Exercise solutions of the Mathematical Thinking.

Study group 2 : From lecture 1 to lecture 10

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<Exercises 2.1.1>

1. How would you show that not every number of the form $N = (p_1 \cdot p_2 \cdot p_3 \cdot \ldots \cdot p_n) + 1$ is prime, where $p_1, p_2, p_3, \ldots, p_n, \ldots$ is the list of all prime numbers?

=> Proof 1) If N is not divisible by any number of our finite set of primes, there are two possibilities:

1. N is itself a prime number.

2. N is divisible by a prime number greater than p_n

If the second possibility is true, then every number of the form N is not prime.

The second possibility follows from the Fundamental Theorem of Arithmetic, which states that any integer can be expressed as a product of primes. Since N is not divisible by any prime less than p_n, it must be divisible by a prime that is greater than p_n. This means the N is a composite number.

=> Proof 2) Show the counter examples

$n = 1 \Rightarrow N = 2 + 1 = 3$

$n = 2 \Rightarrow N = 2*3 + 1 = 7$

$n = 3 \Rightarrow N = 2*3*5 + 1 = 31$

$n = 4 \Rightarrow N = 2*3*5*7 + 1 = 211$

$n = 5 \Rightarrow N = 2*3*5*7*11 + 1 = 2311$

$n = 6 \Rightarrow N = 2*3*5*7*11*13 + 1 = 30031 = 59*509$: not prime number.

2. Find two unambiguous (but natural sounding) sentences equivalent to the sentence : "The man saw the woman with a telescope", the first where the man has the telescope, the second where the woman has the telescope.

=> 1) Where the man has the telescope : The man with a telescope saw the woman.

2) Where the woman has the telescope : The man saw the woman who had got a telescope.

*** Other example of this sentence.**

1) The lady hit the man with an umbrella.

2) He gave her cat good.

3. For each of the four ambiguous newspaper headlines I stated earlier, rewrite it in a way the avoids the amusing second meaning, while retaining the brevity of a typical headline:

(a) Sisters reunited after ten years in checkout line at safeway.

=> 1st : Sisters reunited in checkout line at safeway after ten years being apart.

2nd : The sisters were reunited after they have spent 10 years in checkout line at safe way.

(b) Prostitutes appeal to the Pope.

=> There are two meaning of "appeal" : 1) earnest request for support, 2) man is appealed to woman.

1st : Prostitutes make a request for pope's support.

(c) Large hole appears in High Street. City authorities are looking into it.

=> Large hole appears in High Street. City authorities are looking into fixing it.

(d) Mayor says bus passengers should be belts.

=> Several meaning of "belt" 1) hit something, 2) wear some belt.

=> Mayor says bus passengers should wear a belt.

4. The following notice was posted on the wall of a hospital emergency room:

NO HEAD INJURY IS TOO TRIBIAL TO IGNORE.

Reformulate to avoid the unintended second reading. (The context for this sentence is so strong that many people have difficulty seeing there is an alternative meaning.)

=> 1st : No head injury must be ignore : Too~ to ~.

2nd : No head injury must not be ignore : (Too)~ to ~.

-> If someone didn't have head injury, It is ok to ignore.

5. You often see the following notice posted in elevators:

IN CASE OF FIRE, DO NOT USE ELEVATOR.

This one always amuses me. Comment on the two meanings and reformulate to avoid the unintended second reading. (Again, given the context for this notice, the ambiguity is not problematic.)

=> 1st : When there is a fire in the building, do not use elevator.

2nd : Do not use the elevator, because there might be a fire.

-> as in there is always a possibility there might be a mad people, so do not use the elevator at all. Just "in case"

* Other solutions

⇒ 1) The elevator is to be used only in case there is a fire.

2) If the building catches on fire, the use of the elevator is prohibited.

⇒ Do not use the elevator in case there is a fire.

6. Official documents often contain on or more pages that are empty apart from on sentence at the bottom:

This page intentionally left blank.

Dose the sentence make a true statement? What is the purpose of making such a statement? What reformulation of the sentence would avoid any logical problems about truth? (Once again, the context means that in practice everyone understands the intended meaning and there is no problem. But the formulation of a similar sentence in mathematician's seminal work and led to a major revolution in an entire branch of mathematics.)

=> This statement is False statement. Purpose of statement is to inform the reader that (except for this statements, itself) there is nothing else on the page.

=> Type 1) This page left black intentionally, except for this statement.

=> Type 2) The rest of this page intentionally left blank.

7. Find (and provide citations for) three examples of published sentences whose literal meaning is (clearly) not what the writer intended. [This is much easier than you might think. Ambiguity is very common.]

=> Stop the red light : without context situation, we have to stop when we see the every red light.

=> JuSung killed someone with a gun. : same case of pbm 2.

=> British left waffles on Falkland Islands. : the statement "on Falkland Islands" can be connected with 1) left, and 2) waffles.

=> We build bodies that last a lifetime.

=> The hotel has tennis courts, comfortable beds, and other athletic facilities.

=> Stolen Painting Found by Tree.

=> Include your children when baking cookies.

8. Comment on the sentence "The temperature is hot today." You hear people say things like this all the time, and everyone understands what is meant. But using language in this sloppy way in mathematics would be disastrous.

=> The word 'hot' makes this sentence confused. In common life, we will guess this sentence's meaning by thinking about context. It means that we can compare the temperature with yesterday by using the context. But in mathematics, we couldn't guess the comparison target. This statement could be false or True according to previous condition. There is another problem because we don't know how exactly the temperature is. Because of this reason this statement should be reformed.

*** Other solutions.**

=> The temperature should be a measureable value. The air can be hot, the surrounding structures may be radiating heat, but the temperature can't be hot. So the best way to communicate the idea is "I feel hot today"

9. Provide a context and a sentence within that context, where the word and occurs five times in succession, with no other word between those five occurrences. (You are allowed to use punctuation.)

=> Buffalo buffalo buffalo Buffalo buffalo

⇒ Joe and Marry visited the zoo with their neighbors, the Ands, Mr. And and Mrs. And. So two couples went to the zoo, Joe and Marry and And and And. And after the zoo, the four ate an ice cream.

10. Provide a context and a sentence within that context, where the words and, or, and, or, and occur in that order, with no other word between them. (Again, you can use punctuation.)

=> Kate chose "and", "or" and "or" and she got all the answers right.

=> There are 'AND. OR.' and 'OR. AND' gate.

*** Other solutions.**

=> Peter and Kate were answering the quiz questions concerning the use of grammatical conjunctions in the English language. They were both asked to chose the right answeres to three questions. Peter chose consecutively "and","and" and "and" and he only got one answer right. Kate chose "and", "or" and "or" and she got all the answers right.

<Exercise 2.2.1>

1. The mathematical concept of conjunction captures the meaning of "and" in everyday language. True or false? Explain your answer.

=> I think that it is false. In mathematics, "and" is need for combining two claims into one that asserts both, and also it is independent of order. However, in everyday language, it is not always true.

2. Simplify the following symbolic statements as much as you can, leaving your answer in the standard symbolic form. (In case you are not familiar with the notation, I'll answer the first one for you.)

=> (a) $(\pi > 0) \wedge (\pi < 10)$ $[Answer: 0 < \pi < 10]$

(b) $(p \geq 7) \wedge (p < 12)$: $(7 \leq p < 12)$

(c) $(x > 5) \wedge (x < 7)$: $(5 < x < 7)$

(d) $(x < 4) \wedge (x < 6)$: $x < 4$

(e) $(y < 4) \wedge (y^2 < 9)$: $(y < 4) \wedge (y^2 < 9)$

(f) $(x \geq 0) \wedge (x \leq 0)$: $x = 0$

3. Express each of your simplified statements from question 2 in natural English.

=> a) pi is greater than 0 and less than 10.

b) p is greater than or equal to 7 and p is less than 12.

c) x is greater than 5 and x is less than 7.

d) x is less than 4

e) y is greater than -3 and y is less than 3.

f) x is equal to 0.

4. What strategy would you adopt to show that the conjunction $\phi_1 \wedge \phi_2 \wedge \ldots \wedge \phi_n$ is true?

=> The "and" operation is true only if all conjunctions are true.

5. What strategy would you adopt to show that the conjunction $\phi_1 \wedge \phi_2 \wedge \ldots \wedge \phi_n$ is false?

=> If one or some conjunctions are false then above statement is false.

6. Is it possible for one of $(\phi \wedge \psi) \wedge \theta$ and $\phi \wedge (\psi \wedge \theta)$ to be true and the other false, or does the associative property hold for conjunction? Prove your answer.

=> The conjunction property is if some conjunctions are false, then conjunction is also false. So if $(\phi \wedge \psi) \wedge \theta$ is true, it means that each value is true. Then $\phi \wedge (\psi \wedge \theta)$ also must be true. We can check the truth table for the associative property of the conjunction.

ϕ	φ	θ	$(\phi \wedge \varphi) \wedge \theta$	$\phi \wedge (\varphi \wedge \theta)$
T	T	T	T	T
T	T	F	F	F
T	F	T	F	F
T	F	F	F	F
F	T	T	F	F
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

Both $(\phi \wedge \varphi) \wedge \theta$ and $\phi \wedge (\varphi \wedge \theta)$ are the same. Therefore, associative property holds for the conjunction.

7. Which of the following is more likely?

(a) Alice is a rock star and works in a bank.

(b) Alice is quiet and works in a bank.

(c) Alice is quiet and reserved and works in a bank.

(d) Alice is honest and works in a bank.

(e) Alice works in a bank.

If you believe there is no definite answer, say so.

=> I think (e) is most likely. Because, "and" operation means there are more conditions which have to be true. (e) is most simple statement.

8. In the following table, T denotes 'true' and F denotes 'false'. The first two columns list all the possible combinations of values of T and F that the two statements ϕ and ψ can have. The third column should give the truth value (T or F) $\phi \wedge \psi$ achieves according to each assignment of T or F to ϕ and ψ.

ϕ	ψ	$\phi \wedge \psi$
T	T	T
T	F	F
F	T	F
F	F	F

Fill in the final column. The resulting table is an example of a "propositional truth table".

=> Fill in above table

<Exercise 2.2.2>

1. Simplify the following symbolic statements as much as you can, leaving your answer in a standard symbolic form (assuming you are familiar with the notation) :

=> (a) $(\pi > 3) \vee (\pi > 10)$: $(\pi > 3)$

(b) $(x < 0) \vee (x > 0)$: $(\forall x \neq 0)$

(c) $(x = 0) \vee (x > 0)$: $x \geq 0$

(d) $(x > 0) \vee (x \geq 0)$: $(x \geq 0)$

(e) $(x > 3) \vee (x^2 > 9)$: $(x < -3) \vee (x > 3)$

2 Express each of your simplified statements from question 1 in natural English.

a) pi is greater than 3

b) all x except for x = 0.

c) x is greater than or equal to 0

d) x is greater than or equal to 0

e) x is less than -3 or x is greater than 3

3. What strategy would you adopt to show that the disjunction $\phi_1 \vee \phi_2 \vee \ldots \vee \phi_n$ is true?

=> If one or some of disjunctions are true then above statement is true.

4. What strategy would you adopt to show that the disjunction $\phi_1 \vee \phi_2 \vee \ldots \vee \phi_n$ is false?

=> If all disjunctions are false then the statement is false.

5. Is it possible for one of $(\phi \vee \psi) \vee \theta$ and $\phi \vee (\psi \vee \theta)$ to be true and the other false, or does the associative property hold for disjunction? Prove your answer.

=> $(\phi \vee \psi) \vee \theta$ is false means that each value ϕ, ψ and θ is false. Then $\phi \vee (\psi \vee \theta)$ must be false.

check the truth table for the associative property of the disjunction.

ϕ	φ	θ	$(\phi \vee \varphi) \vee \theta$	$\phi \vee (\varphi \vee \theta)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	T	T
F	F	T	T	T
F	F	F	F	F

Both $(\phi \vee \varphi) \vee \theta$ and $\phi \vee (\varphi \vee \theta)$ are the same. Therefore, associative property holds for the disjunction.

6. Which of the following is more likely?

(a) Alice is a rock star or she works in a bank.

(b) Alice is quiet and works in a bank.

(c) Alice is a rock star.

(d) Alice is honest and works in a bank.

(e) Alice works in a bank.

If you believe there is no definite answer, say so.

=> the operator "or" means that even if one of two proposition is true, then the statement is true. It means that something like a larger chance to be true. So I think that (a) is most likely.

7. Fill in the entries in the final column of the following truth table :

ϕ	ψ	$\phi \vee \psi$
T	T	T
T	F	T
F	T	T
F	F	F

=> Fill in above matrix.

<Exercise 2.2.3>

1. Simplify the following symbolic statements as much as you can, leaving your answer in a standard symbolic form (assuming you are familiar with the notation):

=> (a) $\neg(\pi > 3.2)$: $\pi \leq 3.2$

(b) $\neg(x < 0)$: $x \geq 0$

(c) $\neg(x^2 > 0)$: $x^2 \leq 0$

(d) $\neg(x = 1)$: $x \neq 1$

(e) $\neg\neg\psi$: ψ

2. Express each of your simplified statements from question 1 in natural English.

=> a) pi is less than or equal to 3.2

b) x is greater than or equal to 0

c) x^2 is less than or equal to 0

d) x is not 1

e) ψ

3. Is showing that the negation $\neg\phi$ is true the same as showing that ϕ is false?

=> Yes. If ϕ is false, that means $\neg\phi$ is true.

4. Fill in the entries in the final column of the following truth table :

ϕ	$\neg\phi$
T	F
F	T

=> Fill in above matrix

5. Let D be the statement "The dollar is strong", Y the statement "The Yuan is strong" and T the statement "New US-China trade agreement signed". Express the main content of each of the following (fictitious) newspaper headlines in logical notation. (Note that logical notation captures truth, but not the many nuances and inferences of natural language.) How would you justify and defend your answers?

=> (a) Dollar and Yuan both strong : $D \wedge Y$

(b) Trade agreement fails on news of weak Dollar. : $\neg D \to \neg T$

(c) Dollar weak but Yuan strong, following new trade agreement. : $T \wedge \neg D \wedge Y$

(d) Strong Dollar means a weak Yuan. : $D \to \neg Y$

(e) Yuan weak despite new trade agreement, but Dollar remains strong : $T \wedge \neg Y \wedge D$

(f) Dollar and Yuan can't both be strong at same time. : $(D \wedge \neg Y) \vee (\neg D \wedge Y)$

(g) If new trade agreement is signed, Dollar and Yuan can't both remain strong. : $T \wedge \neg (D \wedge Y)$

(h) New trade agreement does not prevent fall in Dollar and Yuan : $T \wedge (\neg D \wedge \neg Y)$

(i) US-China trade agreement fails but both currencies remain strong : $D \wedge Y \wedge \neg T$

(j) New trade agreement will be good for one side, but no one knows which. : $T \wedge ((\neg D \wedge S) \vee (D \wedge \neg S))$

6. In US law, a trial verdict of "Not guilty" is given when the prosecution fails to prove guilt. This, of course, does not mean the defendant is, as a matter of actual fact, innocent. Is this state of affairs captured accurately when we use "not" in the mathematical sense? (i.e., Do "Not guilty" and "$\neg$guilty" mean the same?) What if we change the question to ask if "Not proven" and "$\neg$ proven" mean the same thing?

