Predicting the Performance of Cooperative Wireless Networking Schemes With Random Network Coding

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***Abstract*—In this paper, we consider a cooperative wireless network in which there are multiple sources and multiple relays. Owing to unreliable wireless channels, the quality of network links between nodes can vary. This results in the failure of intermediate nodes that generate linear combinations of incoming messages in network coding schemes. We propose an analytical framework to evaluate the recovery performance of source messages at the base station. To this end, we consider a random transmission matrix in which each element of the transmission matrix is processed a random variable, where its distribution is a function of the outage probability. We derive an upper bound for the reconstruction performance, i.e., decoding failure probability and nullity. The proposed framework provides an evaluation tool that enables us to investigate the impact of a large number of sources and relays, as well as the field size of the network codes on system performance.**

***Index Terms*—Cooperative network, network coding, upper bound, rank.**

I. Introduction

CHANNEL fading is one of the underlying causes of the performance degradation in wireless networks. One naive approach to combat channel fading is to increase the transmit power. A more advanced approach is to utilize modern diversity techniques, which can be performed without increasing the transmit power. To date, numerous diversity techniques have been proposed and employed in the time, frequency, and space domains. Cooperative networking is one of the current approaches that aim to utilize spatial diversity via user cooperation. Each user participates collaboratively, and shares the benefit of a virtual antenna array in transceiver messages that are available through another user's antenna [1].

Network coding [2] first proposed by Ahlswede *et al.* is shown to achieve maximum information flow in a single source multicast network. Numerous efforts have subsequently been attempted; these efforts focused on elucidating if network coding can provide additional advantages compared to other cooperative networking schemes. For example, network coding over the binary field [3], [4] has shown to improve diversity gain and provide higher spectral efficiency in wireless networks, whereas network coding with a nonbinary field further increase those benefits [5]–[9]. In particular, numerous studies have investigated the extent to which network coding can improve the performance of media access control and routing protocols, in terms of energy efficiency [22], transmission delay [24], and throughput [23], [25], compared to traditional forward-and-relay only based designs [19]–[21]. The performance of cooperative wireless networking schemes with network coding has been analyzed, and compared with erasure channel models [5]–[8], and error propagation models [30]–[33]. We will further address recent cooperative communication schemes in Section II by categorizing them with respect to their decoding techniques, spectral efficiencies, and cooperative strategies.

Xiao and Skoglund [5], [6] recently proposed a network coding scheme called Dynamic Network Codes (DNC) to handle a dynamic network topology. The inherent nature of wireless channels implies that links are unreliable and that link failures will occur randomly in the inter-user channels. DNC is perform successfully over such a dynamic network channel topology, in conjunction with techniques such as enhanced diversity order. In the DNC schemes, multiple network code matrices are used; each one is designed to handle a particular occurrence of link outages. In particular, an intermediate node in a network may fail to decode some of the messages received from the other nodes. The intermediate node creates, and later on forwards it to the base station, a linear combination of the messages which it could only successfully decode, and then forwards it to the base station. That is, a certain occurrence of link outages results in a particular restriction to the elements of the network code matrix. Thus, each network code matrix in the DNC scheme, referred to as the *transmission matrix* in this paper, is designed carefully so as to work effectively for the occurrence of a specific set of link failures. In addition, Rebelatto *et al.* in [7] and [8] extended the two-phase transmission framework of the DNC to multiple phases, in the Generalized Dynamic Network Codes (GDNC), to further enhance the transmission rate and the diversity order.

In this paper, our goal is to focus on the system models of the DNC and GDNC schemes, and provide a novel analysis framework for them. As noted earlier, there are other recently studied cooperative communications schemes, each with an advantage in a different perspective such as spectral efficiency and higher decoding performance. These will be discussed further in Section II. The DNC and GDNC schemes are found to be interesting due at least to the following two aspects: i) DNC is the first network coding schemes designed for dynamic network topology. To the best of the authors' knowledge, DNC is the first attempt to adaptively use different network code

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matrix as the link failure varies, and is designed to achieve the so-called the *min-cut capacity* of randomly changing links [5], ii) GDNC is shown to achieve full diversity order and increases the transmission rate [7]. While a series of performance analyses for DNC and GDNC.

Although a series of performance analyses for DNC and GDNC are provided in [5]–[8], the authors rely on the exhaustive investigation of all individual network code matrices to determine if the resultant transmission matrix at the base station is sufficiently able to decode the source messages. This is an exceedingly time-consuming and tedious process; thus, it cannot be extended to larger and more general networks where the link outage probabilities throughout the networks are, in general, different from each other.

