A study for establishing a sufficient condition for successful joint support set reconstruction

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Abstract

We aim to derive a new sufficient condition for successful joint support set reconstruction in the multiple measurement vectors (MMV) model [1]. Our MMV model consists of S multiple measurement vectors, S multiple K sparse vectors each having the same support set, and a measurement matrix. Our work here aims to give how many measurements per measurement vector are needed for reliable reconstruction of the joint support. We show that if the number of measurements per measurement vector is enough, *i.e.,* $M \geq \Omega\left(\frac{K}{S}\log(N/K)+K\right)$, then reliable reconstruction of the joint support set can be made. Our result is interesting since it gives the exact relation how M is reduced as S is increased. Also, when S is large, *i.e.,* $S > \log(N/K)$, it reduces to $M \geq \Omega(K)$ the result of Duarte *et al.* [3]. Our result can also be extended to the work of Tang and Nehorai [2].

Keywords: compressed sensing, joint support set

1. Introduction

Compressed sensing (CS) [4] considers the problem of solving an under-determined system of linear equations when the solution of the under-determined system is sparse. The under-determined system is called single measurement vector (SMV) because it has single sparse vector in $\mathbb{R}^N$, single measurement matrix in $\mathbb{R}^{M\times N}$, and a measurement vector in $\mathbb{R}^M$. The aim is to correctly reconstruct the support set whose elements are corresponding to the indices of non-zero coefficients of the sparse vector.

An extension to the SMV model is a multiple measurement vectors (MMV) model [1]. The MMV model has S multiple measurement vectors in $\mathbb{R}^M$, S multiple K sparse vectors having the joint support set in $\mathbb{R}^N$ and a measurement matrix in $\mathbb{R}^{M\times N}$. Joint support set reconstruction is to reconstruct the joint support set by simultaneously exploiting all the measurement vectors with the knowledge of the measurement matrix.

There are two sufficient conditions for successful joint support set reconstruction. Duarte *et al.*'s sufficient condition [3] is $M \geq K+1$ for $S \to \infty$. Tang and Nehorai's sufficient condition [2] is $M \geq \Omega(K\log(N/K))$ for $S > \log(N)/\log(\log(N))$. But, they are valid when S is sufficiently large. Donoho [4] has shown that $M \geq \Omega(K\log(N/K))$ is sufficient to reconstruct the support set in the SMV model. Since the MMV model is an extension to the SMV model, Tang and Nehorai's sufficient condition is obvious.

Here, we present a sufficient condition $M \geq \Omega\left(\frac{K}{S}\log(N/K)+K\right)$. From it, we obtain the following results. First, ours is valid regardless of whether S goes to infinity or not. Second, M is reduced as S is increased. Third, $M \geq \Omega(K)$ is sufficient for successful joint support set reconstruction for $S > \log(N/K)$.

What remains are organized as follows: In Sec. 2, notations and system model are described. In Sec. 3, a decoder and a failure probability are presented. Results are presented in Sec. 4. Conclusions are given in Section 5.

2. Notations and system model

Suppose that a capital bold letter $\mathbf{A}$ is a matrix and a small bold letter $\mathbf{a}$ is a vector. The i^{th} element of $\mathbf{a}$ is $a(i)$ and the $(i,j)^{\text{th}}$ element of $\mathbf{A}$ is $A(i,j)$. The probability, the expectation, and the variance are denoted as $\mathbb{P}$, $\mathbb{E}$, and $\mathbb{V}$. A matrix transpose and an inverse operation are denoted as $(\cdot)^T$ and $(\cdot)^{-1}$. $\mathbf{A}^\dagger := \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T$ is the pseudo inverse operation of $\mathbf{A}$. An Euclidean letter $\mathcal{J}$ is a subset of $\{1,\cdots,N\}$ and $\mathbf{A}_{\mathcal{J}}$ is constructed by collecting column vectors of $\mathbf{A}$ corresponding to the indices of $\mathcal{J}$.