=> a) No, in real life, not guilty means we fail to prove guilt. It did not mean innocent. However, in mathematics, not guilty means innocent

b) Different from (a), the statement about proven are only two; proven or not proven. Therefore, we can say that not proven is same meaning with $\neg$ proven.

7. The truth table for $\neg\neg\phi$ is clearly the same as that for ϕ itself, so the two expressions make identical truth assertions. This is not necessarily true for negation in everyday life. For example, you might find yourself saying "I was not displeased with the movie." In terms of formal negation, this has the form $\neg$($\neg$ pleased), but your statement clearly does not mean

that you were pleased with the movie. Indeed, it means something considerably less positive. How would you capture this kind of use of language in the formal framework we have been looking at?

=> this things happen because the statement about pleased are 3 statements; pleased, normal, not pleased. But "negation" in mathematics is only used when the statement has two case.

If we denote that $\neg$ (displeased) = (normal) $\vee$ (pleased), we can understand that statement in mathematical way.

<Exercises 2.3.1>

1. Fill in the second row of the truth table.

ϕ	ψ	$\phi \Rightarrow \psi$
T	T	T
T	F	F
F	T	?
F	F	?

=> Fill in the above matrix

2. Provide a justification of your entry.

=> The truth of phi does not results in the truth of psi. Therefore, "phi implies psi" is False.

<Exercise 2.3.2>

1. Fill in the third and fourth rows of the truth table.

ϕ	ψ	$\phi \Rightarrow \psi$
T	T	T
T	F	F
F	T	T
F	F	T

2. Provide justifications for your entries.

=> As phi is false, the truth of the phi is violated. So, the statement “phi implies psi” is not false.

<Exercise 2.3.3>

1. Which of the following are true and which are false?

=> (a) $(\pi^2 > 2) \Rightarrow (\pi > 1.4)$: T $(\because \phi = T, \psi = T)$

(b) $(\pi^2 < 0) \Rightarrow (\pi = 3)$: T $(\because \phi = F, \psi = F)$

(c) $(\pi^2 > 0) \Rightarrow (1+2=4)$: F $(\because \phi = T, \psi = F)$

(d) $(\pi < \pi^2) \Rightarrow (\pi = 5)$: F $(\because \phi = T, \psi = F)$

(e) $(e^2 \geq 0) \Rightarrow (e < 0)$: F $(\because \phi = F, \psi = T)$

(f) $\neg$(5 is an integer) $\Rightarrow (5^2 \geq 1)$: T $(\because \phi = F, \psi = T)$

(g) (The area of a circle of radius 1 is π) $\Rightarrow$ (3 is prime) : T $(\because \phi = T, \psi = T)$

(h) (Squares have three sides) $\Rightarrow$ (Triangles have four sides) : T $(\because \phi = F, \psi = F)$

(i) (Elephants can climb trees) $\Rightarrow$ (3 is irrational) : T $(\because \phi = F, \psi = F)$

(j) (Euclid’s birthday was July 4) $\Rightarrow$ (Rectangles have four sides) : T $(\because \phi = ?, \psi = T)$

2. As in Exercise 2.2.3(5), let D be the statement “The dollar is strong", Y the statement “The Yuan is strong" and T the statement “New US-China trade agreement signed". Express the main content of each of the following (fictitious) newspaper headlines in logical notation. (Note that logical notation captures truth, but not the many nuances and inferences of natural language.) As before, be prepared to justify and defend your answers.

(a) New trade agreement will lead to strong currencies in both countries.

=> $T \Rightarrow D \wedge Y$

(b) If the trade agreement is signed, a rise in the Yuan will result in a fall in the Dollar.

=> $T \Rightarrow (Y \Rightarrow \neg D)$

(c) Dollar weak but Yuan strong, following new trade agreement.

=> $T \Rightarrow (\neg D \wedge Y)$

(d) Strong Dollar means a weak Yuan

=> $D \Rightarrow \neg Y$

(e) New trade agreement means Dollar and Yuan will be tightly linked.

=> $T \Rightarrow (D \Leftrightarrow Y)$

3. Complete the following truth table

ϕ	$\neg\phi$	ψ	$\phi \Rightarrow \psi$	$\neg\phi \vee \psi$
T	F	T	T	T
T	F	F	F	F
F	T	T	T	T
F	T	F	T	T

Note : $\neg$ has the same binding rules as $-$(minus) in arithmetic and algebra, so $\neg\phi \vee \psi$ is the same as $(\neg\phi) \vee \psi$.

=> Fill in the above table

4. What conclusions can you draw from the above table?

=> $\phi \Rightarrow \psi \equiv \neg\phi \vee \psi$

5. Complete the following truth table. (Recall that $\phi \not\Rightarrow \psi$ is another way of writing $\neg[\phi \Rightarrow \psi]$.)

ϕ	ψ	$\neg\psi$	$\phi \Rightarrow \psi$	$\phi \not\Rightarrow \psi$	$\phi \wedge \neg\psi$
T	T	F	T	F	F
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	F	F

=> Fill in the above table

6. What conclusions can you draw from the above table?

=> $\phi \not\Rightarrow \psi \equiv \phi \wedge \neg\psi$

<Exercise 2.3.4>

1. Build a truth table to prove the claim I made earlier that $\phi \Leftrightarrow \psi$ is true if ϕ and ψ are both true or both false, and $\phi \Leftrightarrow \psi$ is false if exactly one of ϕ, ψ is true and the other false. (To constitute a proof, your table should have columns that show how the entries for $\phi \Leftrightarrow \psi$ are derived, one operator at a time, as in the previous exercises.)

=> Proof: $\phi \Leftrightarrow \psi$ is bi-conditional. It can be written as:

$$(\phi \Rightarrow \psi) \wedge (\psi \Rightarrow \phi)$$

Note that by definition of implication we can write $(\phi \Rightarrow \psi)$ as follows:

$$(\phi \wedge \psi) \vee (\neg\phi)$$

So corresponding Truth Table is:

φ	Ψ	$\neg\varphi$	$\varphi \wedge \Psi$	$(\varphi \wedge \Psi) \vee (\neg\varphi)$	$\varphi \Rightarrow \Psi$	$\Psi \Rightarrow \varphi$	$(\varphi \Rightarrow \Psi) \wedge (\Psi \Rightarrow \varphi)$*	$\phi \Leftrightarrow \psi$ *
T	T	F	T	T	T	T	T	T
T	F	F	F	F	F	T	F	F
F	T	T	T	T	T	F	F	F
F	F	T	F	T	T	T	T	T

Since two columns with * are similar. We see that $\phi \Leftrightarrow \psi$ is true only if ϕ and ψ are both true and if both are false then $\phi \Leftrightarrow \psi$ is also false.

2. Build a truth table to show that

$$(\phi \Rightarrow \psi) \Leftrightarrow (\neg\phi \vee \psi)$$

is true for all truth values of ϕ and ψ. A statement whose truth values are all T is called a logical validity, or sometimes a tautology.

=> Proof:

φ	Ψ	$\neg\varphi$	$\varphi \Rightarrow \Psi$*	$(\neg\varphi \vee \Psi)$*
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Since two columns with * are similar for all possible values of corresponding φ, Ψ, and $\neg\varphi$. So we know bi-conditional statement ($\phi \Leftrightarrow \psi$) can be written as:

$$(\phi \Rightarrow \psi) \wedge (\psi \Rightarrow \phi)$$

Therefore given statement can be written as: $[(\varphi \Rightarrow \Psi) \Rightarrow (\neg\varphi \vee \Psi)] \wedge [(\neg\varphi \vee \Psi) \Rightarrow (\varphi \Rightarrow \Psi)]$

From above truth table we conclude that:

$A \coloneqq (\varphi \Rightarrow \Psi) \Rightarrow (\neg\varphi \vee \Psi)$	$B \coloneqq (\neg\varphi \vee \Psi) \Rightarrow (\varphi \Rightarrow \Psi)$	$A \wedge B$
T	T	T
T	T	T
T	T	T
T	T	T

Since all entries of column $A \wedge B$ are true therefore tautology.

3. Build a truth table to show that

$$(\phi \not\Rightarrow \psi) \Leftrightarrow (\phi \wedge \neg\psi)$$

is a tautology.

=> Answer:

φ	Ψ	$\neg\Psi$	$\varphi \not\Rightarrow \Psi$	$\varphi\wedge\neg\Psi$	$A = (\varphi \not\Rightarrow \Psi) \Rightarrow (\varphi\wedge\neg\Psi)$	$B = (\varphi\wedge\neg\Psi) \Rightarrow (\varphi \not\Rightarrow \Psi)$	A∧B
T	T	F	F	F	T	T	T
F	T	F	F	F	T	T	T
T	F	T	T	T	T	T	T
F	F	T	F	F	T	T	T

Note $\varphi \not\Rightarrow \Psi$ = $\neg(\varphi \Rightarrow \Psi)$

Since A^B is all true so tautology.

4. The ancient Greeks formulated a basic rule of reasoning for proving mathematical statements. Called modus ponens, it says that if you know ϕ and you know $\phi \Rightarrow \psi$, then you can conclude ψ.

(a) Construct a truth table for the logical statement

$$[\phi \wedge (\phi \Rightarrow \psi)] \Rightarrow \psi$$

=> Solution:

Row #	φ	Ψ	$\varphi \Rightarrow \Psi$	$\varphi\wedge(\varphi \Rightarrow \Psi)$	$[\varphi\wedge(\varphi \Rightarrow \Psi)] \Rightarrow \Psi$
1	T	T	T	T	T
2	F	T	T	F	T
3	T	F	F	F	T
4	F	F	T	F	T

(b) Explain how the truth table you obtain demonstrates that modus ponens is a valid rule of inference.

=> Solution: Since basic rule for A ⇒ B means that whenever statement A is true B must also be true for (A ⇒ B) to be true. However if statement A is false then we do not care about truth value of statement of B. In other words if A is false, (A ⇒ B) is always true for whatsoever truth value of B is.

By applying this rule of implication to above truth table for $[\varphi\wedge(\varphi \Rightarrow \Psi)] \Rightarrow \Psi$ we have:

- is true when both are true. So truth value of $\Psi = \mathrm{T}$

Thus from truth table we can conclude that modus ponens is a valid rule of inference

5. Mod-2 arithmetic has just the two numbers 0 and 1 and follows the usual rules of arithmetic together with the additional rule 1 + 1 = 0. (It is the arithmetic the takes place in a single bit location in a digital computer.) Complete the following table:

M	N	$M \times N$	$M + N$
1	1	?	?
1	0	?	?
0	1	?	?
0	0	?	?

=> Solution:

Row #	M	N	MxN	$M + N$
1	1	1	1	0
2	1	0	0	1
3	0	1	0	1
4	0	0	0	0

6. In the table you obtained in the above exercise, interpret 1 as T and 0 as F and view M,N as denoting statements.

(a) Which of the logical combinators $\wedge$, $\vee$ corresponds to $\times$?

=> Answer: logical symbol $\wedge = \times$ (truth table shown below)

(b) Which logical combinator corresponds to +?

=> Answer: logical symbol exclusive-Or is equal to + i.e. $\oplus = +$

(c) Does $\neg$ correspond to $-$ (minus)?

=> Answer: No. for example consider $\neg 1 = 0$ but (-)1 = -1 ≠ 0

7. Repeat the above exercise, but interpret 0 as T and 1 as F. What conclusions can you draw?

=> solution :

Row #	M	N	$M\text{x}N = M \wedge N$	$M + N = \neg(M \vee N)$
1	F	F	F	T
2	F	T	F	F
3	T	F	F	F
4	T	T	T	F

8. The following puzzle was introduced by the psychologist Peter Wason in 1966, and is one of the most famous subject tests in the psychology of reasoning. Most people get it wrong. (So you have been warned!)

Four cards are placed on the table in front of you. You are told (truthfully) that each has a letter printed on one side and a digit on the other, but of course you can only see one face of each. What you see is:

B E 4 7

You are now told that the cards you are looking at were chosen to follow the rule "If there is a vowel on one side, then there is an odd number on the other side." What is the least number of cards you have to turn over to verify this rule, and which cards do you in fact have to turn over?

=> Answer: I have to turn at least two cards. Let $f(x) \coloneqq vowel\ letter\ on\ one\ side$ and $g(x) \coloneqq odd\ digit\ on\ other\ sie$

We need to perform following checking $f(x) \Leftrightarrow g(x) = (f(x) \Rightarrow g(x)\ and\ g(x) \Rightarrow f(x))$ i.e.

- If there is vowel on one side then other side must have odd number or
- If there is odd number on one side then other side must be vowel.

There are two cards that I need to pick

For example (E, 7) ➔ E should turn out to some odd number and 7 should show some vowel.

<Exercise 2.3.5>

1. One way to prove that

$$\neg(\phi \wedge \psi) \text{ and } (\neg\phi) \vee (\neg\psi)$$

are equivalent is to show they have the same truth table:

ϕ	ψ	$\phi \wedge \psi$	$\overset{*}{\neg(\phi \wedge \psi)}$	$\neg\phi$	$\neg\psi$	$\overset{*}{(\neg\phi) \vee (\neg\psi)}$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

Since the two columns marked * are identical, we know that the two expressions are equivalent.

Thus, negation has the effect that it changes $\vee$ into $\wedge$ and changes $\wedge$ into $\vee$. An alternative way to prove this is to argue directly with the meaning of the first statement:

1. $\phi \wedge \psi$ means both ϕ and ψ are true.

2. Thus $\neg(\phi \wedge \psi)$ means it is not the case that both ϕ and ψ are true.

3. If they are not both true, then at least one of ϕ, ψ must be false.

4. This is clearly the same as saying that at least one of $\neg\phi$ and $\neg\psi$ is true. (By the definition of negation).

5. By the meaning of or, this can be expressed as $(\neg\phi) \vee (\neg\psi)$.

Provide an analogous logical argument to show that $\neg(\phi \vee \psi)$ and $(\neg\phi) \wedge (\neg\psi)$ are equivalent.

=> Solution: Proof that $\neg(\varphi \vee \Psi)$ and $(\neg\varphi) \wedge (\neg\Psi)$ are equivalent.

1. $\varphi \vee \Psi$ means that either both $\varphi \; and \; \Psi$ or one of the two $\varphi \; or \; \Psi$ is true.
2. Thus $\neg(\varphi \vee \Psi)$ means that it is true when both of them are false and it is false when either both are true or one of the two $\varphi \; or \; \Psi$ is true.
3. If both of the $\varphi \; and \; \Psi$ are false then by definition of negation the $(\neg\varphi) \; and \; (\neg\Psi)$ are true.

4. If both $(\neg\varphi)$ and $(\neg\Psi)$ are true then $(\neg\varphi)\wedge(\neg\Psi) = true$ which is equal to truth value of $\neg(\varphi \vee \Psi)$.

Then by meaning of and the statement $\neg(\varphi \vee \Psi)$ can be expressed as $(\neg\varphi)\wedge(\neg\Psi)$.

2. By a denial of a statement - we mean any statement equivalent to $\neg\phi$. Give a useful denial of each of the following statements.

