$ ,  $n+4$  is divisible by 3” is true.

**4. Prove that if  $a$  is an odd integer, then  $24 \mid a(a^2-1)$ .**

$\Rightarrow$  Let  $a$  be  $2K+1$ , where  $K = 0, 1, \dots, L, \dots$ . Then, we have

$$a(a^2-1) = (2K+1)(2K)(2K+2) = 4K(K+1)(2K+1).$$

In addition,  $24 = 4 \times 6$ . Thus, we have

$$\frac{a(a^2-1)}{24} = \frac{4K(K+1)(2K+1)}{4 \times 6} = \frac{K(K+1)(2K+1)}{6}.$$

Obviously,  $K(K+1)$  is always divisible by 2. Next, it is easy to show that at least one of the integers  $K$ ,  $K+1$ ,  $2K+1$  is divisible by 3. Thus,  $K(K+1)(2K+1)$  must be divisible by 6. Therefore, the proposition “if  $a$  is an odd integer, then  $24 \mid a(a^2-1)$ ” is true.

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**Exercise 4.3.2**

**1. Prove that if a set  $A$  of integers/rationals/reals has an upper bound, then it has infinitely many different upper bounds.**

Let us  $a$  denote an element of  $A$ ,  $a \in A$ ,  $b$  denote maximum element of  $A$ ,  $b \neq a$ .

Then,  $a < b$  and  $\exists c, \{c \in \mathbb{Z} \text{ or } \mathbb{Q} \text{ or } \mathbb{R} \mid b < c\}$ . Thus, there exist many different upper bounds.

**2. Prove that if a set  $A$  of integers/rationals/reals has a least upper bound, then it is unique.**

Let say  $u$  is a least upper bound of the set  $A$ .

If there exist another upper bound  $x$ ,  $x < u$ ,  $b < x$ . However, it contradicts that  $u$  is a least upper bound.

Thus, the least upper bound is unique.

**3. Let  $A$  be a set of integers, rationals, or reals. Prove that  $b$  is the least upper bound of  $A$  iff:**

(a)  $(\forall a \in A)(a \leq b)$ ; and

(b) whenever  $c < b$  there is an  $a \in A$  such that  $a > c$ .

If the least upper bound is  $c$ ,  $c < b$ . However,  $a > c$  by (b). It contradicts that  $c$  is a least upper bound.

Thus, the least upper bound is  $b$ .

**4. The following variant of the above characterization is often found. Show that  $b$  is the lub of  $A$  iff:**

(a)  $(\forall a \in A)(a \leq b)$ ; and

(b)  $(\forall \varepsilon > 0)(\exists a \in A)(a > b - \varepsilon)$

If the least upper bound is  $b - \varepsilon$ ,  $b - \varepsilon < b$ . However,  $\forall \varepsilon > 0 \exists a \in A (a > b - \varepsilon)$  by (b). It implies that  $b - \varepsilon$  is not the least upper bound. Thus, this contradicts that  $b - \varepsilon$  is a least upper bound.

Thus, the least upper bound is  $b$ .

**5. Give an example of a set of integers that has no upper bound.**

$A = \{x \in \mathbb{Z} \mid x > 0\}$ . The set  $A$  has not upper bound since the maximum of elements in  $A$  is infinity.

**6. Show that any finite set of integers/rationals/reals has a least upper bound.**

Suppose  $A \subseteq \mathfrak{R}$  is nonempty and bounded from above. Among all possible upper bounds for A, we would call L a least upper bound for A if L is an upper bound for A with the additional property that  $L \leq N$  for every upper bound N. that is, among all upper bounds for A, L is no bigger than any of them. The LUB axiom says that every nonempty subset of  $\mathfrak{R}$  that is bounded from above has a least upper bound of  $\mathfrak{R}$ .