In particular, the performance analyses in [5]–[8] to determine the probability of successful decoding at the base station are performed only for small and non-general networks. A successful decoding is assumed to be achieved when the network code matrix at the base station has a sufficient number of linearly independent vectors that at least equals the number of unknown source messages. The successful event begins by determining whether the rank of the network code matrix at the base station is full. Then, the success probability is obtained by adding all individual probabilities of such events over all possible link failures. To achieve this outcome, the authors, using Theorem 1 in [5] and Section V in [7], followed the approach of tracking down each network code matrix individually, and determining if each was full in rank. This is an exhaustive process. When the number of nodes in a network increases, it is evident that this approach becomes intractable, because of the exponential increase in possible combinations. For example, the total number of distinct $N \times N$ random matrices with full rank is $\prod_{i=1}^{N}(q^N - q^{i-1})$ for the finite field of size q [10].

As a result, the analyses performed in [5]–[8] are limited to small and homogeneous networks, i.e., a network of fewer than 10 nodes, with link failure probabilities set to be equal throughout the network. These methods are not suitable for analyzing networks where nodes are randomly deployed in an area of interests, and wireless networks present heterogeneous link outage probabilities. The lack of a general and systematic performance analysis framework to deal with such networks has motivated this study.

In this paper, our main goal is to propose a novel evaluation framework for cooperative network coding schemes. The contributions of this work are summarized as follows.

- (*Design of random transmission matrix*) We model a random transmission matrix with uniform and maximum distance separable (MDS) distributions in (6)–(8) of Section IV. The elements of this matrix are represented with random variables as a function of the outage probability of each wireless link. This new system model, enables us to avoid the exhaustive counting of each network code metric occurrence.
- (*Tight upper bounds to probability of decoding failure*) We derive a series of tight upper bounds on the probability of failure. In particular, the dimension of the nullspace of the random transmission matrix is used to derive an upper bound, as discussed in Proposition 3. It is then linked to the decoding failure probability where the rank of the network code matrix is not full, which is shown in Theorem 4. The upper bounds have proven to be considerably tight in comparison to simulation results.
- (*Generality and scalability*) The developed analysis framework is general and scalable, offering the capability of analyzing large wireless networks with random deployments where all outage probabilities of wireless links are different. For example, consider the network scenario of randomly deployed nodes shown in Fig. 6 in Section VI. In addition, the developed framework can handle a large cooperative network that has more than 100 nodes. To the best of our knowledge, this scale of wireless networks is unprecedented in DNC and GDNC performance evaluations. The proposed framework enables us to investigate its impact on the successful reconstruction of source messages based on varying outage probabilities, and on various key parameters such as the number of relays and the field sizes in DNC and GDNC schemes.

TABLE I
SUMMARY OF RECENT TECHNIQUES FOR COOPERATIVE COMMUNICATIONS

	Recent technique	References
Decoding techniques	Maximum ratio combining	[4], [13]–[15]
	Rank-based decoding	[5]–[8], [10], [11]
High spectral efficiency	Multiuser detection	[17]
	Interference cancellation	[18]
	Non-orthogonal channel	[16]
Cooperative strategies	Amplify-and-Forward	[1], [35]
	Decode-and-Forward	[1], [36]
	Compute-and-Forward	[34]

The remainder of this paper is organized as follows. We review other recent cooperative communications techniques in Section II. We explain our cooperative model for wireless networks in Section III. In Section IV, we model a transmission matrix in terms of an outage probability. In Section V, we calculate upper bounds for the reconstruction performance of cooperative frameworks with various types of link connectivity. Finally, we evaluate the proposed framework in Section VI, and conclude the paper in Section VII.

II. Other Recent Works and Relation to Our Work

In this section, and in Table I, we provide an overview of the prevalent cooperative communications schemes believed to be closely related to the network coding schemes considered in this paper.

Two decoding approaches in cooperative wireless communications have been recently considered. The first is the Maximum Ratio Combining (MRC) decoding scheme [4], [13]–[15]. The main idea is to configure the network to coordinate the transmissions, and repeat a signal with a weak signal-to-noise ratio (SNR) multiple times over independent fading channels. In this manner, MRC allows the destination to maximize SNR. To implement an MRC scheme, all decoding information, as well as the success or failure of each source message at the base station, should be identified and forwarded to the relays. This is required to determine the source of the SNR, for which retransmission was required. To enable its deployment in large-scale networks, the scheduling issue must be resolved.