(a) 34,159 is a prime number.

(b) Roses are red and violets are blue.

(c) If there are no hamburgers, I'll have a hot dog.

(d) Fred will go but he will not play.

(e) The number x is either negative or greater than 10.

(f) We will win the first game or the second.

=> Solution:

a) 34,159 is not a prime number.
b) Roses are not red or violets are not blue.
c) There is no hamburger but I will not have a hot dog. [reason: $\neg(\varphi \Rightarrow \Psi)$: φ^$\neg\Psi$
d) Fred will not go or he will play [original statement:= (A ∧ ¬B) => ¬ (A ∧ ¬B) = (¬ A ∨ B)]
e) Number 'x' is not negative and less than or equal to 10. ¬ [(X<0) ∨ (x>10)] =[(x≥0) ∧ (x≤10)]
f) We will not win first game not the second = we will lose the first two games.

3. Which of the following conditions are necessary for the natural number n to be divisible by 6?

(a) n is divisible by 3.

(b) n is divisible by 9.

(c) n is divisible by 12.

(d) $n = 24$.

(e) n^2 **is divisible by 3.**

(f) n is even and divisible by 3.

=> Solution: Necessary means: $\frac{n}{6} \Rightarrow X$ where X is condition.

Answer: then necessary conditions for given statements are:

a) necessary as: $\frac{n}{6} \Rightarrow \frac{n}{3}$

b) Not necessary as: 6/6 but 6 is not divisible by 9.

c) Not necessary: same reason as for b.

d) Not necessary: 6/6⇒(n=6) but 6 ≠24

e) Necessary as: $\frac{n}{6} \Rightarrow \frac{n}{3} \Rightarrow \frac{n^2}{3}$

Necessary condition as: $\frac{n}{6} \Rightarrow \left(\frac{n}{2} \wedge \frac{n}{3}\right)$

4. In Exercise 3, which conditions are sufficient for n to be divisible by 6?

=> Answer: All following conditions are sufficient for a natural number 'n' to be divisible by 6:

c), d) and f)

5. In exercise 3, which conditions are necessary and sufficient for n to be divisible by 6?

=> Answer: Necessary and sufficient = (Necessary ^ sufficient) = ({d, e, f} ∩ {f}) = f

6. Let m, n **denote any two natural numbers. Prove that** mn **is odd iff** m **and** n **are odd.**

=> Answer: Let T = odd : {T∈ ℕ}, F = even : {F∈ ℕ}; MN = MxN = M∧N. prove that MN is even iff M and N both are even. Proof by truth table:

M	N	$M\text{x}N = M \wedge N$

F	F	F
F	T	F
T	F	F
T	T	T

So MN is even if and only if both are even.

7. With reference to the previous question, is it true that mn is even iff m and n are even?

=> Answer: No; let M = 3(odd) and N = 2 (even) then MxN = 6 (even)

8. Show that $\phi \Leftrightarrow \psi$ is equivalent to $(\neg\phi) \Leftrightarrow (\neg\psi)$. How does this relate to your answers to Questions 6 and 7 above?

=> Solution: We can prove it through truth table as follows:

Since bicondition $(\varphi \Leftrightarrow \Psi)$ is equivalent to $(\varphi \Rightarrow \Psi)$ ^ $(\Psi \Rightarrow \varphi)$ and $(\neg\varphi) \Leftrightarrow (\neg\Psi)$ = $(\neg\varphi \Rightarrow \neg\Psi)$ ^ $(\neg\Psi \Rightarrow \neg\varphi)$:

φ	Ψ	$\varphi \Rightarrow \Psi$	$\Psi \Rightarrow \varphi$	$(\varphi \Rightarrow \Psi)$ ^ $(\Psi \Rightarrow \varphi)$ *	$\neg\varphi$	$\neg\Psi$	$A = (\neg\varphi \Rightarrow \neg\Psi)$	$B = (\neg\Psi \Rightarrow \neg\varphi)$	A^B *
T	T	T	T	T	F	F	T	T	T
F	T	T	F	F	T	F	F	T	F
T	F	F	T	F	F	T	T	F	F
F	F	T	T	T	T	T	T	T	T

Since both columns (*) are equivalent hence $(\varphi \Leftrightarrow \Psi)$ is equivalent to $(\neg\varphi) \Leftrightarrow (\neg\Psi)$

9. Construct truth tables to illustrate the following:

(a) $\phi \Leftrightarrow \psi$

φ	Ψ	$\varphi \Rightarrow \Psi$	$\Psi \Rightarrow \varphi$	$(\varphi \Rightarrow \Psi)$ ^ $(\Psi \Rightarrow \varphi)$
T	T	T	T	T
F	T	T	F	F

=>

T	F	F	T	F
F	F	T	T	T

(b) $\phi \Rightarrow (\psi \vee \theta)$

=>

θ	Ψ	φ	$\theta \vee \Psi$	$\varphi \Rightarrow (\theta \vee \Psi)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	T	T
F	F	T	F	F
F	F	F	F	T

10. Use truth tables to prove that the following are equivalent :

(a) $\neg(\phi \Rightarrow \psi)$ **and** $\phi \wedge (\neg \psi)$

=>

φ	Ψ	$\varphi \Rightarrow \Psi$	$\neg(\varphi \Rightarrow \Psi)$	$\neg\Psi$	φ^$(\neg\Psi)$
T	T	T	F	F	F
F	T	T	F	F	F
T	F	F	T	T	T
F	F	T	F	T	F

From truth table we show those two columns of $\neg(\varphi \Rightarrow \Psi)$ and φ^$(\neg\Psi)$ are equivalent.

(b) $\phi \Rightarrow (\psi \wedge \theta)$ **and** $(\phi \Rightarrow \psi) \wedge (\phi \Rightarrow \theta)$

θ	Ψ	φ	$\theta \wedge \Psi$	$\varphi \Rightarrow (\theta \wedge \Psi) *$	$A = (\varphi \Rightarrow \Psi)$	$B = (\varphi \Rightarrow \theta)$	A ∧ B *
T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T
T	F	T	F	F	F	T	F
T	F	F	F	T	T	T	T
F	T	T	F	F	T	F	F
F	T	F	F	T	T	T	T
F	F	T	F	F	F	F	F
F	F	F	F	T	T	T	T

Since two columns (*) in truth table are equivalent hence given statements are equivalent.

(c) $(\phi \vee \psi) \Rightarrow \theta$ **and** $(\phi \Rightarrow \theta) \wedge (\psi \Rightarrow \theta)$

θ	Ψ	φ	$\varphi \vee \Psi$	$(\varphi \vee \Psi) \Rightarrow \theta *$	$A = (\varphi \Rightarrow \theta)$	$B = (\Psi \Rightarrow \theta)$	A ∧ B *
T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T
T	F	T	T	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	F	F	F	F
F	T	F	T	F	T	F	F
F	F	T	T	F	F	T	F
F	F	F	F	T	T	T	T

Since two columns (*) in truth table are equivalent hence given statements are equivalent.

11. Verify the equivalence in (b) and (c) in the previous question by means of a logical argument. (So, in the case of (b), for example, you must show that assuming ϕ and deducing $\psi \wedge \theta$ is the same as both deducing ψ from ϕ and θ from ϕ.)

=> Solution:

For Part b): [⇒] we can deduce $\theta \wedge \Psi$ from φ. But we can deduce $\theta \; and \; \Psi$ from $\Psi \wedge \theta$. It follows that we can deduce $\theta \; and \; \Psi$ from φ i.e. $\varphi \Rightarrow \Psi$ and $\varphi \Rightarrow \theta$. Hence $(\varphi \Rightarrow \Psi) \wedge (\varphi \Rightarrow \theta)$

[⇐] we can deduce $(\varphi \Rightarrow \Psi)$ and $(\varphi \Rightarrow \theta)$. Hence we can deduce Ψ from φ or we can deduce θ from φ. Hence we can deduce $(\Psi \wedge \varphi)$ from φ. So $\varphi \Rightarrow (\Psi \wedge \varphi)$

For Part c):

[⇒] We can deduce $\theta\ from\ (\varphi \vee \Psi)$ and $(\varphi \vee \Psi) from\ (\varphi \wedge \Psi)$. It means that we can deduce θ from both φ and Ψ. Both of them means that $\varphi \Rightarrow \theta\ and\ \Psi \Rightarrow \theta$ thus $\varphi \Rightarrow \theta\ \wedge\ \Psi \Rightarrow \theta$

[⇐] we can deduce $\varphi \Rightarrow \theta\ and\ \Psi \Rightarrow \theta$ so we can deduce θ from both φ and Ψ. Also we can deduce $\varphi\ and\ \Psi$ from $\varphi \vee \Psi$. So we can deduce θ from $\varphi \vee \Psi$ i.e. $(\varphi \vee \Psi) \Rightarrow \theta$

12. Use truth tables to prove the equivalence of $\phi \Rightarrow \psi$ and $(\neg\psi) \Rightarrow (\neg\phi)$.

=> $(\neg\psi) \Rightarrow (\neg\phi)$ is called the contrapositive of $\phi \Rightarrow \psi$. The logical equivalence of a conditional and its contrapositive means that one way to prove an implication it is to verify the contrapositive. This is a common form of proof in mathematics that we'll encounter later.

φ	Ψ	$\varphi \Rightarrow \Psi$ *	$\neg\Psi$	$\neg\varphi$	$(\neg\Psi) \Rightarrow (\neg\varphi)$ *
T	T	T	F	F	T
F	T	T	F	T	T
T	F	F	T	F	F
F	F	T	T	T	T

Since two columns (*) in truth table are equivalent hence given statements are equivalent.

13. Write down the contrapositives of the following statements:

(a) If two rectangles are congruent, they have the same area.

(b) If a triangle with sides a, b, c (c largest) is right-angled, then $a^2 + b^2 = c^2$.

(c) If $2^n - 1$ is prime, then n is prime.

(d) If the Yuan rises, the Dollar will fall.

=> Answer:

a) Let $\varphi \coloneqq$ Two rectangles are congruent and $\Psi \coloneqq$ rectangles have same area then $\varphi \Rightarrow \Psi \coloneqq$ if two rectangles are congruent then they must have same area.
Its contrapositive $((\neg\varphi) \Rightarrow (\neg\Psi) \coloneqq$ if rectangles do not have the same area, they are not congruent.

b) Let $\varphi \coloneqq$ triangle with sides a, b, c (c>a and c >b) is right-angled triangle and $\Psi \coloneqq a^2 + b^2 = c^2$ then $\varphi \Rightarrow \Psi \coloneqq$ if triangle with sides a, b, c (c>a and c >b) is right-angled triangle then $a^2 + b^2 = c^2$
Its contrapositive $(\neg\varphi) \Rightarrow (\neg\Psi) \coloneqq$ If in a triangle with sides a, b, c (c>a and c >b); $a^2 + b^2 \neq c^2$, then triangle is not right-angled triangle.

c) Let $\varphi \coloneqq 2^n - 1$ is prime and $\Psi \coloneqq$ n is prime then $\varphi \Rightarrow \Psi \coloneqq$ if $2^n - 1$ is prime then n is prime.
Its contrapositive $(\neg\varphi) \Rightarrow (\neg\Psi) \coloneqq$ If n is not prime then $2^n - 1$ is not prime.

d) Let $\varphi \coloneqq$ Yuan rises and $\Psi \coloneqq$dollar falls then $\varphi \Rightarrow \Psi \coloneqq$ if Yuan rises, Dollar will fall.
Its contrapositive $(\neg\varphi) \Rightarrow (\neg\Psi) \coloneqq$ if dollar does not fall, then Yuan will not rise.

Note that a contrapositive is consistent with its original statement.

14. It is important not to confuse the contrapositive of a conditional $\phi \Rightarrow \psi$ with its converse $\psi \Rightarrow \phi$. Use truth tables to show that the contrapositive and the converse of $\phi \Rightarrow \psi$ are not equivalent.

=> Solution:

φ	Ψ	$\varphi \Rightarrow \Psi$	$\Psi \Rightarrow \varphi$*	$\neg\Psi$	$\neg\varphi$	$(\neg\Psi) \Rightarrow (\neg\varphi)$ *
T	T	T	T	F	F	T
F	T	T	F	F	T	T
T	F	F	T	T	F	F
F	F	T	T	T	T	T

Converse of $\varphi \Rightarrow \Psi$ is defined as: $\Psi \Rightarrow \varphi$ and contrapositive is defined as: $(\neg\Psi) \Rightarrow (\neg\varphi)$

Since two columns (*) in truth table are not equivalent hence contrapositive and converse of $\varphi \Rightarrow \Psi$ are not equivalent.

15. Write down the converses of the four statements in question 14.

=> Solution : Note converse of $\varphi \Rightarrow \Psi$ is defined as: $\Psi \Rightarrow \varphi$

a) Let $\varphi :=$ Two rectangles are congruent and $\Psi :=$ rectangles have same area then $\varphi \Rightarrow \Psi :=$ if two rectangles are congruent then they must have same area.

   Its converse is $\Psi \Rightarrow \varphi :=$ if two rectangles have the same area, they are congruent.

b) Let $\varphi :=$ triangle with sides a, b, c (c>a and c >b) is right-angled triangle and $\Psi := a^2 + b^2 = c^2$ then $\varphi \Rightarrow \Psi :=$ if triangle with sides a, b, c (c>a and c >b) is right-angled triangle then $a^2 + b^2 = c^2$

Its converse is $\Psi \Rightarrow \varphi :=$ If in a triangle with sides a, b, c (c>a and c >b), $a^2 + b^2 \neq c^2$, then triangle is right-angled.

c) Let $\varphi :=$ $2^n - 1$ is prime and $\Psi :=$ n is prime then $\varphi \Rightarrow \Psi :=$ if $2^n - 1$ is prime then n is prime.

Its converse is $\Psi \Rightarrow \varphi :=$ If n is prime then $2^n - 1$ is also prime.

d) Let $\varphi :=$ Yuan rises and $\Psi :=$dollar falls then $\varphi \Rightarrow \Psi :=$ if Yuan rises, Dollar will fall.

Its converse is $\Psi \Rightarrow \varphi :=$ if dollar falls, then Yuan will rise.

Note that a converse may not be consistent with its original statement. For example statement in a) is True but its converse is false.

16. Show that for any two statements ϕ and ψ either $\phi \Rightarrow \psi$ or its converse is true (or both). This is another reminder that the conditional is not the same as implication.

=> Solution:

For any two statements φ or Ψ, either their implication $(\varphi \Rightarrow \Psi)$ is true or its converse $(\Psi \Rightarrow \varphi)$ is true or both of them are true. For example consider following statements:

Let $\varphi :=$ Two rectangles are congruent and $\Psi :=$ rectangles have same area then the implication is defined as: $\varphi \Rightarrow \Psi :=$ if two rectangles are congruent then they must have same area. = True

Its converse is $\Psi \Rightarrow \varphi \coloneqq$ if two rectangles have the same area, they are congruent which is false.

Example 2: A triangle with sides a, b, c (c>a and c >b) is given. Let $\varphi \coloneqq$ right-angled triangle and $\Psi \coloneqq a^2 + b^2 = c^2$ then $\varphi \Rightarrow \Psi \coloneqq$ if triangle is right-angled triangle then $a^2 + b^2 = c^2$ which is true (Pythagoras theorem)

Its converse is $\Psi \Rightarrow \varphi \coloneqq$ If $a^2 + b^2 \neq c^2$, then triangle is right-angled which is also true (Pythagoras theorem).