**7. What is  $\text{lub}(a,b)$ ? what is  $\text{lub}[a,b]$ ? what is  $\text{max}(a,b)$ ? what is  $\text{max}[a,b]$ ?**

$$\text{lub}(a,b) = \{x \in \mathfrak{R} : a < x < b\}$$

$$\text{lub}[a,b] = \{x \in \mathfrak{R} : a \leq x \leq b\}$$

$$\text{max}(a,b) = \{x \in \mathfrak{R} : a < x < b\}$$

$$\text{max}[a,b] = \{x \in \mathfrak{R} : a \leq x \leq b\}$$

**9. Define the notion of a lower bound of a set of integers/rationales/reals.**

The term lower bound is defined dually as an element of K which is less than or equal to every element of S. A set with a lower bound is said to be bounded from below by that bound.

Ex:- 2 and 5 are both lower bounds for the set  $\{5,10,34,13934\}$  but 8 is not.

Every subset of the natural numbers has a lower bound, since the natural numbers have at least element (0, or 1 depending on the exact definition of natural numbers).

**10. Define the notion of a greatest lower bound (glb) of a set of integers/rationales'/real's by analogy with our original definition of lub.**

A number  $G \in \mathfrak{R}$  is said to be greatest lower bound (GLB) for a set A if G has the following properties:

(G1) for every  $a \in A$ , we have that  $a \geq G$ , and

(G2) If N is any lower bound of A, it must be that  $N \leq G$

**11. Let A be a set of integers, rationals or reals. Prove that b is the greatest lower bound of A if and only if.**

a)  $(\forall a \in A)(a \leq b)$

b) whenever  $c < b$  there is an a belongs to A such that  $a > c$

b is a greatest lower bound of A

iff there is c such that  $c > b$  is not lower bound.

Iff for any  $c > b$  there is an a such that  $a < c$



**12. The following variant of the above characterization is often found. Show that  $b$  is the greatest lower bound of  $A$  iff.**

$$a) (\forall a \in A)(a \leq b);$$

$$b) (\forall \varepsilon > 0)(\exists a \in A)(a > b - \varepsilon)$$

$b$  is the greatest lower bound

$b$  is a glb iff there is no  $a$  such that  $a < b$

iff for any  $c > b$  there is  $a < c$

**13. Show that the Completeness Property for the real number system could equally well have been defined by the statement, "Any nonempty set of reals that has a lower bound has a greatest lower bound."**

Let's form a new set  $A'$  by negating all the elements of  $A$ . Since this set is a non-empty set of real numbers, it must be complete. Hence  $A'$  has a lub that is also real. All negative upper bounds of  $A$  becomes the lower bounds of  $A'$ , and the negative lub of  $A$  becomes the glb of  $A'$ .

**14. The integers satisfy the completeness property, but for a trivial reason. What is that reason?**

Every non-empty set of real numbers that is bounded above has a lub that is also real

Every non-empty set of real numbers that is bounded below has a glb that is also real

**Exercises 4.4.1.**

**1. Formulate both in symbols and in words what it means to say that  $a_n$  not  $\rightarrow a$  as  $n \rightarrow \infty$**

In symbols  $a_n$  not  $\rightarrow a$  as  $n \rightarrow \infty$  is simply  $\lim_{n \rightarrow \infty} a_n = a$  iff  $(\exists n \in \mathbb{N})(\forall m \geq n)(|a_m - a| < \varepsilon)$ .

This says that there is an  $n$  such that for all  $m$  greater than or equal to  $n$ , the distance from  $a_m$  to  $a$  is less than  $\varepsilon$ .

