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# Asymptotic Analysis of Marginal Posterior PDF via Belief Propagation in Noisy Sparse Estimation

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## 1. Introduction

Recently, belief propagation (BP)-based sparse recovery algorithms have received significant attention due to the availability of low complexity [1]. The aim of such algorithms is to estimate a sparse vector  $\mathbf{X} \in \mathbb{R}^N$  from noisy measurements  $\mathbf{Y} \in \mathbb{R}^M$  which is a linear projection of an underdetermined system ( $M < N$ ), represented as

$$\mathbf{Y} = \Phi \mathbf{X} + \mathbf{W} \quad (1)$$

where the measurements are corrupted by an additive Gaussian noise vector  $\mathbf{W} \in \mathbb{R}^M$ .

In particular, we are interested in the BP-based signal recovery in conjunction with sparse measurement matrices  $\Phi \in \{0, -1, 1\}^{M \times N}$  which is tree-like structured in factor graph representation. Such BP algorithms iteratively approximate the marginal posterior corresponding to each element  $X_i$  by exchanging probability messages over the factor graph associated with  $\Phi$ . The iterative behavior of BP have been well investigated in applications to channel coding [3], but still remains elusive in the sparse estimation problem.

In this paper, we asymptotically analyze such BP behavior in the sparse estimation problem using the *decoupling principle* [3-5]. According to this principle, the vector measurement channel using BP, shown in Fig.1-(a), can be asymptotically decoupled to a sequence of scalar Gaussian channels, shown in Fig.1-(b), if the system is a *large-sparse-system* (LSS) satisfying two conditions below:

1. *Large-system-limit*: The system size is very large, i.e.,  $(N, M \rightarrow \infty)$  where  $M/N$  remains a fixed constant.

2. *No-short-cycle*: The bipartite graph of the sparse measurement matrix  $\Phi$  does not include cycles shorter than the number of the BP-iterations denoted by  $l$ .

In this work, we consider spike-and-slab PDF as the prior for the sparse signal estimation. Our result shows that the marginal posterior PDF takes the form of a spike-and-slab PDF, being a function of the signal magnitude and noise level.

## 2 Main Result

We aim to derive an expression of the marginal posterior PDF  $f_{X_i}(x | Z_i^{(l)}, \Phi)$  from the scalar Gaussian

channel using the concept of the relaxed BP [3]. Let  $V_{X_i \rightarrow Y_j}^{(l)}$  and  $U_{Y_j \rightarrow X_i}^{(l)}$  be RVs representing the BP-messages passed from  $X_i$  to  $Y_j$  and from  $Y_j$  to  $X_i$ , respectively, for all pairs of  $(i, j): \phi_{ji} \neq 0$  at the  $l$ -th iteration. Our derivation starts from the update rule of  $U_{Y_j \rightarrow X_i}^{(l)}$ , expressed as

$$\begin{aligned} U_{Y_j \rightarrow X_i}^{(l)} &:= \phi_{ji} \left( Y_j - \sum_{k \neq i} \phi_{jk} E\{X_k | V_{X_k \rightarrow Y_j}^{(l)}, \Phi\} \right) \\ &\stackrel{(a)}{=} \phi_{ji}^2 X_i + \text{CEI}_{ji}^{(l)} + \phi_{ji} W_j, \\ &\stackrel{(b)}{=} X_i + \text{CEI}_{ji}^{(l)} + W_j, \end{aligned} \quad (2)$$

where (a) holds since we know  $Y_j = \sum_k \phi_{jk} X_k$  and define the cross-element-interference (CEI) term passed through  $Y_j$  to  $X_i$  as

$$\text{CEI}_{ji}^{(l)} := \phi_{ji} \sum_{k \neq i} \phi_{jk} \left( X_k - E\{X_k | V_{X_k \rightarrow Y_j}^{(l)}, \Phi\} \right), \quad (3)$$