17. Express the combinatory

$$\phi \text{ unless } \psi$$

in terms of the standard logical combinators.

=> Answer: $\neg\Psi \Rightarrow \varphi$

Example: let $\varphi \coloneqq$ apple is red and Ψ is apple is ready to eat. Then φ unless Ψ $(\neg\Psi \Rightarrow \neg\varphi)$ means: apple is not ready to eat unless it is red.

18. Identify the antecedent and the consequent in each of the following conditionals :

(a) If the apples are red, they are ready to eat.

(b) The differentiability of a function f is sufficient for f to be continuous.

(c) A function f is bounded if f is integrable.

(d) A sequence s is bounded whenever s is convergent.

(e) It is necessary that n is prime in order for $2^n - 1$ to be prime.

(f) The team wins only when Karl is playing.

(g) When Karl plays the team wins.

(h) The team wins when Karl plays.

=> Solution: An antecedent is first half of a given hypothetical proposition and consequent is second half. For example if following is hypothetical proposition " if φ then Ψ" then φ is antecedent and Ψ is consequent.

a) Antecedent: apples are red

   Consequent: apples are ready to eat.

b) Sufficiency $\Rightarrow$ X (conclusion)

   Antecedent: f is differentiable

   Consequent: f is continuous

c) Antecedent: f is integrable.

   Consequent: f is bounded.

d) Antecedent: S is convergent (not whenever is equal to 'if' clause)

   Consequent: S is bounded

e) X $\Rightarrow$ Necessity

   Antecedent: 'n' is prime

   Consequent: 2^n-1 is prime

f) Only when refers to conclusion

   Antecedent: the team wins

   Consequent: karl is playing

g) When is equal to 'if'

   Antecedent: karl is playing

   Consequent: the team wins

h) When is equal to 'if'

   Antecedent: karl is playing

Consequent: the team wins

19. Write the converse and aontrapositive of each conditional in the previous question.

=> Solution: Converse and contrapositive of each conditional in question 18 are as follows:

Converse:

1. If apples are ready to eat then they are red.
2. If f is continuous then f is differentiable

3. If f is bounded then f is integrable
4. If S is bounded then S is convergent
5. If 2^n-1 is prime then 'n' is prime
6. If karl is playing then the team wins
7. If the team wins then karl is playing
8. If the team wins then karl is playing

Contrapositive:

1. If apples are not ready to eat then they are not red.
2. If f is not continuous then f is not differentiable
3. If f is not bounded then f is not integrable
4. If S is not bounded then S is not convergent
5. If 2^n-1 is not prime then 'n' is not prime
6. If karl is not playing then the team does not win
7. If the team does not win then karl is not playing

If the team does not win then karl is not playing

20. Let $\dot{\vee}$ denote the `exclusive or' that corresponds to the English expression "either one or the other but not both". Construct a truth table for this connective.

=>

φ	Ψ	$\varphi \dot{\vee} \Psi$
T	T	F
F	T	T
T	F	T
F	F	F

21. Express $\phi \dot{\vee} \psi$ it terms of the basic combinators $\wedge, \vee, \neg$.

=> $(\varphi \vee \Psi) \Rightarrow (\neg\varphi \wedge \Psi) \vee (\varphi \wedge \neg \Psi)$

22. Which of the following pairs of propositions are equivalent?

(a) $\neg(P \vee Q)$, $\neg P \wedge \neg Q$

(b) $\neg P \vee \neg Q$, $\neg(P \vee \neg Q)$

(c) $\neg(P \wedge Q)$, $\neg P \vee \neg Q$

(d) $\neg(P \Rightarrow (Q \wedge R))$, $\neg(P \Rightarrow Q) \vee \neg(P \Rightarrow R)$

(e) $P \Rightarrow (Q \Rightarrow R)$, $(P \wedge Q) \Rightarrow R$

=> Solution:

a)

P	Q	$\neg(P \vee Q)$ *	$\neg P$	$\neg Q$	$(\neg P) \wedge (\neg Q)$ *
T	T	F	F	F	F
F	T	F	T	F	F
T	F	F	F	T	F
F	F	T	T	T	T

Since two columns (*) in truth table are equivalent hence given propositions pair is equivalent.

b)

P	Q	$\neg P$	$\neg Q$	$\neg P \vee \neg Q$ *	$\neg(P \vee \neg Q)$ *
T	T	F	F	F	F
F	T	T	F	F	T
T	F	F	T	F	F
F	F	T	T	T	F

Since two columns (*) in truth table are not equivalent hence given propositions pair is not equivalent.

c)

P	Q	$\neg(P \wedge Q)$ *	$\neg P$	$\neg Q$	$\neg P \vee \neg Q$ *
T	T	F	F	F	F
F	T	T	T	F	T
T	F	T	F	T	T
F	F	T	T	T	T

Since two columns (*) in truth table are equivalent hence given propositions pair is equivalent.

d)

P	Q	R	$Q \land R$	$\neg(P \Rightarrow (Q \land R))*$	$A := \neg(P \Rightarrow Q)$	B:=$\neg(P \Rightarrow R)$	A ∨ B *
T	T	T	T	F	F	F	F
T	T	F	F	T	F	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	F	F	F	F
F	T	F	F	F	F	F	F
F	F	T	F	F	F	F	F
F	F	F	F	F	F	F	F

Since two columns (*) in truth table are equivalent hence given propositions pair is equivalent.

23. Give, if possible, an example of a true conditional sentence for which

(a) the converse is true.

(b) the converse is false.

(c) the contrapositive is true.

(d) the contrapositive is false.

=> Solution:

a) Statement is: = if 2^n-1 is prime then n is prime. This statement is true.
   It converse is: if n is prime then 2^n-1 is also prime. The converse is also true.
b) Statement is: = if a shape is a triangle then it is polygon. The statement is true.
   Its converse is: = if a shape is polygon then it is triangle. The converse is false as a square is polygon but not triangle.
c) Statement is: = if 2^n-1 is prime then n is prime. This statement is true.
   It contrapositive is: if 2^n-1 is not prime then n is not prime. The contrapositive is also true.
d) Statement is: = if there is bat then it is mammal. This statement is true as all bats are considered as mammal.
   If a statement is true then its contrapositive is also logically true. So there is no false contrapositive for a true statement.

The contrapositive of above statement is: if there is no mammal then there is no bat. It is also true.

24. You are in charge of a party where there are young people. Some are drinking alcohol, others soft drinks. Some are old enough to drink alcohol legally, others are under age. You are responsible for ensuring that the drinking laws are not broken, so you have asked each person to put his or her photo ID on the table. At one table are four young people. One person has a beer, another has a Coke, but their IDs happen to be face down so you cannot see their ages. You can, however, see the IDs of the other two people. One is under the drinking age, the other is above it. Unfortunately, you are not sure if they are drinking Seven-up or vodka and tonic. Which IDs and/or drinks do you need to check to make sure that no one is breaking the law?

=> Solution: Information clause:

- Some are drinking alcohol and others soft drink.
- Some are old others are younger
- 4 younger people are sitting on a table with following setting:

	A	B	C	D
Drink	Beer	Coke	Seven up or Vodka	Vodka or seven up
Age	?	?	Underage	Not underage

Which ID attender should check to ensure that no one is breaking drinking law? where law states that underage should not drink alcohol.

=> Answer: Since attender can see ID of C type of person so he needs to check his drink type. And for person A attender need to check ID. Persons B and D are not breaking laws so they do not need to be checked.

25. Compare the logical structure of the previous question with Wason's problem (Exercise 2.3.4(8)). Comment on your answers to those two questions. In particular, identify any logical rules you used in solving each problem, say which one was easier, and why you felt it was

easier.

=> Let rule A's ID check is defined as: $p \Longrightarrow q$

And rule C's drink check is defined as: $\neg q \Rightarrow \neg p$

So question 24 answer can be stated as:

$p \Longrightarrow q\ is\ true$ and $\neg q \Rightarrow \neg p\ also\ true$ i.e. $[(p \Longrightarrow q)\ \wedge (\neg q \Rightarrow \neg p)] = True$

<Exercise 2.4.1>

1. The same kind of argument I just outlined to show that the cubic equation $y = x^3 + 3x + 1$ has a real root, can be used to prove the "Wobbly Table Theorem." Suppose you are sitting in a restaurant at a perfectly square table, with four identical legs, one at each corner. Because the floor is uneven, the table wobbles. One solution is to fold a small piece of paper and insert it under one leg until the table is stable. But there is another solution. Simply by rotating the table you will be able to position it so it does not wobble. Prove this. [WARNING : This is a thinking-outside-the-box question. The solution is simple, but it can take a lot of effort before you find it. This would be an unfair question on a timed exam but is a great puzzle to keep thinking about until you hit upon the right idea.]

=> Solution:

=> Objective: "By rotating a square table on uneven surface it can be ensured that all four legs touch the surface i.e. the distance between surface and any leg is zero".

=> Method: consider a three dimensional space as shown in the figure. Suppose that table is positioned on such 3-D surface. Let's consider four legs of the table as four vertices of the square named as A, B, C, D. Then perform following steps to balance it:

Wobble the table around the z-axis until vertices A, B, C keep touching the surface and only vertex D has distance from surface. Let's consider the distance of D as function: $f(x)$ where 'x' represents angle between two opposite vertices i.e. direction of vertices (say A,C) from top of table. Apply Intermediate Value Theorem to ensure that there is an angle 'x' such that fourth vertex (D) is also touching the surface.

So objective in mathematical form we can write as:

"$\exists x: f(x) = 0$"

=> Steps:

- Assume that center of vertices (A, B, C, D) is at z-axis.
- Starting from initial stage such that three vertices (say A, B,C) have zero distance from the surface (x-axis).
- Define function $f(x)$ which represents signed vertical (y-axis) distance of vertex 'D' from the x-axis
- Start with x=0, let $f(0) > 0$ i.e. vertex D is above surface
- Rotate the vertices say (A, C) around the z-axis such that an opposite pair of vertices (say B, D) has equal vertical (y-axis) distance from surface (x-axis).

Note that initially vertices pair (A, C) has zero y-axis with respect to x-axis. But after this rotation pair (A, C) has y-axis distance < 0.

=> Now rotate the vertices say (B, D) around the z-axis such that pair of vertices (say B, D) has equal vertical (y-axis) distance from surface (x-axis) and vertex 'C' is at surface (vertex 'C' has y-axis value =0). Now three vertices (say B,C, D) have y-axis =0 while vertex 'A' has y-axis value < 0. This situation is same if we rotate vertices by 90^0 i.e. $f\left(\frac{\pi}{2}\right) < 0$.

Thus mathematically we can express the situation as:

$$\left\{if\ f(0) > 0\ then\ f\left(\frac{\pi}{2}\right) < 0\ or\ if\ f(0) < 0\ then\ f\left(\frac{\pi}{2}\right) > 0\right\}$$

The expression results us that if we choose $0 < x < \frac{\pi}{2}$ for both pairs of vertices then objective "$\exists x: f(x) = 0$" is achieved.

<Exercise 2.4.2>

1. Express the following as existence assertions. (Feel free to use a mix of symbols and words.)

(a) The equation $x^3 = 27$ has a natural number solution.

(b) 1,000,000 is not the largest natural number.

(c) The natural number n is not a prime.

=> Solution:

a) $\exists x \in \mathbb{N}: \{x^3 = 27\}$

b) $\exists x \in \mathbb{N}: \{x > 1{,}000{,}000\}$

c) $(\exists p \in \mathbb{N}) and (\exists q \in \mathbb{N}): [if\,(p > 1 \wedge q > 1 \wedge n = pq) \text{ then n is not prime}]$

2. Express the following as `for all' assertions (using symbols and words):

(a) The equation $x^3 = 28$ does not have a natural number solution.

(b) 0 is less than every natural number.

(c) The natural number n is a prime.

=> Solution:

a) $\neg(\exists x \in \mathbb{N}): \{x^3 = 28\}$

b) $(\forall x \in \mathbb{N}): \{0 < x\}$ (note natural number starts are 1,2,3,...

c) $\neg(\exists p \in \mathbb{N}) and (\exists q \in \mathbb{N}): [if\,(p > 1 \wedge q > 1 \wedge n = pq) \text{ then n is not prime}]$ here negation symbol makes 'n' as prime

3. Express the following in symbolic form, using quantifiers for people:

(a) Everybody loves somebody.

(b) Everyone is tall or short.

(c) Everyone is tall or everyone is short.

(d) Nobody is at home.

(e) If John comes, all the women will leave.

(f) If a man comes, all the women will leave.

=> Solution:

a) $(\forall x)(\exists y): L(x, y)$, where L(x,y) denotes that x loves y

b) $(\forall x): \{Tall(x) \vee Short(x)\}$

c) $(\forall x)Tall(x) \vee (\forall x)Short(x)$

d) $(\forall x)\neg H(x)$ where H(x) means 'x' is at home

e) $if\ comes(John) \Rightarrow (\forall x)\{Women(x) \Rightarrow Leaves(x)\}$

f) $(\exists x)\{Man(x) \wedge Comes(x)\} \Rightarrow (\forall y)\{Women(y) \Rightarrow Leaves(y)\}$

4. Express the following using quantifiers that refer (only) to the sets R (real numbers) and N (natural numbers):

(a) The equation $x^2 + a = 0$ has a real root for any real number a.

(b) The equation $x^2 + a = 0$ has a real root for any negative real number a.

(c) Every real number is rational.

(d) There is an irrational number.

(e) There is no largest irrational number. (This one looks quite complicated.)

=> Solution:

a) $(\forall a \in \mathbb{R})(\exists x \in \mathbb{R}): \{x^2 + a = 0\}$

b) $(\forall a \in \mathbb{R})[(a < 0) \Rightarrow (\exists x \in \mathbb{R})(x^2 + a = 0)]$

c) $(\forall x \in \mathbb{R})(\exists n \in \mathbb{N})(\exists m \in \mathbb{N}): \{m = nx \vee m = -nx\}$ (note natural numbers 'n' and 'm' are written in this way as natural numbers can only be added or subtracted. So m=nx means x=m/n.)

d) $(\exists x \in \mathbb{R})(\forall n \in \mathbb{N})(\forall m \in \mathbb{N}): \{m \neq nx \wedge m \neq -nx\}$

e) $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R}): \{(y > x) \wedge [(\forall m \in \mathbb{N})(\forall m \in \mathbb{N})(m \neq -ny)]\}$ where $(\forall x \in \mathbb{R})$ means given a real number, $(\exists y \in \mathbb{R})(y > x)$ means there is a real number such that y is not quotient of 'm' and 'n'

5. Let C be the set of all cars, let D(x) mean that x is domestic, and let M(x) mean that x is badly made. Express the following in symbolic form using these symbols:

(a) All domestic cars are badly made.

(b) All foreign cars are badly made.

(c) All badly made cars are domestic.

(d) There is a domestic car that is not badly made.

(e) There is a foreign car that is badly made.