Recently, two research groups have proposed advanced cooperative network schemes to achieve high spectral efficiency. In [16], Youssef and Amat have proposed the use of non-orthogonal channel allocation to improve spectral efficiency. For wireless networks where a multiuser detection receiver is utilized at the base station, cooperative transmission protocols with high spectral efficiency have been developed [17]. Furthermore, the work describe in [18] has aimed at improving the spectral efficiency of cooperative systems using superposition coding and iterative detection methods.

We believe our analysis framework could be utilized in [16] with a necessary but simple modification. The first aspect to consider is that all wireless channels should not be modeled independently from each other anymore. The outage probabilities are not independent from each other. This can be achieved by designing a joint probability distribution for the random matrix. The probability that the random transmission matrix is not full rank can be obtained by considering such an event over the joint probability distribution. For more details, we will show Example 3 for channel correlation cases in Section V-B of this paper.

Using network coding on lattice codes, Nazer and Gastpar recently proposed the compute-and-forward (CF) relaying scheme [34]. In CF, a relay is configured with a linear combination of multiple codeword signals, which are simultaneously transmitted and superposed in the air. The key idea of using CF relaying in network coding is to utilize the property of the lattice code property stating that the *integer combination* of lattice codewords remains a lattice codeword. Thus, the relay receives a codeword with additive noise. After a denoising step, the relay retransmits the decoded lattice codeword to the base station. Therefore, the benefit of using CF relaying in network coding is evident. Because the transmissions from all sources to the relay are performed simultaneously, the spectral efficiency is significantly enhanced.

There are two widely recognized cooperative relaying strategies, referred to as amplify-and-forward (AF) [1], [35] and decode-and-forward (DF) [1], [36]. AF and DF cooperative relaying strategies perform effectively in either low or high SNR regimes, while CF approach offers advantages in moderate SNR regimes where both interference and noise are significant factors [34]. The DNC and GDNC schemes considered in this paper are categorized as DF based network coding strategies.

Cooperative wireless communications with multiple sources and multiple relays closely related to our work have attracted significant attention because of their higher achievability rate [26], better error performance [28], [29] and diversity-multiplexing tradeoffs [9], [27]. There are two types of error propagation models worth that should be considered here. The authors in [30]–[33] assume a network channel model where erroneous messages are permitted to propagate throughout the network. For the erasure channel model, [5]–[8], erroneous messages at the relays are discarded, to avoid unnecessary error propagation caused by encoding and forwarding operations.

Recent studies [5]–[12] closely related to this paper have focused on the performance analysis and the design of network code matrices for cooperative networks with erasure channel models. In particular, in order to maximize diversity order in a multiple-access network, the problem of designing network code matrices subject to link failures was studied [5]–[8], to maximized diversity order in a multiple-access network. In [7], it is shown that the design of network code matrices is equivalent to the design of linear block codes for erasure correction coding. It was shown that maximum diversity order is guaranteed if an MDS code generator matrix of MDS codes is utilized as the network code matrix. In addition, Nguyen *et al.* [11] have defined upper and lower bounds on GDNC scheme recovery performance. For random linear network coding schemes, Trullols-Cruces *et al.* [10] have derived the exact decoding probability of obtaining network codes of full rank.

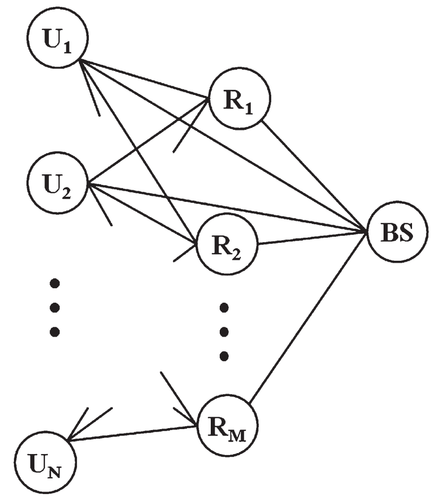


Fig. 1. An (N, M) cooperative network with N sources and M relays.

III. Cooperative Network

We consider an (N, M) cooperative scheme for wireless networks as shown in Fig. 1, in which there are N sources, $\{U_1, U_2, \ldots, U_N\}$, and M relays, $\{R_1, R_2, \ldots, R_M\}$. There are two cooperating transmission phases: *broadcasting* and *relaying*. In the broadcasting phase, each source transmits its message to the base station (BS). Owing to the nature of wireless channels, the relays in this phase can, in general, receive and successfully decode the messages from the sources. In the relay phase, each relay can generate a parity message constructed from a linear combination of these messages, and forward it to the BS. In this work, we assume that the received message for a single channel is considered either completely corrupted—an outage and therefore not available at the receiver—or error-free, i.e., no outage. For more complicated cooperative communications error models that have been studied in [30]–[33], we have included a discussion in Section II.