A model considered in here is

$$\mathbf{y}_s = \mathbf{F}\mathbf{x}_s + \mathbf{n}_s \in \mathbb{R}^M \text{ for } s = 1,2,\cdots,S, \quad (1)$$

where $n_i(j) \sim \mathcal{N}(0,\sigma_n^2)$, $\mathbf{x}_i \in \mathbb{R}^N$ with $\|\mathbf{x}_i\|_0 = K$.

Also, the joint support set is modeled as

$$\mathcal{I} := \mathcal{I}(\mathbf{x}_1) = \cdots = \mathcal{I}(\mathbf{x}_S),$$

where $\mathcal{I}(\mathbf{x}) := \{j | x(j) \neq 0\}$ is the support set of $\mathbf{x}$ and $j \in \{1,\cdots,N\}$. All the noise vectors and all the column vectors of $\mathbf{F}$ are mutually independent and we use $\mathbf{Y} := [\mathbf{y}_1 \ \cdots \ \mathbf{y}_S] \in \mathbb{R}^{M\times S}$.

3. Joint typical decoder

We introduce our joint typical (JT) decoder. The JT decoder is defined as $Decoder: \{\mathbf{Y},\mathbf{F}\} \mapsto \mathcal{J}$, where $\mathcal{J} \subset \{1,\cdots,N\}$ with $|\mathcal{J}| = K$. The JT decoder is to find the correct support set by using $\delta-$joint typicality.

Definition 1: We say an $M\times S$ matrix $\mathbf{Y}$ and a set $\mathcal{J} \subset \{1,2,\cdots,N\}$ with $|\mathcal{J}| = K$ are $\delta-$ jointly typical if both $rank(\mathbf{F}_{\mathcal{J}}) = K$ and

$$\left| \left(\sum_s \frac{\|\mathbf{Q}(\mathbf{F}_{\mathcal{J}})\mathbf{y}_s\|^2}{SM} \right) - \frac{(M-K)\sigma_n^2}{M} \right| < \delta \quad (2)$$

are satisfied, where $\mathbf{Q}(\mathbf{F}) := \mathbf{I} - \mathbf{F}\mathbf{F}^\dagger$, $\delta > 0$ and $rank(\mathbf{F})$ is the rank of $\mathbf{F}$.

Let $\mathrm{E}(D_{failure})$ be a failure event. Then, it is

$$\mathrm{E}(D_{failure}) = \mathrm{E}(\mathbf{Y},\mathcal{I},\delta)^c \bigcup_{\forall \mathcal{J}\neq\mathcal{I},\mathcal{J}=K} \mathrm{E}(\mathbf{Y},\mathcal{J},\delta), \quad (3)$$

where $\mathrm{E}(\mathbf{Y},\mathcal{I},\delta)^c$ is a failure event where the correct support set $\mathcal{I}$ is not $\delta-$ jointly typical with $\mathbf{Y}$ and $\mathrm{E}(\mathbf{Y},\mathcal{J}\neq\mathcal{I},\delta)$ is a failure event where an incorrect support set $\mathcal{J}$ is $\delta-$jointly typical with $\mathbf{Y}$.

By using the union bound, we have an upper bound on the probability of $\mathrm{E}(D_{failure})$. It is

$$\mathbb{P}\{\mathrm{E}(D_{failure})\} \leq \mathbb{P}\{\mathrm{E}(\mathbf{Y},\mathcal{I},\delta)^c\} + \sum_{\forall \mathcal{J}\neq\mathcal{I},|\mathcal{J}|=K} \mathbb{P}\{\mathrm{E}(\mathbf{Y},\mathcal{J},\delta)\}. \quad (4)$$

We present two lemmas. They describe the statistical information of the random variable of each probability given at the right hand side of (4).