=> Solution:

a) $(\forall x \in C)[D(x) \Longrightarrow M(x)]$

b) $(\forall x \in C)[\neg D(x) \Longrightarrow M(x)]$

c) $(\forall x \in C)[M(x) \Longrightarrow D(x)]$

e) $(\exists x \in C)[D(x) \wedge \neg M(x)]$

f) $(\exists x \in C)[\neg D(x) \wedge M(x)]$

6. Express the following sentence symbolically, using only quantifiers for real numbers, logical connectives, the order relation $<$, and the symbol $Q(x)$ having the meaning `x is rational':

You can find a rational number between any two unequal real numbers.

=> Solution:

$$(\forall a \in \mathbb{R})(\forall b \in \mathbb{R}): \{(a < b) \ \Longrightarrow \exists x: \{Q(x) \wedge \ a < x < b\}\}$$

7. Express the following famous statement (by Abraham Lincoln) using quantifiers for people and times: "You may fool all the people some of the time, you can even fool some of the people all of the time, but you cannot fool all of the people all the time."

=> Solution: Let $F(p,t)$ means that "You can make fool a person 'p' at time t" then:

$$(\exists t)(\forall p)F(p,t) \wedge (\exists p)(\forall t)F(p,t) \wedge \neg[(\forall p)(\forall t)F(p,t)]$$

8. A US newspaper headline read, "A driver is involved in an accident every six seconds." Let x be a variable to denote a driver, t a variable for a six second interval, and let A(x; t) be the property that x is in an accident during interval t. Express the headline (as written) in logical notation.

=> Answer: "A driver is involved in an accident every six seconds"

$$(\exists x)(\forall t)A(x,t)$$

"For every six second a driver is involved in an accident"

$$(\forall t)(\exists x)A(x,t)$$

Exercises 2.4.3

1. Show that $\neg[\exists x A(x)]$ is equivalent to $\forall x[\neg A(x)]$.

Ex1) We begin by assuming that $\neg[\exists x A(x)]$. That is, we assume it is not the case that $\exists x A(x)$ is true. If it is not the case that at least one x satisfies $A(x)$, what must happen is that all x must fail to satisfy $A(x)$. In other words, for all x, $\neg A(x)$ must be true. In symbols, this can be written $\forall x[\neg A(x)]$.

Ex2) Give an everyday example to illustrate this equivalence, and verify it by an argument specific to your example.

It is not the case that at least one signal lamp run red lights ⇔ All signal lamps do not run red lights.

Exercise 2.4.4

Prove that the statement

***There is an even prime bigger than* 2 is false.**

So1)

Let $P(x)$ denotes the property " x is a prime" and O(x) the property " x is odd"

$(\exists x > 2)[P(x) \wedge \neg O(x)]$

(This is False.)

$\because$) x is even bigger than 2, then x is divided by 2. The definition of prime is a natural number that has no positive divisors other than 1 and itself. In this case, x always has divisor 2.

Exercises 2.4.5

1. Translate the following sentences into symbolic form using quantifiers. In each case the assumed domain is given in parentheses.

(a) All students like pizza. (All people)

$\Rightarrow$ Let P be the set of all people, $S(x)$: x is student, $L(x)$: x likes pizza.

$(\forall x \in P)(S(x) \Rightarrow L(x))$.

(b) One of my friends does not have a car. (All people)

$\Rightarrow$ Let P be the set of all people, $O(x)$: x is one of my friends, $C(x)$: x has a car.

$(\exists x \in P)(O(x) \wedge \neg C(x))$.

(c) Some elephants do not like muffins. (All animals)

Sol) Let A be the set of all animals, $E(x)$: x is elephants, $M(x)$: x likes muffins.

$(\exists x \in A)(E(x) \wedge \neg M(x))$.

(d) Every triangle is isosceles. (All geometric figures)

Sol) Let G be the set of geometric figures, T(x) means x is triangle, I(x) means x is isosceles.

$\Rightarrow$ Let G be the set of geometric figures, $T(x)$: x is triangle, $I(x)$: x is isosceles.

$(\forall x \in G)(T(x) \Rightarrow I(x))$.

(e) Some of the students in the class are not here today. (All people)

⇒ Let P be the set of all people, $S(x)$: x is student in the class, $T(x)$: x are here today.

$(\exists x \in P)(S(x) \wedge \neg T(x))$.

(f) Everyone loves somebody. (All people)

⇒ x : people, y : people, $L(x, y)$: x loves y.

$(\forall x)(\exists y)L(x, y)$.

(g) Nobody loves everybody. (All people)

⇒ x : people, y : people, $L(x, y)$: x loves y.

$\neg(\exists x)(\forall y)L(x, y)$. Where L(x,y) denotes " x loves y"

(h) If a man comes, all the women will leave. (All people)

⇒ $(\exists x)[Man(x) \wedge Comes(x)] \Rightarrow (\forall x)[Woman(x) \Rightarrow Leaves(x)]$

(i) All people are tall or short. (All people)

⇒ $(\forall x)[Tall(x) \vee Short(x)]$

(j) All people are tall or all people are short. (All people)

⇒ $(\forall x)[Tall(x)] \vee (\forall x)[Short(x)]$

(k) Not all precious stones are beautiful. (All stones)

⇒ $(\exists x)[\Pr ecious(x) \wedge \neg Beautiful(x)]$

(l) Nobody loves me.

$(\forall x)\neg Loveme(x)$

(m) At least one American snake is poisonous. (All snakes)

$(\exists x)[American(x) \wedge poisonous(x)]$

(n) At least one American snake is poisonous. (All animals)

$(\exists x)[American(x) \wedge snake(x) \wedge poisonous(x)]$

2. Which of the following are true? The domain for each is given in parentheses.

a) $\forall x(x+1 \geq x)$ (Real numbers)

True

b) $\exists x(2x+3=5x+1)$ (Natural numbers)

False (Not exist)

c) $\exists x(x^2+1=2^x)$ (Real numbers)

True $x=0,1$

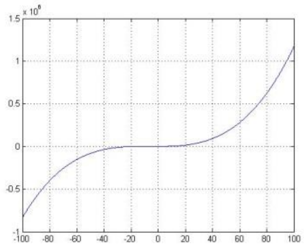
d) $\exists x(x^2=2)$ **(Rational numbers)**

False. $x=\pm\sqrt{2}$ is irrational number.

e) $\exists x(x^2=2)$ **(Real numbers)**

$x=\pm\sqrt{2}$. True.

f) $\forall x(x^3+17x^2+6x+100 \geq 0)$ **(Real numbers)**

False. 

(g) $\exists x(x^3+x^2+x+1 \geq 0)$ **(Real numbers)**

Sol) True. When $x=1$, $4 \geq 0$.

(h) $\forall x \exists y(x+y=0)$ **(Real numbers)**

Sol) True. For every real number x there is a real number y such that x+y=0. This states that every real number has an additive inverse. $y=-x$

(i) $\exists x \forall y(x+y=0)$ **(Real numbers)**

Sol) False. All real number y, x+y=0 is not exist. There is a real number x such that for every real number y, x+y=0.

(j) $\forall x \exists ! y(y = x^2)$ **(Real numbers)**

Sol) $\exists ! y$ means " There exists a unique something such that ..".

True. For all real number x there is unique y. Because x^2 is positive.

(k) $\forall x \exists ! y(y = x^2)$ **(Natural numbers)**

Sol) $\exists ! y$ means " There exists a unique something such that ..".

True. For all natural number x there is unique (only one) y.

(l) $\forall x \exists y \forall z(xy = xz)$ **(Real numbers)**

Sol) false

$\exists y \forall z(y = z)$.

(m) $\forall x \exists y \forall z(xy = xz)$ **(Prime numbers)**

Sol) false

$\exists y \forall z(y = z)$.

(n) $\forall x \exists y(x \geq 0 \Rightarrow y^2 = x)$ **(Real numbers)**

Sol) True.

For every real number x, there is a real number y such that if $x \geq 0$, then $y^2 = x$.

$y = \sqrt{x}(x \geq 0)$.

(o) $\forall x[x < 0 \Rightarrow \exists y(y^2 = x)]$ **(Real numbers)**

Sol) False

$x = -1$ $y = i$ not real number.

(p) $\forall x[x < 0 \Rightarrow \exists y(y^2 = x)]$ **(Positive real numbers)**

Sol) True. Because for positive real numbers there are not $x < 0$.

So antecedent equal is false therefore conditional is always true.

3. Negate each of the symbolic statements you wrote in Question 1, putting your answers in positive form. Express each negation in natural, idiomatic English.

(a) All students like pizza. (All people)

⇒ Let P be the set of all people, $S(x)$: x is student, $L(x)$: x likes pizza.

$\neg(\forall x \in P)(S(x) \Rightarrow L(x)) = (\exists x \in P)[S(x) \wedge \neg L(x)]$.

(b) One of my friends does not have a car. (All people)

⇒ Let P be the set of all people, $O(x)$: x is one of my friends, $C(x)$: x has a car.

$\neg(\exists x \in P)(O(x) \wedge \neg C(x)) = (\forall x \in P)(O(x) \Rightarrow C(x))$.

(c) Some elephants do not like muffins. (All animals)

Sol) Let A be the set of all animals, $E(x)$: x is elephants, $M(x)$: x likes muffins.

$(\forall x \in A)(E(x) \Rightarrow M(x))$.

(d) Every triangle is isosceles. (All geometric figures)

Sol) Let G be the set of geometric figures, T(x) means x is triangle, I(x) means x is isosceles.

⇒ Let G be the set of geometric figures, $T(x)$: x is triangle, $I(x)$: x is isosceles.

$\neg(\forall x \in G)(T(x) \Rightarrow I(x)) = (\exists x \in G)(T(x) \wedge \neg I(x))$.

(e) Some of the students in the class are not here today. (All people)

⇒ Let P be the set of all people, $S(x)$: x is student in the class, $T(x)$: x are here today.

$\neg(\exists x \in P)(S(x) \wedge \neg T(x)) = (\forall x \in P)(\neg S(x) \vee T(x))$.

(f) Everyone loves somebody. (All people)

⇒ x : people, y : people, $L(x, y)$: x loves y.

$\neg(\forall x)(\exists y)L(x, y) = (\exists x)(\forall y)\neg L(x, y)$.

(g) Nobody loves everybody. (All people)

$\Rightarrow$ x : people, y : people, $L(x,y)$: x loves y.

$\neg(\exists x)(\forall y)L(x,y)$. Where L(x,y) denotes " x loves y"

$(\exists x)(\forall y)L(x,y)$

(h) If a man comes, all the women will leave. (All people)

$\Rightarrow$ $\begin{array}{l}\neg\{(\exists x)[Man(x)\wedge Comes(x)]\Rightarrow(\forall x)[Woman(x)\Rightarrow Leaves(x)]\}\\=(\forall x)[Man(x)\wedge Comes(x)]\wedge(\exists x)[Woman(x)\wedge\neg Leaves(x)]\end{array}$

(i) All people are tall or short. (All people)

$\Rightarrow$ $\neg(\forall x)[Tall(x)\vee Short(x)]=(\exists x)[Short(x)\wedge Tall(x)]$

(j) All people are tall or all people are short. (All people)

$\Rightarrow$ $(\forall x)[Tall(x)]\vee(\forall x)[Short(x)]=(\exists x)[Short(x)]\wedge(\exists x)[Tall(x)]$

(k) Not all precious stones are beautiful. (All stones)

$\Rightarrow$ $\neg(\exists x)[\Pr ecious(x)\wedge\neg Beautiful(x)]=(\forall x)[\Pr ecious(x)\Rightarrow Beautiful(x)]$

(l) Nobody loves me.

$\neg[(\forall x)\neg Loveme(x)]=(\exists x)Loveme(x)$

(m) At least one American snake is poisonous. (All snakes)

$\neg(\exists x)[American(x)\wedge poisonous(x)]=(\forall x)[American(x)\Rightarrow\neg poisonous(x)]$

(n) At least one American snake is poisonous. (All animals)

$\neg(\exists x)[American(x)\wedge snake(x)\wedge poisonous(x)]=(\forall x)[\neg American(x)\vee\neg snake(x)\vee\neg poisonous(x)]$

4. Negate each of the statements in Question 2, putting your answers in positive form.

Sol)

(a) $\exists x(x+1<x)$ (Real numbers)

(b) $\forall x(2x+3\neq 5x+1)$ (Natural numbers)

(c) $\forall x(x^2+1\neq 2^x)$ (Real numbers)

(d) $\forall x(x^2\neq 2)$ (Rational numbers)

(e) $\forall x(x^2\neq 2)$ (Real numbers)

(f) $\exists x(x^3+17x^2+6x+100<0)$ (Real numbers)

(g) $\forall x(x^3+x^2+x+1<0)$ (Real numbers)

(h) $\exists x\forall y(x+y\neq 0)$ (Real numbers)

(i) $\forall x\exists y(x+y\neq 0)$ (Real numbers)

(j) $\exists x\forall !y(y\neq x^2)$ (Real numbers)

(k) $\exists x\forall !y(y\neq x^2)$ (Natural numbers)

(l) $\exists x\forall y\exists z(xy\neq xz)$ (Real numbers)

(m) $\exists x\forall y\exists z(xy\neq xz)$ (Prime numbers)

(n) $\exists x\forall y(x\geq 0\wedge y^2\neq x)$ (Real numbers)

(o) $\exists x[x<0\wedge\forall y(y^2\neq x)]$ (Real numbers)

(p) $\exists x[x<0\wedge\forall y(y^2\neq x)]$ (Positive real numbers)

5. Negate the following statements and put each answer into positive form:

(a) $(\forall x\in\mathrm{N})(\exists y\in\mathrm{N})$ **(x+y=1)**

Sol) $(\exists x\in\mathrm{N})(\forall y\in\mathrm{N})(x+y\neq 1)$.

(b) $(\forall x>0)(\exists y<0)(x+y=0)$ **(where x, y are real number variables)**

Sol) $(\exists x>0)(\forall y<0)(x+y\neq 0)$

(c) $\exists x(\forall\varepsilon>0)(-\varepsilon<x<\varepsilon)$ **(where x, ε are real number variables)**

Sol) $\forall x(\exists\varepsilon>0)(x\leq-\varepsilon\vee x\geq\varepsilon)$.

(d) $(\forall x \in N)(\forall y \in N)(\exists z \in N)(x+y=z^2)$ **(x+y=** z^2 **)**

Sol) $(\exists x \in N)(\exists y \in N)(\forall z \in N)(x+y \neq z^2)$.

6. Give a negation (in positive form) of the quotation which you met in Exercise 2.4.2(7): "You may fool all the people some of the time, you can even fool some of the people all of the time, but you cannot fool all of the people all the time."

Sol) Let F(x,t) mean " You can fool person p at the time t."

Quotation is :

$\exists t \forall p F(p,t) \wedge \exists p \forall t F(p,t) \wedge \neg \forall p \forall t F(p,t)$.

Negation

$\forall t \exists p \neg F(p,t) \vee \forall p \exists t \neg F(p,t) \vee \forall p \forall t F(p,t)$.

Quotation is :

$\exists t \forall p F(p,t) \wedge \forall t \exists p F(p,t) \wedge \neg \forall t \forall p F(p,t)$.