For both transmission types, we assume that in both transmissions all transmitters send their signals through orthogonal channels using either time- or frequency-division multiple access, and that all channels are spatially independent, flat-faded, and perturbed by additive white Gaussian noise (AWGN). We further assume that the channel gains are independent in both the broadcasting and relay phases. A discussion is included in

Section II in which no orthogonal transmissions are utilized and inter-user channels are not independent.

In the broadcasting phase, the signal y_{u,d_1} received at node d_1 for $d_1 \in \{R_1, R_2, \ldots, R_M, BS\}$ is given by $y_{u,d_1} = \sqrt{P_u} h_{u,d_1} x_{u,d_1} + n_{u,d_1}$, where u denotes the transmitter node, i.e., $u \in \{U_1, U_2, \ldots, U_N\}$; P_u denotes the transmit power at node u; h_{u,d_1} denotes the channel gain between the two nodes u and d_1, which is a circular symmetric complex-valued Gaussian random variable with zero mean and variance $\sigma^2_{u,d_1}/2$ per dimension; x_{u,d_1} is the signal transmitted from node u; and noise n_{u,d_1} denotes the complex-valued AWGN with zero mean and variance $N_0/2$ per dimension. In the relay phase, the signal y_{r,d_2} received at the BS is $y_{r,d_2} = \sqrt{P_r} h_{r,d_2} x_{r,d_2} + n_{r,d_2}$, where r denotes the relay node, i.e., $r \in \{R_1, R_2, \ldots, R_M\}$; d_2 denotes the BS; P_r denotes the transmit power at relay node r; h_{r,d_2} denotes the channel gain between the relay node r and the BS, which is a circular symmetric complex-valued Gaussian random variable with zero mean and variance $\sigma^2_{r,d_2}/2$ per dimension; x_{r,d_2} is the signal transmitted from relay node r; and noise n_{r,d_2} denotes the same AWGN as in the broadcasting phase. For Rayleigh fading channels, the variances of channel gains are defined as $\sigma^2_{u,d_1} := \rho^{-\eta}_{u,d_1}$ and $\sigma^2_{r,d_2} := \rho^{-\eta}_{r,d_2}$, letting ρ_{u,d_1} and ρ_{r,d_2} be the distances for u-to-d_1 and r-to-d_2, respectively, and η be the path-loss exponent, i.e., $2 \le \eta \le 6$ [37]. Throughout this paper, we use $\eta = 3$. The instantaneous SNRs of the two channels are denoted as $\gamma_{u,d_1} := |h_{u,d_1}|^2 P_u/N_0$ and $\gamma_{r,d_2} := |h_{r,d_2}|^2 P_r/N_0$.

Let R_{th} be the predefined threshold of the spectral efficiency in bits/s/Hz. For both phases, we utilize the following outage probabilities based on [1] and [38],

$$\delta_{u,d_1} = \Pr\{\log(1+\gamma_{u,d_1}) < R_{th}\}, \quad (1)$$

and

$$\delta_{r,d_2} = \Pr\{\log(1+\gamma_{r,d_2}) < R_{th}\}. \quad (2)$$

Throughout this paper, we use $R_{th} = 1$ bit/s/Hz. Each of the outage probabilities is a function of the instantaneous SNR and the distance between two nodes. We can use these outage probabilities to model the elements of a transmission matrix.

IV. Modeling of Transmission Matrices

A. Transmission Matrix

We utilize the outage probabilities defined in Section III to model the elements of the transmission matrix. A random transmission matrix can be used to represent a family of network coding matrices for an (N, M) cooperative scheme shown in Fig. 1 in which two transmissions occur over a multiple access network with N sources and M relays. Let $\mathbb{F}_q$ be a finite field of size q. Let $\mathbf{x} \in \mathbb{F}_q^{N\times 1}$ denote the $N \times 1$ vector of transmitted messages, $\mathbf{y} \in \mathbb{F}_q^{(N+M)\times 1}$ denote the $(N+M) \times 1$ vector of messages received at the BS, and $\mathbf{A} \in \mathbb{F}_q^{(N+M)\times N}$ denote the $(N+M) \times N$ transmission matrix. The vector $\mathbf{y}$ received at the BS is then given by