Lemma 2: Let $\mathcal{J}$ be an incorrect support set, $M > K$, $Z_1 := \sum_{s=1}^S \|\mathbf{Q}(\mathbf{F}_{\mathcal{J}})\mathbf{y}_s\|^2$, and $\mathbf{R}$ be the covariance matrix of $\mathbf{L} := [c_1(1)\cdots c_1(M-K)\cdots c_S(1)\cdots c_S(M-K)]^T$, where $\mathbf{c}_s := \left(\sum_{u\in\mathcal{I}\setminus\mathcal{J}} x_s(u)\mathbf{f}_u\right) + \mathbf{n}_s$. Then, $\mathbb{E}[Z_1] = \mathrm{tr}[\mathbf{R}]$, $\mathbb{V}[Z_1] = 2\mathrm{tr}[\mathbf{R}^2]$, $\mathbb{E}[\exp(tZ_1)] = \prod_{n=1}^{S(M-K)} (1-2t\lambda_n)^{-\frac{1}{2}}$ and λ_n is the n^{th} eigen value of $\mathbf{R}$.

Lemma 3: Let $\mathcal{I}$ be the correct support set, $M > K$, $Z_2 := \sum_{s=1}^S \left(\|\mathbf{Q}(\mathbf{F}_{\mathcal{I}})\mathbf{y}_s\|^2 / \sigma_n^2\right)$, Then, $\mathbb{E}[Z_2] = S(M-K)$, $\mathbb{V}[Z_2] = 2S(M-K)$ and $\mathbb{E}[\exp(tZ_2)] = (1-2t)^{-\frac{S(M-K)}{2}}$.

Now, we present each further upper bound.

Lemma 4: Let $\mathcal{J}$ be an incorrect support set, $0 \leq |\mathcal{I}\cap\mathcal{J}| < K$ and $rank(\mathbf{F}_{\mathcal{J}}) = K$. Then, for any $\delta > 0$, we have

$$\mathbb{P}\{\mathrm{E}(\mathbf{Y},\mathcal{J},\delta)\} \leq p(\mathcal{T},\mathcal{J},\lambda_{min}) := \exp\left(-\frac{S(M-K)}{2}(d_1(M)-1)\right) \times d_1(M)^{\frac{S(M-K)}{2}}, \quad (5)$$

where $d_1(M) := \frac{\sigma_n^2}{\lambda_{min}} + \frac{M\delta}{(M-K)\lambda_{min}}$, $\mathcal{T} := \{S,M,K,\delta,\sigma_n^2\}$, and λ_{min} is the smallest eigen value of $\mathbf{R}$ defined in Lemma 2.

Lemma 5: Let $\mathcal{I}$ be the correct support set and $rank(\mathbf{F}_{\mathcal{I}}) = K$. Then, for any $\delta > 0$, we have

$$\mathbb{P}\{\mathrm{E}(\mathbf{Y},\mathcal{I},\delta)^c\} \leq 2p(\mathcal{T},\mathcal{I}) := 2\exp\left(-\frac{S(M-K)}{2}d_2(M)\right) \times (1+d_2(M))^{\frac{S(M-K)}{2}}, \quad (6)$$

where $d_2(M) := \frac{M\delta}{(M-K)\sigma_n^2}$ and $\mathcal{T} := \{S,M,K,\delta,\sigma_n^2\}$.

Now, we provide the upper bound on. It is

$$\mathbb{P}\{\mathrm{E}(D_{failure})\} \leq 2p(\mathcal{T},\mathcal{I}) + \binom{N}{K} p(\mathcal{T},\mathcal{J}^*,\lambda_{min}), \quad (7)$$

where $p(\mathcal{T},\mathcal{J}^*,\lambda_{min}) := \max_{\forall \mathcal{J}\neq\mathcal{I},|\mathcal{J}|=K} (p(\mathcal{T},\mathcal{J},\lambda_{min}))$.

All the proofs of the lemmas are given in [5] and all the explanations of the lemmas are given in [5].

4. Main results

We prove that $M = \Omega\left(\frac{K}{S}\log\left(N/K\right)+K\right)$ is sufficient for successful joint support set reconstruction as either $S\to\infty$ or $N\to\infty$.

Thereom 6: If $\lambda_{min} > \sigma_n^2$, then $M = \Omega\left(\frac{K}{S}\log\left(N/K\right)+K\right)$ is sufficient for successful joint support set reconstruction when $S\to\infty$ or $N\to\infty$.