Negation

$\forall t \exists p \neg F(p,t) \vee \exists t \forall p \neg F(p,t) \vee \forall p \forall t F(p,t)$.

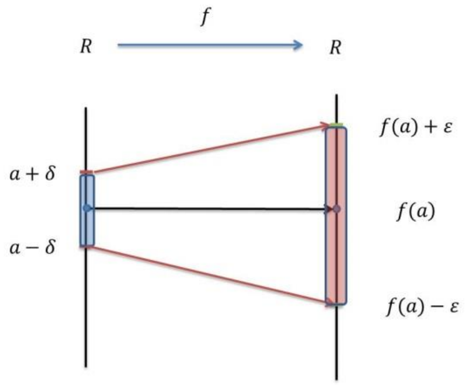
At any time, there is someone you can't fool or For every person, you can't always fool them. Or you can fool all the people, all the time.

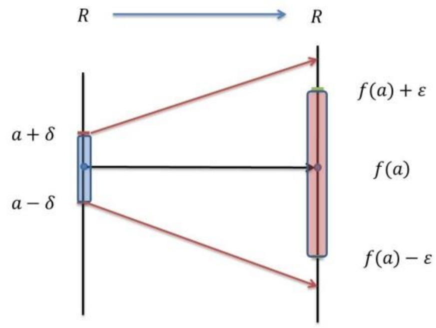
7. The standard definition of a real function f being continuous at a point x=a is

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x)[|x-a| < \delta \Rightarrow |f(x)-f(a)| < \varepsilon]$$

Write down a formal definition for f being discontinuous at a. Your definition should be in positive form.

Sol) $(\exists \varepsilon > 0)(\forall \delta > 0)(\exists x)[|x-a| < \delta \wedge \neg [|f(x)-f(a)| < \varepsilon]]$





Exercises 3.2.1

1. Prove that $\sqrt{3}$ is irrational.

Sol)

Assume that $\sqrt{3}$ is rational.

$\sqrt{3} = \frac{p}{q}$ $, where\ p\ and\ q$ have no common factors and both are natural numbers.

$$\sqrt{3} = \frac{p}{q} \leftrightarrow 3 = \frac{p^2}{q^2}$$
$$\leftrightarrow p^2 = 3q^2$$

p^2 divide by 3 but 3 is prime
$\rightarrow p$ divide by 3

so $p = 3r \rightarrow p^2 = 9r^2$

From $p^2 = 3q^2$, $9r^2 = 3q^2$ so $q^2 = 3r^2$

q^2 divide by 3 but 3 is prime
$\rightarrow q$ divide by 3

Since, $p\ and\ q$ have no common factors contradiction!

Hence, our assumption that $\sqrt{3}$ was rational must be false. It means $\sqrt{3}$ must be irrational.

2. Is it true that $\sqrt{N}$ is irrational for every natural number N?

Sol)

No, there is counter example, which is N=4.

Clearly $\sqrt{4} = 2$ is rational

3. If not, then for what N is $\sqrt{N}$ irrational? Formulate and prove a result of the form " $\sqrt{N}$ irrational if and only if N ..."]

Prove)

If $\sqrt{N}$ is a rational number, then N is perfect square.

Assume that $\sqrt{N}$ is a rational number B/A ,which is the lowest terms.

$$\sqrt{N} = \frac{B}{A} \leftrightarrow \sqrt{N}\sqrt{N} = \frac{B}{A}\sqrt{N} \leftrightarrow \frac{NA}{B} = \sqrt{N}$$
$$\therefore \frac{B}{A} = \frac{NA}{B}$$

Since B/A is the lowest terms, there is an integer C such that BC=NA and AC=B

Since AC=B, C=B/A, that is, B/A is an integer, so $\sqrt{N}$ is an integer and N is a perfect square.

Taking the contrapositive,

If N is not a perfect square, $\sqrt{N}$ is irrational.

Exercises 3.3.1

Let *r, s* be irrationals. For each of the following, say whether the given number is necessarily irrational, and prove your answer.

1. $r+3$

sol) Yes. Suppose $r+3$ is rational. Then $\begin{array}{l} r+3=\dfrac{p}{q} \text{ where, } p \text{ and } q \in N \\ r=\dfrac{p}{q}-3=\dfrac{p-3q}{q} \in Q \quad \text{Contradiction!} \end{array}$

2. $5r$

sol) Yes. Suppose $5r$ is rational. Then $\begin{array}{l} 5r=\dfrac{p}{q} \text{ where, } p \text{ and } q \in N \\ r=\dfrac{p}{5q} \in Q \quad \text{Contradiction!} \end{array}$

3. $r+s$

If $r=\sqrt{2}$, $s=10-\sqrt{2}$ then $r+s=10$ rational!

4. rs

If $r=\sqrt{2}$, $s=\sqrt{2}$ then $rs=2$ rational!

5. $\sqrt{r}$

Sol) Yes. Suppose $\sqrt{r}$ is rational. Then $\begin{array}{l} \sqrt{r}=\dfrac{p}{q} \text{ where, } p \text{ and } q \in N \\ r=\left(\dfrac{p}{q}\right)^2 \in Q \quad \text{Contradiction!} \end{array}$

6. r^s

Sol) r, s is irrational $\rightarrow r^s$ is irrational

If $\sqrt{2}^{\sqrt{2}}$ is rational $\rightarrow \text{let } r=s=\sqrt{2},$ is disproved

If $\sqrt{2}^{\sqrt{2}}$ is irrational $\rightarrow r=\sqrt{2}^{\sqrt{2}}, s=\sqrt{2} \Rightarrow r^s=\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}=\sqrt{2}^2=2$ hence, disproved

So, r, s is irrational $\rightarrow r^s$ is rational

Exercises 3.3.2

1. Explain why proving $\phi \Rightarrow \varphi$ and $\varphi \Rightarrow \phi$ establishes the truth of $\phi \Leftrightarrow \varphi$.

Sol) if both ϕ and φ are true, then $\phi \Rightarrow \varphi$ and $\varphi \Rightarrow \phi$ is true.

Also, if both ϕ and φ are false, then $\phi \Rightarrow \varphi$ and $\varphi \Rightarrow \phi$ is true.

Thus, proving $\phi \Rightarrow \varphi$ and $\varphi \Rightarrow \phi$ establish the truth of $\phi \Leftrightarrow \varphi$.

2. Explain why proving $\phi \Rightarrow \varphi$ and $(\neg\varphi) \Rightarrow (\neg\phi)$ establishes the truth of $\phi \Leftrightarrow \varphi$.

Sol) if both ϕ and φ are true, then $\phi \Rightarrow \varphi$ and $(\neg\varphi) \Rightarrow (\neg\phi)$ is true.

Also, if both ϕ and φ are false, then $\phi \Rightarrow \varphi$ and $(\neg\varphi) \Rightarrow (\neg\phi)$ is true.

In addition, if ϕ is false and φ is true, then $\phi \Rightarrow \varphi$ and $(\neg\varphi) \Rightarrow (\neg\phi)$ is true

Thus, proving $\phi \Rightarrow \varphi$ and $(\neg\varphi) \Rightarrow (\neg\phi)$ cannot establish the truth of $\phi \Leftrightarrow \varphi$.

3. Prove that if five investors split a payout of $2 million, at least one investor receives at least $400,000.

sol) all five investors receive at most $400,000 then cannot split $2 million dollar.

4. Write down the converse of the following condition statements:

If the Dollar falls, the Yuan will rise

If the Yuan will rise, the Dollar falls

If $x < y$ then $-y < -x$. (For x, y real numbers.) T

If $-y < -x$, then $x < y$. T

If two triangles are congruent, they have the same area.T

If two triangles have the same area, then they are congruent.F

The quadratic equation $ax^2 + bx + c = 0$ has a solution whenever $b^2 \geq 4a$. (Where a, b, c, x denote real number and $a \neq 0$.)T

$b^2 \geq 4a$ whenever the quadratic equation $ax^2 + bx + c = 0$ has a solution.F

Let ABCD be a quadrilateral. If the opposite sides of ABCD are pairwise equal, then the opposite angles are pairwise equal.T

If the opposite angles of ABCD are pairwise equal, then the opposite sides are pairwise equal.T

Let ABCD be a quadrilateral. If all four sides of ABCD are pairwise equal, then all four angles are equal.T

If all four angles of ABCE are equal, then all four sides of ABCD are pairwise equal.

If n is not divisible by 3, then $n^2 + 5$ is divisible by 3. (For n a natural number.)T

If $n^2 + 5$ is divisible by 3, then n is not divisible by 3.

6. Let *m* and *n* be integers. Prove that:

(a) If *m* and *n* are even, then *m*+*n* is even.

$m = 2s, n = 2r$ then,

$m + n = 2(s + r) = 2k$

$\therefore$ even

(b) If *m* and *n* are even, then *mn* is divisible by 4.

$m = 2s, n = 2r$ then,

$mn = 4sr$

$\therefore mn$ is dividible by 4

(c) If *m* and *n* are odd, then *m*+*n* is even.

$m = 2s + 1, n = 2r + 1$ then,

$m + n = 2(s + r) + 2 = 2\,(s + r + 1)$

$\therefore$ even

(d) If one of *m* and *n* is even and the other is odd, then *m*+*n* is odd.

$m = 2k, n = 2l + 1$ then,

$m + n = 2(k + l) + 1$

$\therefore$ odd

(e) If one of *m* and *n* is even and the other is odd, then *mn* is even.

$m = 2s, n = 2r + 1$ then,
$mn = 4sr + 2s = 2(2sr + s)$
$\therefore$ even

7. Prove or disprove the statement " An integer *n* is divisible by 12 if and only if n^3 is divisible by 12."

$[\Rightarrow]\ n = 12s \rightarrow n^3 = 12^3 s^3 = 12(12^2 s^3)$ proved
$[\Leftarrow] n^3 = 12k \rightarrow n = 12s$
if $n = 6$ then $6^3 = 216 = 12 \bullet 18 \rightarrow 6 \neq 12s$ disproved

Exercise 3.4.1

1. Prove or disprove the statement "All birds can fly."

Ans)

Disprove: counter example: duck is a bird, and duck cannot fly.

2. Prove or disprove the claim $(\forall x, y \in \Re)[(x - y)^2 > 0]$.

Ans)

Disprove: counter example: when x=y.

3. Prove that between any two unequal rationals there is a third rational.

Ans)

Prove:

put x=p/q, y=r/s

put z=(x+y)/2

it is always true that x<z<y.

4. Say whether each of the following is true or false, and support your decision by a proof:

(a) There exist real numbers x and y such that x+y=y.

Ans) T (when x=0)

(b) $\forall x \exists y (x+y=0)$ **(where x, y are real number variables).**

Ans) T (x=n, y=-n)

(c) $(\exists m \in \mathbb{N})(\exists n \in \mathbb{N})(3m+5n=12)$

Ans) F (It is shown in the following that all the cases becomes false: F when m=1,n=1, F when m=1,n=2, F when m=1,n>2, F when m=2,n=1, F when m=3,n=1, F when m>3,n=1: for all cases, it is F)

(d) For all intergers a,b,c, if a divides bc (without remainder), then either a divides b or a divides c.

Ans) F (a=6,b=4,c=9)

(e) The sum of any five consecutive integers is divisible by 5 (without remainder).

Ans) T ($(\forall n)[n+(n+1)+(n+2)+(n+3)+(n+4)] = (\forall n)[5(n+2)]$ and $5 | 5(n+2)$)

(f) For any integer n, the number n^2+n+1 **is odd.**

Ans) T ($n^2+n+1 = n(n+1)+1$. Here, n(n+1) is even for any n integer n. Hence, n(n+1)+1 is odd.)

(g) Between any two distinct rational numbers there is a third rational number.

Ans) T (see the problem # 3)

(h) For any real numbers x,y, if x is rational and y is irrational, then x+y is irrational.

Ans) T.

We use proof by contradiction.

This is expressed as $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})[Rational(x) \wedge \neg Rational(y) \Rightarrow \neg Rational(x+y)]$

First, assume that $(\exists x \in \mathbb{R})(\exists y \in \mathbb{R})[Rational(x) \wedge \neg Rational(y) \Rightarrow Rational(x+y)]$

Let x=p/q and x+y=r/s, then we found that y is rational. This contradict the assumption. Therefore, the argument is True.

(i) For any real numbers x,y, if x+y is irrational, then at least one of x,y is irrational.

Ans) T

Proof by contradiction

$(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})[\neg Rational(x+y) \Rightarrow \{\neg Rational(x) \vee \neg Rational(y)\}]$

Assume $(\exists x \in \mathbb{R})(\exists y \in \mathbb{R})[\neg Rational(x+y) \Rightarrow \{Rational(x) \wedge Rational(y)\}]$.

Let x=p/q, y=r/s. We find that x+y is not any form of a/b. It contradicts the assumption.

(j) For any real numbers x,y, if x+y is rational, then at least one of x,y is rational.

Ans) F (counter example: put x=-y=sqrt(2))

5. Prove or disprove the claim that there are integers m,n such that m^2+mn+n^2 is a perfect square.

Ans) T (m=0 or n=0)

6. Prove that for any positive m there is a positive integer n such that mn+1 is a perfect square.

Ans) T (n=m-2 or n=m+2)

7. Show that there is a quadratic $f(n)=n^2+bn+c$, with positive integer coefficients b,c, such that f(n) is composite (i.e., not prime) for all positive integers n.

Ans) $(n+k)(n+l)=n^2+(k+l)n+kl$ for positive k and l.

8. Prove that for any finite collection of points in the plane, not all collinear, there is a triangle having three of the points as its vertices, which contains none of the other points in its interior.

Ans) Proof by contradiction

Assume that all the triangles contain a point in its interior. Then, any triangle containing one point inside, can become three triangles by connecting lines from the three vertices to the point inside. And the tree triangles do not contain any point inside.

9. Prove that if every even natural number greater than 2 is a sum of two primes (the Goldbach Conjecture), then every odd natural number greater than 5 is a sum of three primes.

Ans)

2n=p+q (n>1) => 2n+3=p+q+3 (here, 3 is a prime number).

Exercise 3.5.1

In the above proof:

1. Write own the statement A(n) which is being proved by induction.

$$A(n): 1+2+...+n = \frac{1}{2}n(n+1)$$

2. Write down A(1), the initial step.

$$A(1): 1 = \frac{1}{2}1 \cdot 2$$

3. Write down the statement $(\forall n \in \mathbb{N})[A(n) \Rightarrow A(n+1)]$**, the induction step.**

$$\sum_{1}^{n} k = \frac{1}{2}n(n+1) \Rightarrow \sum_{1}^{n+1} k = \frac{1}{2}(n+1)(n+2)$$

Exercise 3.5.2

1. Use the method of induction to prove that the sum of the first n odd numbers is equal to n^2.

Ans)

$$A(n): \sum_{k=1}^{n} 2k-1 = n^2$$

Then $A(n+1): \sum_{k=1}^{n+1} 2k-1 = \sum_{k=1}^{n} 2k-1+2\left(n+1\right)-1 = n^2+2n+1 = (n+1)^2$

2. Prove the following by induction:

(a) 4^n-1 is divisible by 3.