$$\mathbf{y} = \mathbf{A}\mathbf{x}, \quad (3)$$

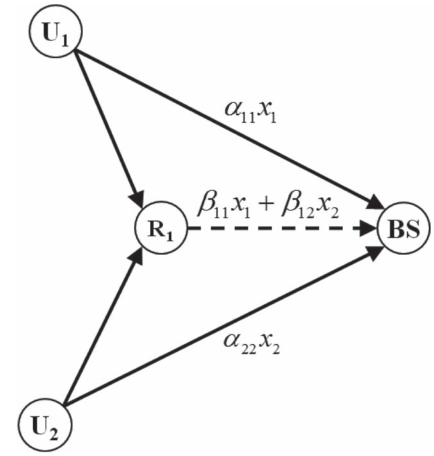


Fig. 2. The (2,1) cooperative wireless network with $N = 2$ and $M = 1$. The solid lines indicate the broadcasting phase, and the dashed line indicates the relay phase.

where the transmission matrix $\mathbf{A}$ consists of the $N \times N$ direct matrix $\mathbf{D}$ and the $M \times N$ combination matrix $\mathbf{P}$; i.e., $\mathbf{A} := \begin{bmatrix}\mathbf{D}\\ \mathbf{P}\end{bmatrix}$. Note that all of the arithmetic operations are performed over finite fields.

The direct matrix $\mathbf{D}$ can be modeled as a diagonal matrix, i.e., one with zeroes for all off-diagonal elements. If there are no outage events for the channel links between the sources and the BS, the diagonal elements of this direct matrix are all set to one; otherwise, the corresponding elements are set to zero. Let α_{ii} denote the ith diagonal element of $\mathbf{D}$, i.e., $\alpha_{ii} \in \{0, 1\}$ for $i \in \{1, 2, \ldots, N\}$, then the ith element $y_{i,1}$ of $\mathbf{y}$ is represented as $y_{i,1} = \alpha_{ii} x_i$, where $x_i \in \mathbb{F}_q$ denotes the ith element of $\mathbf{x}$, and 1 in the subscript of $y_{j,1}$ indicates the broadcasting phase for $j \in \{1, 2, \ldots, M\}$. Let $\beta_{ji} \in \mathbb{F}_q$ denote an element of $\mathbf{P}$, then the $(N+j)$th element $y_{j,2}$ of $\mathbf{y}$ is represented by $y_{j,2} = \sum_{i=1}^{N} \beta_{ji} x_i$, where 2 in the subscript of $y_{j,2}$ indicates the relay phase. As the BS receives $N+M$ messages from N sources and M relays, we can represent the transmission matrix as

$$\begin{bmatrix} y_{1,1} \\ \vdots \\ y_{N,1} \\ y_{1,2} \\ \vdots \\ y_{M,2} \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha_{NN} \\ \beta_{11} & \cdots & \beta_{1N} \\ \vdots & \ddots & \vdots \\ \beta_{M1} & \cdots & \beta_{MN} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}. \quad (4)$$

Note that all elements of the transmission matrix $\mathbf{A}$ are random variables except for the off—diagonal terms of $\mathbf{D}$. The following simple example illustrates the method for determining the elements of the transmission matrix.

Example 1: Consider two sources (U_1 and U_2) and one relay (R_1) in a (2, 1) cooperative wireless network shown in Fig. 2. Let the size of the finite field for the network coding be 2, $q = 2$. In the broadcasting phase, source U_1 transmits message x_1 to the BS and relay R_1, while U_2 transmits message x_2 to the BS and R_1. The relay overhears, decodes, and then linearly combines the decoded messages to generate a parity message that is forwarded to the BS. Thus, the BS receives three messages: x_1 and x_2 from the respective sources and a

TABLE II
DETERMINATION OF THE TRANSMISSION MATRIX FOR ALL CASES OF FAILURES FOR $q = 2$, WHERE "O" INDICATES NO OUTAGE, "×" INDICATES AN OUTAGE, AND "−" INDICATES DON'T CARE

U_1-BS	U_2-BS	**D**	U_1-R_1	U_2-R_1	R_1-BS	**P**
O	O	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	O	O	O	$(1\,1)$
O	×	$\begin{pmatrix}1&0\\0&0\end{pmatrix}$	O	×	O	$(1\,0)$
×	O	$\begin{pmatrix}0&0\\0&1\end{pmatrix}$	×	O	O	$(0\,1)$
×	×	$\begin{pmatrix}0&0\\0&0\end{pmatrix}$	×/–	×/–	O/×	$(0\,0)$

parity message from the relay. This transmission mechanism is depicted in Fig. 2.