**2. Prove that  $(n/(n+1))^2 \rightarrow 1$  as  $n \rightarrow \infty$**

$\varepsilon > 0$ , we seek on  $N$  such that

$$n \geq N \Rightarrow \left| \left( \frac{n}{n+1} \right)^2 - 1 \right| < \varepsilon,$$

$$n \geq N \Rightarrow \left| \left( \frac{n^2 - n^2 - 2n - 1}{(n+1)^2} \right)^2 - 1 \right| < \varepsilon$$

$$n \geq N \Rightarrow \frac{2n+1}{(n+1)^2} < \varepsilon$$

if we pick big  $N$ , then  $\frac{(N+1)^2}{2N+1} > \frac{1}{\varepsilon}$ .

$$\text{Then } n \geq N \Rightarrow \frac{2n+1}{(n+1)^2} \leq \frac{2N+1}{(N+1)^2} < \varepsilon$$

**3. Prove that  $\{n/(n+1)\} \rightarrow 1$  as  $n \rightarrow \infty$**

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots$$

Let  $\varepsilon > 0$ . We must find an  $n \in \mathbb{N}$  such that  $m \geq n$

$$\left| \frac{m}{m+1} - 1 \right| < \varepsilon$$

Pick  $n$  so large that  $n > 1/\varepsilon$ . Then, for  $m \geq n$ ,

$$\left| \frac{m}{m+1} - 1 \right| = \left| \frac{-1}{m+1} \right| = \frac{1}{m+1} < \frac{1}{m} \leq \frac{1}{n} < \varepsilon$$

as required.

**4. Prove that  $1/2^n \rightarrow 0$  as  $n \rightarrow \infty$ .**

$$\left\{\frac{1}{2^n}\right\}_{n=1}^{\infty} = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots, \frac{1}{1024}, \dots, \frac{1}{2^n}, \dots$$

We should show that  $(\forall \varepsilon > 0)(\exists n \in \mathbb{N})(\forall m \geq n)\left(\left|\frac{1}{2^m}\right| < \varepsilon\right)$ .

Let choose  $n$  large enough so that  $\frac{1}{2^n} < \varepsilon$  for arbitrary  $\varepsilon > 0$ . If now  $m \geq n$ ,  $\frac{1}{2^m} \leq \frac{1}{2^n} < \varepsilon$ .

In other words,  $(\forall m \geq n)\left(\frac{1}{2^m} < \varepsilon\right)$ . Thus,  $(\forall \varepsilon > 0)(\exists n \in \mathbb{N})(\forall m \geq n)\left(\left|\frac{1}{2^m}\right| < \varepsilon\right)$ .

**5. We say a sequence  $\{a_n\}_{n=1}^{\infty}$  tends to infinity if, as  $n$  increases,  $a_n$  increases without bound. For instance, the sequence  $\{n\}_{n=1}^{\infty}$  tends to infinity, as does the sequence  $\{2^n\}_{n=1}^{\infty}$ . Formulate a precise definition of this notion, and prove that both of these examples fulfill the definition.**

A precise definition of this notion:

$$a_n \rightarrow \infty \text{ as } n \rightarrow \infty \text{ if and only if } (\forall \varepsilon > 0)(\exists n \in \mathbb{N})(\forall m \geq n)(|a_m| \geq \varepsilon)$$

Let us consider the sequence  $\{n\}_{n=1}^{\infty}$ . We aim to prove  $(\forall \varepsilon > 0)(\exists n \in \mathbb{N})(\forall m \geq n)(m \geq \varepsilon)$ .

Let  $\varepsilon > 0$  be any value. We have to find an  $n$  such that  $m \geq n \Rightarrow m \geq \varepsilon$ . Pick  $n$  large enough so that  $n \geq \varepsilon$ . For any  $m \geq n$ ,  $m \geq n \geq \varepsilon$  is required.

Let us consider the sequence  $\{2^n\}_{n=1}^{\infty}$ . We aim to prove  $(\forall \varepsilon > 0)(\exists n \in \mathbb{N})(\forall m \geq n)(2^m \geq \varepsilon)$ . Similar to the first sequence, we have to find an  $n$  such that  $m \geq n \Rightarrow 2^m \geq \varepsilon$ . Pick  $n$  large enough so that  $n \geq \log_2(\varepsilon)$ .

For any  $m \geq n$ ,  $2^m \geq 2^n \geq 2^{\log_2(\varepsilon)} = \varepsilon$  is required.