Ans)

$$A(n): 4^n - 1 = 3k$$

Then, $A(n+1): 4^{n+1}-1=4\cdot 4^{n}-1=4(3k+1)-1=3(4k+1)$

(b) (n+1)!>2^n+3 for all $n\geq 5$**.**

Ans)

$$((n+1)+1)!=(n+2)!=(n+2)(n+1)!>(n+2)2^{n+3}=2^{n+4}+n\cdot 2^{n+3}>2^{n+4}$$

3. Prove the following by induction.

(a) $\forall n\in\mathbb{N}:\sum_{r=1}^{n}r^{2}=\frac{1}{6}n(n+1)(2n+1)$

$$\sum_{r=1}^{n+1}r^{2}=\sum_{r=1}^{n}r^{2}+(n+1)^{2}=\frac{1}{6}n(n+1)(2n+1)+(n+1)^{2}=\frac{1}{6}(n+1)\{n(2n+1)+6(n+1)\}$$
$$=\frac{1}{6}(n+1)(n+2)(2n+3)$$

(b) $\forall n\in\mathbb{N}:\sum_{r=1}^{n}2^{r}=2^{n+1}-2$

$$\sum_{r=1}^{n}2^{r}=\sum_{r=1}^{n}2^{r}+2^{n+1}=2^{n+1}-2+2^{n+1}=2^{n+2}-2$$

(c) $\forall n\in\mathbb{N}:\sum_{r=1}^{n}r\cdot r!=(n+1)!-1$

$$\sum_{r=1}^{n+1}r\cdot r!=\sum_{r=1}^{n}r\cdot r!+(n+1)(n+1)!=(n+1)!-1+(n+1)(n+1)!=(n+2)!-1$$

4. Generalize Gauss's idea to prove the theorem without recourse to the method of induction.

Ans)

1+2+…+n=N

$n+(n-1)+\ldots+1=N$

$(n+1)+(n+1)+\ldots+(n+1)=2N$

$N=n(n+1)/2$

Exercises 4.1.1

1. The Hilbert Hotel scenario is as before, but this time, two guests arrive at the already full hotel. How can they be accommodated (in separate rooms) without anyone having to be ejected?

The clerk moves everyone into the next room. So the occupant of Room 1 moves into Room 2, the occupant of Room 2 moves into Room 3, and so on throughout the hotel. And then repeat this one more again. When the clerk has done that, Room 1 and Room 2 are empty. The clerk puts two guests in those rooms.

2. This time, the desk clerk faces an even worse headache. The hotel is full, but an infinite tour group arrives, each group member wearing a badge that says "HELLO, I'M N", for N=1,2,3,... Can the clerk find a way to give all the new guests a room to themselves, without having to eject any of the existing guests? How?

The clerk moves everyone into the next room. So the occupant of Room 1 moves into Room 2, the occupant of Room 2 moves into Room 3, and so on throughout the hotel. And then repeat this infinite time. When the clerk has done that, the number of empty room is infinite. The clerk puts infinite guests in those rooms.

Exercises 4.1.2

1. Express as concisely and accurately as you can the relationship between $b|a$ and a/b.

⇒ $b|a$ is something to do with two integers, tells you whether two integers are in a certain relationship. The result is true or false. On the other hand, a/b is a notation for a specific number.

2. Determine whether each of the following is true or false. Prove your answers.

Remember the condition we have to check is $b|a$ iff $\exists q[a=bq]$ where $b \neq 0$.

(a) 0|7 $\Rightarrow$ We can't consider the visibility by a zero number. Thus, this one doesn't work.

(b) 9|0 $\Rightarrow$ True. q is 0.

(c) 0|0 $\Rightarrow$ Likewise (a), this one also doesn't work.

(d) 1|1 $\Rightarrow$ True q is 1.

(e) 7|44 $\Rightarrow$ This one doesn't work. Because 7 simply doesn't divide into 44.

(f) 7|(-42) $\Rightarrow$ True q is (-6)

(g) (-7)|49 $\Rightarrow$ True q is (-7)

(h) (-7)|(-56) $\Rightarrow$ True q is 8.

(i) 2708|569401 $\Rightarrow$ 2708 is even number and 569401 is odd number. Thus, we cannot have an even number dividing an odd number.

(j) ($(\forall n \in N)[2n|n^2] \Rightarrow$ Assume n is 1. Then 2|1 is not true.

(k) $(\forall n \in Z)[2n|n^2]$ $\Rightarrow$ Same (j).

(l) $(\forall n \in Z)[1|n]$ $\Rightarrow$ True q is n.

(m) $(\forall n \in N)[n|0]$ $\Rightarrow$ True q is 0.

(n) $(\forall n \in Z)[n|0]$ $\Rightarrow$ Likewise (a), if $n=0$, then this one also doesn't work.

(o) $(\forall n \in N)[n|n]$ $\Rightarrow$ True.

(p) $(\forall n \in Z)[n|n]$ $\Rightarrow$ Likewise (n), if $n=0$, then this one also doesn't work.

Exercises 4.1.3

1. Prove all the parts of Theorem 4.1.3.

(i) $a|0$, $a|a$

=> Using $b|a$ iff $\exists q[a=bq]$ where $b\neq 0$.

$a|0$ iff $\exists q[0=aq]$. Since $a\neq 0$, $q=0$. $a|a$ iff $\exists q[a=aq]$. $q=1$.

(ii) $a|1$ **iff** $a=\pm 1$**;**

=> $a|1$ iff $\exists q[1=aq]$. If a is not ± 1, then q will be $1/q$. This is fraction number. Thus, $a=1$ and $q=1$ or $a=-1$ and $q=-1$.

(iii) if $a|b$ **and** $c|d$, **then** $ac|bd$ **for** $c\neq 0$

=> $a|b \Leftrightarrow b=a\times q_1$ and $c|d \Leftrightarrow d=c\times q_2$. Then, $bd=ac\times(q_1q_2)=ac\times q_3 \Leftrightarrow ac|bd$

(iv) if $a|b$ **and** $b|c$, **then** $a|c$ **for** $b\neq 0$

=> $a|b \Leftrightarrow b=a\times q_1$ and $b|c \Leftrightarrow c=b\times q_2$. Then, $c=b\times q_2=a\times q_1\times q_2=a\times q_3 \Leftrightarrow a|c$

(v) [$a|b$ **and** $b|a$ **] if and only if** $a=\pm b$**.**

2. Prove that every odd number is of one of the forms $4n+1$ **or** $4n+3$**.**

⇒ Let p be an odd number. Then, $p=2K+1$, where $K=0,1,\cdots,L,\cdots$. obviously, we have

$$p=2K+1=2\left(\frac{2K}{2}\right)+1=4\left(\frac{K}{2}\right)+1=4K_1+1$$

or

$$p=2K+1=2\left(\frac{2K-2}{2}+1\right)+1=4\left(\frac{K-1}{2}\right)+3=4K_2+3.$$

Thus, the proposition "every odd number is of one of the forms $4n+1$ or $4n+3$" is true.

3. Prove that for any integer n, $n+2$, $n+4$ is divisible by 3.

$\Rightarrow$ Let $n=3K$, then n is divisible by 3. Let $n=3K+1$, then $n+2=3(K+1)$ is divisible by 3. Let $n=3K+2$, then $n+4=3K+6=3(K+2)$ is divisible by 3. Therefore, the propositions "for any integer n, at least one of the integers n, $n+2$, $n+4$ is divisible by 3" is true.

4. Prove that if a is an odd integer, then $24\,|\,a(a^2-1)$. [Hint: Look at the example that followed Theorem 4.1.2.]

=> Let a be $2K+1$, where $K=0,1,\cdots,L,\cdots$. Then, we have

$$a\left(a^2-1\right)=(2K+1)(2K)(2K+2)=4K(K+1)(2K+1).$$

In addition, $24=4\times 6$. Thus, we have

$$\frac{a\left(a^2-1\right)}{24}=\frac{4K(K+1)(2K+1)}{4\times 6}=\frac{K(K+1)(2K+1)}{6}.$$

Obviously, $K(K+1)$ is always divisible by 2. Next, it is easy to show that at least one of the integers K, $K+1$, $2K+1$ is divisible by 3. Thus, $K(K+1)(2K+1)$ must be divisible by 6. Therefore, the proposition "if a is an odd integer, then $24\left|a\left(a^2-1\right)\right.$" is true.

5. Prove the following version of the Division Theorem. Given integers a,b with $b\neq 0$, there are unique integers q and r such that

$$a=qb+r \textbf{ and } -\frac{1}{2}|b|<r\leq\frac{1}{2}|b|$$

[Hint: Write $a=q'b+r'$ where $0\leq r'<|b|$. If $0\leq r'\leq\frac{1}{2}|b|$, let $r=r'$, $q=q'$. If $\frac{1}{2}|b|<r'<|b|$, let $r=r'-|b|$, and set $q=q'+1$ if $b>0$ and $q=q'-1$ if $b<0$.]

Generalized Division Theorem

Let a,b be integers, $b\neq 0$. Then there are unique integer q,r such that $a=q\cdot b+r$ and $0\leq r<|b|$

Exercises 4.1.4

1. Does the following statement accurately define prime numbers? Explain your answer. If the statement does not define the primes, modify it so it does.

$$p \text{ **is prime iff** } (\forall n \in N)[(n \mid p) \Rightarrow (n = 1| \vee n = p)]$$

2. A classic unsolved problem in number theory asks if there are infinitely many pairs of 'twin prime', pairs of primes separated by 2, such as 3 and 5, 11 and 13, or 71 and 73. Prove that the only prime triple (i.e. three primes, each 2 from the next) is 3, 5, 7.

⇒ $(p, p+2, p+4)$

Any integer n, $n+2$, $n+4$ is divisible by 3 [Reference exercise 4.1.3 problem 3]. Hence, the only prime triple is (3,5,7)

3. It is a standard result about primes (known as Euclid's Lemma) that if p is prime, then whenever p divides a product ab, p divides at least one of a, b. Prove the converse, that any natural number having this property (for any pair a, b) must be prime.

⇒ Converse : If $p \mid ab$ implies that $p \mid a$ or $p \mid b$, then p is prime.

Let me use contrapositive. If p is not prime, then it must be composite, i.e., p can be ab. Then $p \nmid a$ and $p \nmid b$.

Exercises 4.1.5

1. Try to prove Euclid's Lemma. If you do not succeed, move on to the following exercise.

If p is prime and $p \mid ab$, then $p \mid a$ or $p \mid b$.

(1) If $ab = 0$, then $a = 0$ or $b = 0$, and $p \mid 0$, so the theorem is true for this case.

(2) For the case where $ab \neq 0$, I will prove the theorem by contradiction. So, let's assume that p

neither divides a nor b.

(3) Now, if $p|ab$, then ab can be written as a product which contains p as factor. If p is neither a divisor of a nor of b, then the only way to do this is to find integer factorizations such $a = cd, b = ef$, and $de = p$. If such a factorization were not possible, then ab would not be divisible by p. Since p is prime and $de = p$, then either is $d = \pm 1 \wedge e = \pm p$, or $d = \pm p \wedge e = \pm 1$. In the first case, $b = ef = \pm pf$ and therefore $p|b$. In the second case, $a = cd = \pm cp$ and therefore $p|a$. In both cases, we have a contradiction to our assumption (2) that p divides neither a nor b. So, the assumption is false and therefore the theorem is proved.

2. You can find proofs of Euclid's Lemma in most textbooks on elementary number theory, and on the Web. Find a proof and make sure you understand it. If you find a proof on the Web, you will need to check that it is correct. There are false mathematical proofs all over the Internet. Though false proofs on Wikipedia usually get corrected fairly quickly, they also occasionally become corrupted when a well-intentioned contributor makes an attempted simplification that renders the proof incorrect. Learning how to make good use of Web resources is an important part of being a good mathematical thinker.

3. A fascinating and, it turns out, useful (both within mathematics and for real world applications) result about prime numbers is Fermat's Little Theorem: If p is prime and a is natural number that is not a multiple of p, then $p|(a^{p-1}-1)$. Find (in a textbook or on the Web) and understand a proof of this result. (Again, be wary of mathematics you find on websites of unknown or non-accredited authorship.)

Exercise 4.2.1

1. Take the integers, Z, as a given system of numbers. You want to define a larger system, Q, that extends Z by having a quotient a/b for every pair a, b of integers, $b \neq 0$. How would you go about defining such a system? In particular, how would you respond to the question, "What is the quotient a/b?" (You cannot answer in terms of actual quotients, since until Q has been defined, you don't have quotients.)

=> The definition of rational number is like this :

$$(\forall a \in Z)(\forall b \in Z)((\gcd(a,b)=1) \wedge (b \neq 0) \to (\frac{a}{b} \, is \, rational \; number(Q)))$$

So, if there are quotient c about $\frac{a}{b}$, it can be represented as $\frac{a/c}{b/c}$ and again it is rational number. That is, $\frac{a}{b} = \frac{a/c}{b/c}$ and it is rational number.

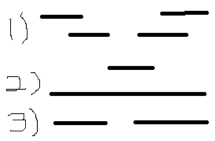
2. Find an account of the construction of the rationals from the integers and understand it, once again being cautious about mathematics found on the Internet.

=> See the answer of problem 1 (Exercise 4.2.1)

Exercise 4.3.1

1. Prove that the intersection of two intervals is again an interval. Is the same true for unions?

=> we can think about using diagram : there are 3 cases



=> Let $A=(a,b)$ and $C=(c,d)$. Then

$$\begin{aligned} A \cap C &= \{x \mid a < x < b\} \cap \{x \mid c < x < d\} \\ &= \{x \mid \max(a,c) < x < \min(b,d)\} \\ &= (\max(a,c), \min(b,d)) \end{aligned}$$

[On video] It may be empty set(3rd case), the empty set is an interval, it still the set of numbers between two points.

Similarly for closed intervals and for half open intervals.

=> False for union, eg : $(0,1) \cup (3,4)$ is not an interval (also it is an counterexample on video)

Problem 2. Taking R as the universal set, express the following as simply as possible in terms of intervals and unions of intervals. (Note that A' denotes the complement of the set A relative to the given universal set, which in this case is R. See the appendix.)

(a) $[1,3]' = (-\infty,1)\cup(3,\infty)$

(b) $(1,7]' = (-\infty,1]\cup(7,\infty)$

(c) $(5,8]' = (-\infty,5]\cup(8,\infty)$

(d) $(3,7)\cup[6,8] = (3,8]$

(e) $(-\infty,3)'\cup(6,\infty) = [3,\infty)\cup(6,\infty) = [3,\infty)$

(f) $\{\pi\}' = (-\infty,\pi)\cup(\pi,\infty)$

(g) $(1,4]\cap[4,10] = (1,10]$

(h) $(1,2)\cap[2,3) = (1,3)$

(i) A', were $A = (6,8)\cap(7,9]$, $A' = (6,9]' = (-\infty,6]\cup(9,\infty)$

(j) A', where $A = (-\infty,5]\cup(7,\infty)$, $A' = ((-\infty,5]\cup(7,\infty))' = (5,7]'$

Exercise 4.3.2

1. Prove that if a set A of integers/rationals/reals has an upper bound, then it has infinitely many different upper bounds.