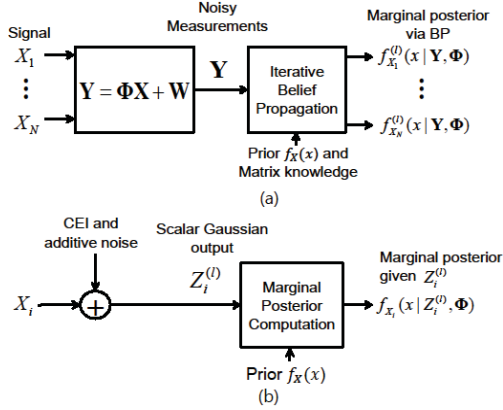
and (b) also holds since we know  $\phi_{ji}^2 = 1$  and scaling the zero-mean Gaussian RV  $W_j$  by  $\phi_{ji} \in \{1, -1\}$  does not change its statistics. By the central limit theorem (CLT), as a sum of *i.i.d.* RVs,  $\text{CEI}_{ji}^{(l)}$  is asymptotically Gaussian distributed with zero-mean and the variance

$$\begin{aligned} (\sigma_{\text{CEI}_{ji}}^2)^{(l)} &= (\sigma_{\text{CEI}_j}^2)^{(l)} \\ &\stackrel{(a)}{\approx} (R-1) \mathbf{Var} \left\{ X_k | V_{X_k \rightarrow Y_j}^{(l)}, \Phi \right\}, \end{aligned} \quad (4)$$

under the LSS setup, where  $R$  is the number of nonzero in a row of the matrix  $\Phi$ , and the approximation of (a) is valid by the weak law of large numbers. Therefore, from (2)-(4), it turns out that the message  $U_{Y_j \rightarrow X_i}^{(l)}$  is Gaussian distributed with the mean  $X_i$  and the variance  $\eta_j^{(l)} := \sigma_w^2 + (\sigma_{\text{CEI}_j}^2)^{(l)}$ , i.e.,

$$f_{U_{Y_j \rightarrow X_i}^{(l)}}(x | X_i, \mathbf{Y}, \Phi) \xrightarrow{\text{CLT}} \mathcal{N}(x; X_i, \eta_j^{(l)}). \quad (5)$$

The marginal posterior of  $X_i$  is obtained after a certain number of the BP-iterations  $l^*$ . Let  $\mathcal{U}_i^{(l)}$  denote a set of the messages from the measurement side with respect to  $X_i$ , i.e.,  $\mathcal{U}_i^{(l)} := \{U_{Y_j \rightarrow X_i}^{(l)} | \phi_{ji} \neq 0\}$ . Then, using the Bayesian rule (Posterior  $\propto$  Prior  $\times$  Likelihood), the



**Fig. 1** (a) Vector measurement channel with BP, (b) The asymptotically equivalent scalar Gaussian channel

marginal posterior is given as

$$f_{X_i}^{(l)}(x | \mathbf{Y}, \Phi) \stackrel{\text{BP}}{\propto} f_X(x) \times f_{U_i}^{(l)}(x | X_i, \mathbf{Y}, \Phi) \quad (6)$$

$$= f_X(x) \times \prod_{k: \phi_{ji} \neq 0} f_{U_j}^{(l)}(x | X_i, \mathbf{Y}, \Phi),$$

where the likelihood PDF  $f_{U_i}^{(l)}(x | X_i, \mathbf{Y}, \Phi)$  can be represented by the product of the message PDFs in  $\mathcal{U}_i^{(l)}$  since the *no-short-cycle* property ensures the statistical independency among  $\mathcal{U}_i^{(l)}$ . By applying the fact that the product of Gaussian PDFs results in a scaled Gaussian PDF, the likelihood PDF is given as

$$f_{U_i}^{(l)}(x | X_i, \mathbf{Y}, \Phi) = \mathcal{N}(x; X_i, (\sum_j \frac{1}{\eta_j^{(l)}})^{-1}). \quad (7)$$