The transmission matrix in this example is given by

$$\begin{bmatrix} y_{1,1} \\ y_{2,1} \\ y_{1,2} \end{bmatrix} = \begin{bmatrix} \alpha_{11} & 0 \\ 0 & \alpha_{22} \\ \beta_{11} & \beta_{12} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{5}$$

Here, the connectivities of the channel links (U_1-BS and U_2-BS) are represented by α_{11} and α_{22} in the transmission matrix. If a channel link (U_1-BS) or (U_2-BS) incurs an outage, then its associated connectivity, α_{11} or α_{22}, will be set to zero; otherwise, this element is set to one. Similarly, two elements β_{11} and β_{12} represent the joint factors of the qualities of the three channel links: $(U_1 - R_1)$, $(U_2 - R_1)$, and (R_1-BS). If both links, $(U_2 - R_1)$ and (R_1-BS), do not simultaneously incurs outages, β_{12} is set one; conversely, if either of the two links undergoes an outage, β_{11} will be set to zero. Thus, the transmission matrix will be determined by the condition of all five channel links in the wireless network, and if the transmission matrix at a given condition has full rank, the BS can successfully decode the two source messages x_1 and x_2. Table II summarizes all outage events of the transmission matrix for the (2,1) cooperative wireless network in which there are two sources and one relay. ■

B. Modeling of Random Elements

In this subsection, we provide techniques for defining the elements of the transmission matrix as random variables. We assume that all outage events are mutually independent from each other, which is reasonable for typical wireless networks. Then, the probability distribution of the elements can be determined based on the outage probabilities of the wireless channels as follows. First, the probability distribution for each diagonal element of $\mathbf{D}$ is modeled using the outage probability of the source-to-BS channels. Second, by simultaneously considering the outage events in both channels (i.e., source-relay and relay-BS), we determine the probability distribution for each element of $\mathbf{P}$.

To model each diagonal element of $\mathbf{D}$, the probability of the ith diagonal element α_{ii}, $i \in \{1, 2, \ldots, N\}$, can be defined from the set of possible outage events between the sources and the BS in the broadcasting phase as:

$$\Pr\{\alpha_{ii} = \theta\} = \begin{cases} \delta_{U_i,BS} & \text{if } \theta = 0, \\ 1 - \delta_{U_i,BS} & \text{if } \theta = 1, \end{cases} \tag{6}$$

where $\delta_{U_i,BS}$ the outage probability defined in (1) where an outage occurs in the single link between the ith source U_i and the BS.

Next, to model each element of the combination matrix $\mathbf{P}$, we consider two types of probability distributions. The first is to permit the nonzero values of each element in $\mathbf{P}$ to be uniformly distributed. This distribution is reasonable, considering the recent result in [39] where it is acknowledged that a uniform distribution for linear network coding provides various benefits, inlcuding decentralized operation and robustness to network changes or link failures in multisource, multicast networks. The second is to allow the nonzero value to be predetermined. This specific value can be set using MDS codes [40] in coding theory. It is well known that MDS codes achieve the Singleton bounds. This supports the consideration of MDS codes for optimum reconstruction performance. In the latest literature, Rebelatto *et al.* [7] have proved that a systematic MDS code generator matrix, operating over sufficiently large finite fields such as the transmission matrix, is sufficient for obtaining full diversity in cooperative networks. However, MDS codes use a large field size so that may result in excessive complexity, especially in the cases where the dimension of the code is large.

1) Uniform Distribution: When modeling the elements of the combination matrix $\mathbf{P}$, we have to consider the outage events in both the source-to-relay and relay-to-BS links, as the occurrence of either or both of these events will prevent the relay from delivering the source message to the BS. Let $\bar{\mathcal{E}}_j$ and $\mathcal{E}_j$ denote the nonoccurrence and occurrence of an outage from the jth relay R_j, $j \in \{1, 2, \ldots, M\}$, to the BS, respectively. Thus, both probabilities are: $\Pr\{\bar{\mathcal{E}}_j\} = 1 - \delta_{R_j,BS}$ and $\Pr\{\mathcal{E}_j\} = \delta_{R_j,BS}$. Because the outage event from a source to a relay is independent of any other outage events, the conditional probability $\Pr\{\beta_{ji} = \theta | \bar{\mathcal{E}}_j\}$ of this element of the combination matrix $\mathbf{P}$ can be modeled as

$$\Pr\{\beta_{ji} = \theta | \bar{\mathcal{E}}_j\} = \begin{cases} \delta_{U_i,R_j} & \text{if } \theta = 0, \\ \left(1 - \delta_{U_i,R_j}\right)/(q-1) & \text{if } \theta \neq 0, \end{cases} \tag{7}$$

where δ_{U_i,R_j} denotes the probability that the outage occurs from the ith source U_i to the jth relay R_j. Each outage probability δ_{U_i,R_j} can be determined independently in (1).