=> upper bound b means $(\forall a \in A)(a \le b)$ or $(\forall a \in A)(a < b)$

=> so, if we think about upperbound $(\forall a \in A)(a \le b)$, no matter what domain is choosen (eg : integers, rationals, reals), there are infinitely many other numbers which is larger than b and all of them can be upper bound ($b < c_i$).

=> i.e: $(\forall a \in A)(a \le b < c_1 < c_2 < c_3 ....)$, all b and c_i can be upper bound.

2. Prove that if a set A of integers/rationals/reals has a least upper bound, then it is unique.

=> To be a least upper bound, it has to be satisfied two conditions, one thing is $(\forall a \in A)(a \leq b)$ and second thing is $(\forall \varepsilon > 0)(\exists a \in A)(a > b - \varepsilon)$. So, in case of lub, there must be an a which is equal with b.

=> if there is $(\forall a \in A)(a \leq c)$ and $b \neq c$, then both $b < c$ and $b > c$ cases meet the contradiction.

=> In case of $b < c$, than $(\forall a \in A)(a \leq b < c)$ and it violate the $(\forall a \in A)(a \leq c)$ condition. (equal is impossible)

=> In case of $b > c$, than $(\forall a \in A)(a \leq c < b)$ and it violate the $(\forall \varepsilon > 0)(\exists a \in A)(a > b - \varepsilon)$ condition (there are $b > c$ but there is not exist such that $a > c$). And it also violate the $(\forall a \in A)(a \leq b)$ condition.

=> So, it must be $b = c$: it's unique.

3. Let A be a set of integers, rationals, or reals. Prove that b is the least upper bound of A iff:

(a) $(\forall a \in A)(a \leq b)$; and

(b) whenever $c < b$ there is an a $a \in A$ such that $a > c$.

=> [on video]

Condition (a) says that b is an upper bound.

Condition (b) means that b is a lub iff no $c < b$ is an upper bound

Iff, for any $c < b$, c is not an upper bound.

Iff, for any $c < b$, there is an $a \in A$ such that $\neg(a \leq c)$

* in here $\neg$ could means "not the case"

Iff, for any $c < b$, there is an $a \in A$, such that $a > c$

4. The following variant of the above characterization is often found. Show that b is the lub

of A iff:

(a) $(\forall a \in A)(a \le b)$ **; and**

(b) $(\forall \varepsilon > 0)(\exists a \in A)(a > b - \varepsilon)$ **.**

=> in this problem, if we let $b - \varepsilon = c$, then, $(\forall \varepsilon > 0) -> (\forall c > b)$, and $(a > b - \varepsilon) -> (a > c)$. Then, this problem is equal to problem 3.

5. Give an example of a set of integers that has no upper bound.

=> $A = \{x \mid x \in N\}$ has no upper bound.

6. Show that any finite set of integers/rationals/reals has a least upper bound.

=> Let A is some finite set ('no matter what domain is'). Then, we can express like this :

$(\forall a \in A)(a \le \max(A))$

=> now, if we let the $\max(A) = b$, then automatically, condition of pbm 3 or 4 is satisfied.

7. Intervals: What is lub (a, b)? What is lub [a, b]? What is max (a, b)? What is max [a, b]?

=> because of the completeness property,

lub (a, b) = $b-1$(integer cases)

lub (a, b) = lub $[a,b]$ = b (other cases / source : Wikipedia 'supremum' explain)

lub [a, b] = b

max (a, b) = $b-1$ (integer cases) / not defined (other cases)

max [a, b] = b

8. Let $A = \{|x - y| \mid x, y \in (a,b)\}$. Prove that A has an upper bound. What is lub A?

=> $x, y \in (a,b) = (a < x < b) and\, (a < y < b)$, so

$a - b < x - y < b - a$ and $0 \leq | x - y | < b - a$.

So, $| x - y |$'s upper bound is $b - a$.

9. Define the notion of a lower bound of a set of integers/rationals/reals.

=> <lower bound>

=> $(\forall a \in A)((a \geq b) or (a > b))$

10. Define the notion of a greatest lower bound (glb) of a set of integers/rationals/reals by analogy with our original definition of lub.

=><greatest lower bound>

=> (a) $(\forall a \in A)(a \geq b)$; and

(b) $(\forall \varepsilon > 0)(\exists a \in A)(a < b + \varepsilon)$.

11. State and prove the analog of question 3 for greatest lower bounds.

=> (a) $(\forall a \in A)((a \geq b) or (a > b))$; and

(b) whenever $c > b$ there is an $a \in A$ such that $a < c$.

=> Condition (a) says that b is an lower bound.

Condition (b) means that b is a glb iff no $c > b$ is an lower bound

Iff, for any $c > b$, c is not an lower bound.

Iff, for any $c > b$, there is an $a \in A$ such that $\neg(a \geq c)$

* in here $\neg$ could means "not the case"

Iff, for any $c > b$, there is an $a \in A$, such that $a < c$

12. State and prove the analog of question 4 for greatest lower bounds.

=> (a) $(\forall a \in A)((a \geq b) or (a > b))$; and

(b) $(\forall \varepsilon > 0)(\exists a \in A)(a < b + \varepsilon)$.

=> in this problem, if we let $b + \varepsilon = c$, then, $(\forall \varepsilon > 0) -> (\forall c > b)$, and $(a < b + \varepsilon) -> (a < c)$. Then, this problem is equal to problem 11.

13. Show that the Completeness Property for the real number system could equally well have been defined by the statement, "Any nonempty set of reals that has a lower bound has a greatest lower bound."

=> Let A is "a nonempty set of reals that has a lower bound". So, let $(\forall a \in A)(a \geq \min(A))$. So, if we let the $\min(A) = b$, then automatically, condition of pbm 11 or 12 is satisfied.

14. The integers satisfy the Completeness Property, but for a trivial reason. What is the reason?

=> "Any nonempty set of integers that has a lower bound has a greatest lower bound."

=> It is trivial. If there is a finite integer set A, than definitely there is an $(\forall a \in A)(a \geq \min(A))$. So, if we let the $\min(A) = b$, then automatically, condition of pbm 11 or 12 is satisfied. So, the above statement is valid.

Exercise 4.3.3

1. Let $A = \{r \in Q | r > 0 \wedge r^2 > 3\}$. Show that A has a lower bound in Q but no greatest lower bound in Q. Give all details of the proof along the lines of Theorem 4.3.1

=> Let $x = \dfrac{p}{q} \in Q$ be any upper bound of A, where $p, q \in N$.

Suppose first that $x^2 < 3$. Thus $3q^2 > p^2$. Now, as n gets large, the expression $\frac{n^2}{2n+1}$ increases without bound, so we can pick $n \in N$ so large that

$$\frac{n^2}{2n+1} > \frac{p^2}{3q^2 - p^2}$$

Rearranging, this gives

$$3n^2q^2 > p^2(n+1)^2$$

Hence

$$(\frac{n+1}{n})^2 \frac{p^2}{q^2} < 3$$

Let

$$y = (\frac{n+1}{n}) \frac{p}{q}$$

Thus $y^2 < 3$. Now, since $\frac{n+1}{n} > 1$, we have $x < y$. But y is rational and we have just seen that $y^2 < 3$, so $y \in A$(from definition of A). This contradicts the fact that x is an upper bound for A.

It follows that we must have $x^2 \geq 3$. Since there is no rational whose square is equal to 3, this means that $x^2 > 3$. Thus $p^2 > 3q^2$, and we can pick $n \in N$ so large now that

$$\frac{n^2}{2n+1} > \frac{3p^2}{p^2 - 3q^2}$$

Rearranging, this gives

$$n^2p^2 > 3q^2(n+1)^2$$

Hence

$$(\frac{n}{n+1})^2 \frac{p^2}{q^2} > 3$$

Let

$$y = (\frac{n}{n+1})\frac{p}{q}$$

Then $y^2 > 3$. Since $n/(n+1) < 1$, $y < x$. But for any $a \in A$, $a^2 < 3 < y^2$, so $a < y$. Thus y is an upper bound of A less than x, as we set out to prove.

From these prove, the completeness property is not hold in rational line Q.

2. In addition to the completeness property, the Archimedean property is an important fundamental property of R. It says is that if $x, y \in R$ and $x, y > 0$, there is an $n \in N$ such that $nx > y$.

Use the Archimedean property to show that if $r, s \in R$ and $r < s$, there is a $q \in Q$ such that $r < q < s$. (Hint: pick $n \in N$, $n > \frac{1}{s-r}$, and find an $m \in N$ such that $r < \frac{m}{n} < s$).

=> Because $s - r > 0$, so we can use the Archimedean property at $s - r$ and 1. So, if we apply the Archimedean property at $s - r$ and 1, then there must be an $n \in N$ such that $n(s-r) > 1$. It also means that $nr < ns$ and because $ns - nr > 1$ (the distance of two number in larger than 1), there must be an m such that $nr < m < ns$. Finally, $r < \frac{m}{n} < s$ and $\frac{m}{n}$ is rational number (Q).

Exercise 4.4.1

1. Formulate both in symbols and in words what it means to say that $a_n \not\to a$ as $n \to \infty$.

=> $\neg(\lim_{n\to\infty} a_n = a)$ or $\lim_{n\to\infty} a_n \neq a$

=> a_n don't get arbitrarily closer and closer to a

=> $\neg(\forall \varepsilon > 0)(\exists n \in N)(\forall m \geq n)(|a_m - a| < \varepsilon)$

$(\exists \varepsilon > 0)(\forall n \in N)(\exists m \geq n)(|a_m - a| \geq \varepsilon)$

2. Prove that $(\frac{n}{n+1})^2 \to 1$ **as** $n \to \infty$.

=>[On video]

Given $\varepsilon > 0$, we seek the N s.t. : $n \geq N -> |(\frac{n}{n+1})^2 - 1| < \varepsilon$

i.e. s.t. $n \geq N -> |\frac{n^2 - n^2 - 2n - 1}{(n+1)^2}| < \varepsilon$

i.e. s.t. $n \geq N -> \frac{2n+1}{(n+1)^2} < \varepsilon$

Pick N so big, that $\frac{(N+1)^2}{2N+1} > \frac{1}{\varepsilon}$

then, $n \geq N -> \frac{2n+1}{(n+1)^2} \leq \frac{2N+1}{(N+1)^2} < \varepsilon$

3. Prove that $\frac{1}{n^2} \to 0$ **as** $n \to \infty$.

=> Given $\varepsilon > 0$, we seek the N s.t. : $n \geq N -> |\frac{1}{n^2} - 0| < \varepsilon$

i.e. s.t. $n \geq N -> \frac{1}{n^2} < \varepsilon$

Pick N so big, that $N^2 > \frac{1}{\varepsilon}$

then, $n \geq N \to \frac{1}{n^2} \leq \frac{1}{N^2} < \varepsilon$

4. Prove that $\frac{1}{2^n} \to 0$ **as** $n \to \infty$.

=> Given $\varepsilon > 0$, we seek the N s.t. : $n \geq N -> |\frac{1}{2^n} - 0| < \varepsilon$

i.e. s.t. $n \geq N -> \frac{1}{2^n} < \varepsilon$

Pick N so big, that $2^N > \frac{1}{\varepsilon}$

then, $n \geq N \rightarrow \frac{1}{2^n} \leq \frac{1}{2^N} < \varepsilon$

5. We say a sequence $\{a_n\}^{\infty}{}_{n=1}$ tends to infinity if, as n increases, a_n increases without bound. For instance, the sequence $\{n\}^{\infty}{}_{n=1}$ tends to infinity, as does the sequence $\{2^n\}^{\infty}{}_{n=1}$. Formulate a precise definition of this notion, and prove that both of these examples fulfill the definition.

=> $\lim_{n \to \infty} a_n = \infty$

=> a_n get arbitrarily closer and closer to infinity

=> $(\forall \varepsilon > 0)(\exists n \in N)(\forall m \geq n)(| a_m - \infty | < \varepsilon)$

=> We can also represent the 'tends to infinity' as like this : A sequence $\{a_n\}^{\infty}{}_{n=1}$ tends to infinity if for any $M > 0$ there exists N such that $a_n > M$ whenever $n > N$

i.e. $(\forall M > 0)(\exists N)((\forall n \geq N) \rightarrow (a_n > M))$

=> For $\{n\}^{\infty}{}_{n=1}$, let $m > 0$ be arbitrary and set $N = M$. Clearly $a_n = n > M$ whenever $n > M$. Thus $\{n\}^{\infty}{}_{n=1}$ tends to infinity.

=> For $\{2^n\}^{\infty}{}_{n=1}$, let $m > 0$ be arbitrary and set $N = M$. Clearly, $a_n = 2^n > M$ whenever $n > M$. Thus $\{2^n\}^{\infty}{}_{n=1}$ tends to infinity.

6. Let $\{a_n\}^{\infty}{}_{n=1}$ be an increasing sequence (i.e. $a_n < a_{n+1}$ for each n). Suppose that $a_n \to a$ as $n \to \infty$. Prove that $a = \text{lub}\{a_1, a_2, a_3, ...\}_{n=1}^{\infty}$.

=> suppose that $\lim_{n \to \infty} a_n = a$, and $(\forall \varepsilon > 0)(\exists n \in N)(\forall m \geq n)(| a_m - a | < \varepsilon)$

=> the conditions of lub is like this :

(a) $(\forall a \in A)(a \le b)$; and

(b) $(\forall \varepsilon > 0)(\exists a \in A)(a > b - \varepsilon)$.

=> To prove this sentence easily, let a is element of set, and b is value of lub

=> Than we can suppose that $\lim_{n\to\infty} a_n = b$, and $(\forall \varepsilon > 0)(\exists n \in N)(\forall m \ge n)(|a_m - b| < \varepsilon)$ and the conditions of lub is $(\forall a \in A)(a \le b)$ and $(\forall \varepsilon > 0)(\exists a \in A)(a > b - \varepsilon)$.

=> If we rearrange the statement $|a_m - b| < \varepsilon$ than we can obtain the condition $b - \varepsilon < a_m < b + \varepsilon$. So, if we let the set $A = \{a_n\}^{\infty}{}_{n=1}$, than we can obtain $(\forall a \in A)(a \le b)$ and $(\forall \varepsilon > 0)(\exists a \in A)(a > b - \varepsilon)$.

=> So we can say that $\lim_{n\to\infty} a_n = a = \text{lub}\{a_1, a_2, a_3, \ldots\}_{n=1}^{\infty}$

7. Prove that if $\{a_n\}^{\infty}{}_{n=1}$ is increasing and bounded above, then it tends to a limit.

=> if $\{a_n\}^{\infty}{}_{n=1}$ is increasing (i.e. $a_n < a_{n+1}$ for each n), and $\{a_n\}^{\infty}{}_{n=1}$ is bounded(i.e. $(\forall a \in A)(a \le b)$, all of element in set A is smaller or equal to b), than if we pick n so large, than it must be $|a_n - b| < \varepsilon$ and ε is some positive small number. Also Because A is increasing sequence, than for all m such that $m \ge n$, also it must be $|a_m - a| < \varepsilon$

=> From these reason, we can make the statement $(\forall \varepsilon > 0)(\exists n \in N)(\forall m \ge n)(|a_m - a| < \varepsilon)$ and we can say that $\lim_{n\to\infty} a_n = b$ and it tends to a limit.