It is important to note from (6) that the PDF of the scalar channel output  $Z_i^{(l)}$  corresponds to the likelihood PDF  $f_{U_i}^{(l)}(x | X_i, \mathbf{Y}, \Phi)$  in the *decoupling principle* [3-5], i.e.,

$$f_{U_i}^{(l)}(x | X_i, \mathbf{Y}, \Phi) \stackrel{\text{LSS}}{\rightarrow} f_{Z_i}^{(l)}(x | X_i, \Phi), \quad (8)$$

in probability. Hence, the scalar output  $Z_i^{(l)}$  is Gaussian distributed with the mean  $X_i$  and the variance  $\eta^{(l)}$  following

$$\left( \sum_j \frac{1}{\eta_j^{(l)}} \right)^{-1} \stackrel{\text{LSS}}{\rightarrow} \eta^{(l)}. \quad (9)$$

In addition, we can describe the iterative behavior of  $\eta^{(l)}$  by an fixed point equation resulting from (4) and (9), given as

$$\eta^{(l)} = \frac{\sigma_w^2}{L} + \frac{R}{L} \mathbf{Var}\{X_k | Z_i^{(l-1)}, \eta^{(l-1)}, \Phi\}, \quad (10)$$

where the number of nonzeros  $L$  in a column of the matrix  $\Phi$  should be properly chosen to satisfy the weak law of large numbers while holding the *no-short-cycle* property. The variance  $\eta^{(l)}$  consists of two factors: the additive noise and CEI. The CEI factor is

represented as the variance of  $X_k$  at the previous iteration, obtained from

$$\mathbf{Var}\{X_k | Z_i^{(l-1)}, \eta^{(l-1)}, \Phi\} = \int (x - \mathbf{E}\{X_k | Z_i^{(l-1)}, \eta^{(l-1)}, \Phi\})^2 f_{X_i}(x | Z_i^{(l)}, \Phi) dx, \quad (11)$$

and it will be continuously approaches to zero as  $l \rightarrow \infty$  if the SNR level is sufficiently high [5]. Such a fixed point equation in (10) was originally discussed by Guo and Verdu [4], and has widely investigated to analyze performance of the BP-based algorithms.

In the sparse estimation, the marginal posterior is computed by imposing a sparsifying prior PDF. In this work, we consider the spike-and-slab PDF as the prior, given as

$$f_X(x) := q\mathcal{N}(x; 0, \sigma_x^2) + (1-q)\delta_0. \quad (12)$$

In addition, it is obvious from (8) that

$$f_{X_i}^{(l)}(x | \mathbf{Y}, \Phi) \stackrel{\text{LSS}}{\rightarrow} f_{X_i}^{(l)}(x | Z_i^{(l)}, \Phi) \quad (13)$$

for every  $i$  and  $x$ , in probability under the LSS setup.

Then, the marginal posterior PDF is expressed as

$$f_{X_i}^{(l)}(x | Z_i^{(l)}, \Phi) \quad (14)$$

$$\propto qc_2 \mathcal{N}(x; \frac{X_i \sigma_x^2}{\sigma_x^2 + \eta^{(l)}}, \frac{\sigma_x^2 \eta^{(l)}}{\sigma_x^2 + \eta^{(l)}}) + (1-q)c_1 \delta_0,$$

where  $c_1, c_2$  are some constants such that the marginal posterior is valid as a PDF, i.e.,  $\int f_{X_i}^{(l)}(x | Z_i^{(l)}, \Phi) dx = 1$ .

### 3 Conclusion

From (14), it turns out that the marginal posterior PDF is a spike-and-slab PDF whose parameters are functions of the signal value  $X_i$  and  $\eta^{(l)}$  which includes the noise variance. For the calculation of (11), we refer to the paper by Krzakala *et al.* [6]. This analysis result will be very useful to evaluate the BP performance in the sparse estimation problems.

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