In (7), the elements of $\mathbf{P}$ are nonzero when both outage events do not occur simultaneously; however, when an outage event from the jth relay to the BS occurs, i.e., when $\mathcal{E}_j$ is true, the conditional probability is set as $\Pr\{\beta_{ji} = 0 | \mathcal{E}_j\} = 1$, regardless of the condition of the outage event (source-relay).

2) MDS Distribution: Next, we consider modeling the elements of $\mathbf{P}$ based on the systematic generator matrix of MDS codes. The difference from the aforementioned uniform distribution is that the nonzero value of each element should be taken from the pertinent value of a predefined MDS code. In this subsection, we refer to this as the MDS distribution. By considering the MDS distribution, we can compare its reconstruction performance to that of the uniform distribution given in (7). For the MDS distribution, the conditional probability $\Pr\{\beta_{ji} = \theta | \bar{\mathcal{E}}_j\}$ defined similarly to that in (7) is given by

$$\Pr\{\beta_{ji} = \theta | \bar{\mathcal{E}}_j\} = \begin{cases} \delta_{U_i,R_j} & \text{if } \theta = 0, \\ 1 - \delta_{U_i,R_j} & \text{if } \theta = \chi, \\ 0 & \text{otherwise}, \end{cases} \tag{8}$$

where χ denotes the coefficient that is predefined from the systematic generator matrix of MDS codes. To generate this code, we used the software application SAGE [41]. For $N = 8$ and $M = 4$, for example, the 4×8 submatrix of the systematic MDS code is:

$$\begin{bmatrix} 9 & 13 & 14 & 7 & 2 & 15 & 13 & 12 \\ 15 & 3 & 9 & 12 & 12 & 10 & 12 & 2 \\ 14 & 9 & 12 & 7 & 8 & 1 & 3 & 7 \\ 4 & 5 & 5 & 10 & 9 & 3 & 4 & 1 \end{bmatrix}. \quad (9)$$

In example (9), the conditional probability $\Pr\{\beta_{11} = 9|\bar{\mathcal{E}}_1\}$ is $1 - \delta_{U_1,R_1}$, and for any $\theta \in \mathbb{F}_{16} \setminus \{0, 9\}$, it is set to zero, i.e., $\Pr\{\beta_{11} = \theta|\bar{\mathcal{E}}_1\} = 0$. Based on this, we can investigate the improvement in the reconstruction performance that is achieved when using MDS codes in a cooperative wireless network.

Remark 1: In this work, we assume that all the inter-node channels are independent from each other. Thereby, probability distributions of random elements are defined independently. If channel correlations are considered, the distributions, i.e., (6)–(8), should be modeled as a joint distribution corresponding to the channel correlation. Using joint distributions, we can evaluate the performance of correlated wireless network coding schemes. Example 3 in Section V-B of this paper shows that the proposed framework can be extended to correlation cases. A generalized version of the proposed framework for channel correlations will be another direction of future research.

V. Upper Bound on Reconstruction of Messages

If a transmission matrix for a dynamic network topology randomly generated using the probability distributions given in (6)–(8) has full rank, the BS can uniquely decode all messages from all sources. In this section, we aim to derive an upper bound on the decoding failure probability, and the dimension of the nullspace of the random transmission matrix of an (N, M) cooperative wireless network. We then connect them to investigate the manner in which network coding performance varies based on wireless channel conditions, the number of relays, field sizes, and the positions of nodes deployed in a 2D space. Throughout this paper, we use the random transmission matrix as a bold face, i.e., $\mathbf{A}$, while the realized transmission matrix in sans-serif style, i.e., A.

We define the dimension of the nullspace within the column space of a transmission matrix as follows. Let A be an $(N+M)\times N$ matrix over the finite field with size q as $\mathbb{F}_q$. Based on linear algebra theory, the columns $A_1, \ldots, A_N$ of A are linearly dependent if and only if a vector $\mathbf{c} = (c_1, \ldots, c_N) \in \mathbb{F}_q^N$ such that exists, with at least one nonzero c_i, such that

$$\sum_{i=1}^{N} c_i A_i = 0. \quad (10)$$

Definition 1. (Number of Nonzero Coefficient Vectors): Let $L(\mathsf{A})$ be the number of all such nonzero vectors $\mathbf{c}$ belonging to the nullspace of the given matrix A. Let the column rank of a realized transmission matrix be $\mathsf{rank}(\mathsf{A})$. Thus, $L(\mathsf{A})$ can be represented as

$$L(\mathsf{A}) = q^{N-\mathsf{rank}(\mathsf{A})} - 1. \quad (11)$$

Definition 2. (Nullity): Let $\mathsf{nullity}(\mathsf{A})$ be the dimension of the nullspace in the column space of A.