Proof: First, we show that $p(\mathcal{T},\mathcal{I})\to 0$ by either $S\to\infty$ or $N\to\infty$. We investigate the term

$$\log\left(p\left(\mathcal{T},\mathcal{I}\right)\right) = \alpha_1 \times \left(\underbrace{\log\left(1+d_2\left(M\right)\right)-d_2\left(M\right)}_{:=\alpha_2}\right), \quad (8)$$

where $\alpha_1 := \frac{S(M-K)}{2}$. By letting $M = c\left(\frac{K}{S}\log\left(N/K\right)+K\right)$, we have

$$\alpha_1 = \left(cK\log\left(N/K\right)+ScK-SK\right)/2, \quad (9)$$

where $c>1$ is a constant. It is easy to see that $\alpha_2<0$ for any positive $d_2\left(M\right)$. Also, we ensure that $\alpha_1\to\infty$ as either $S\to\infty$ or $N\to\infty$ in (9). Thus, we conclude that $\log\left(p\left(\mathcal{T},\mathcal{I}\right)\right)\to-\infty$ implying that $p\left(\mathcal{T},\mathcal{I}\right)\to 0$ by either $S\to\infty$ or $N\to\infty$. Second, we show that β converges to zero as either $S\to\infty$ or $N\to\infty$, where $\beta := \binom{N}{K}p\left(\mathcal{T},\mathcal{J}^*,\lambda_{min}\right)$. By taking the logarithm to β with the bound $\binom{N}{K}\le\exp\left(K\log\left(Ne/K\right)\right)$, we have

$$\log\left(\beta\right)\le\alpha_1\underbrace{\left(\log\left(d_1\left(M\right)\right)-d_1\left(M\right)+1\right)}_{:=\beta_1}+\beta_2, \quad (10)$$

where $\beta_2 := K\log\left(Ne/K\right)$ and $\beta_1<0$ for any positive $d_1(M)$. It is obvious that $\beta\to 0$ as $S\to\infty$. Last, we investigate the behavior of term given at the right hand side of (10) when $N\to\infty$. By rearranging the term, it becomes

$$\beta_3 K\log\left(N/K\right)+\beta_4, \quad (11)$$

where $\beta_3 := cK\beta_1/2+K$ and $\beta_4 := K+\beta_1 SK\left(c-1\right)/2$. Since $\beta_1<0$, there exists $c>1$ such that $\beta_3<0$. Since β_4 is fixed, it is obvious that $\beta\to 0$ as $N\to\infty$. Therefore, $\mathbb{P}\left\{\mathrm{E}\left(D_{failure}\right)\right\}\to 0$ as either $S\to\infty$ or $N\to\infty$ for both $\lambda_{min}>\sigma_n^2$ and $M=\Omega\left(\frac{K}{S}\log\left(N/K\right)+K\right)$.

Comments: First, we have shown that a relationship between M and S, *i.e.,* M is inversely proportional to S. Namely, M is reduced as S is increased. However, both Tang and Nehorai's sufficient condition [2] and Duarte *et al.*'s sufficient condition [3] do not give insights of the relationship. Second, ours is valid whether S goes to infinity or not. However, the two sufficient conditions are valid when S is sufficiently large. Last, ours becomes $M=\Omega\left(K\right)$ for $S>\log\left(N/K\right)$. It matches with Duarte *et al.*'s sufficient condition [3].

5. Conclusions

We have established a sufficient condition for successful joint support set reconstruction. We have used the JT decoder and we have obtained the upper bound on the probability that joint support set reconstruction of the JT decoder is a failure. The upper bound led us make the sufficient condition $M=\Omega\left(\frac{K}{S}\log\left(N/K\right)+K\right)$. Unlike to Tang and Nehorai's sufficient condition [2] and Duarte *et al.*'s sufficient condition [3], we have shown that ours is valid whether S goes to infinity or not. In addition, we have shown that M is inversely proportional to S. Namely, we can decrease M by increasing S. Last, similar to Duarte *et al.*'s sufficient condition, we have shown that $M=\Omega\left(K\right)$ is sufficient for successful joint support set reconstruction for $S>\log\left(N/K\right)$.

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