Proposition 3: For a random matrix $\mathbf{A}$, the expectation of the nullity of $\mathbf{A}$ is upper bounded by $\mathbb{E}[\mathsf{nullity}(\mathbf{A})] \le \log_q(\mathbb{E}[L(\mathbf{A})] + 1)$, where $\mathbb{E}[\cdot]$ denotes the expectation.

Proof: For any $(N + M) \times N$ matrix A, we follow $\mathsf{nullity}(\mathsf{A}) = N - \mathsf{rank}(\mathsf{A})$, known as the rank-nullity theorem of linear algebra [42]. Considering the expectation for a random transmission matrix in both sides of (11), we obtain the following upper bound using Jensen's inequality:

$$\begin{aligned} \mathbb{E}\left[\mathsf{nullity}(\mathbf{A})\right] &:= N - \mathbb{E}\left[\mathsf{rank}(\mathbf{A})\right] \\ &= \mathbb{E}\left[\log_q\left(L(\mathbf{A}) + 1\right)\right] \\ &\le \log_q\left(\mathbb{E}\left[L(\mathbf{A})\right] + 1\right). \end{aligned} \quad (12)$$

The proof of Proposition 3 is complete. ■

Theorem 4: Let P_{fail} be the decoding failure probability for the reconstruction of source messages. Then, $P_{fail} \le \min\{1, 1/(q-1)\mathbb{E}[L(\mathbf{A})]\}$.

Proof: The probability P_{fail} is defined and upper bounded by

$$\begin{aligned} P_{fail} &= \Pr\left\{\mathsf{rank}(\mathbf{A}) < N\right\} \\ &= \Pr\left\{\exists \mathbf{c} : \sum_{i=1}^{N} c_i A_i = 0\right\} \\ &\overset{(a)}{\le} \sum_{\mathbf{c}\in\mathbb{F}_q^N\setminus\{0^T\}} \Pr\{\mathbf{A}\mathbf{c} = 0^T\} \\ &= \mathbb{E}\left[L(\mathbf{A})\right]. \end{aligned} \quad (13)$$

where inequality (a) is due to the union bound; note that $\mathbb{E}[L(\mathbf{A})] = \sum_{\mathbf{c}\in\mathbb{F}_q^N\setminus\{0^T\}} \Pr\{\mathbf{A}\mathbf{c} = 0^T\}$. Then, the upper bound on the probability P_{fail} can be tightened as

$$P_{fail} \le \min\left\{1, \frac{1}{q-1}\mathbb{E}\left[L(\mathbf{A})\right]\right\}, \quad (14)$$

where the $1/(q-1)$ factor is due to the following reason. Suppose a nonzero vector $\mathbf{c}$ exists such that $\mathbf{A}\mathbf{c} = 0^T$. Then, other $q - 2$ nonzero vectors $\theta\mathbf{c}, \theta^2\mathbf{c}, \ldots, \theta^{q-2}\mathbf{c}$ certainly exist for a primitive element $\theta \in \mathbb{F}_q \setminus \{0\}$, where each satisfies $\mathbf{A}\theta^i\mathbf{c} = 0^T$ for $i \in \{1, \ldots, q-2\}$. Then, we note $\bigcup_{\mathbf{c_1}\in\{\theta\mathbf{c},\ldots,\theta^{q-2}\mathbf{c}\}}\{\mathbf{A} : \mathbf{A}\mathbf{c_1} = 0^T\} = \{\mathbf{A} : \mathbf{A}\mathbf{c} = 0^T\}$. ■

Remark 2: Proposition 3 and Theorem 4 provide a groundwork novel performance evaluation framework of cooperative wireless network coding schemes. These results enable us to calculate the decoding failure probability without an exhaustive search of all possible individual cases. They are new key steps that enable the evaluation framework to be computationally efficient, are not available in the literature, for example [5]–[8].

Next, we will derive $\mathbb{E}[L(\mathbf{A})]$ for three types of cooperative wireless networks.

A. Homogeneous and Heterogeneous Connectivity

In this subsection, we aim to find $\mathbb{E}[L(\mathbf{A})]$ for two cases: i) *homogeneous connectivity* in which all outage probabilities in the wireless network are equal; i.e., $\delta = \delta_{U_i,BS} = \delta_{U_i,R_j} = \delta_{R_j,BS}$ given in (1) and (2) for $i \in \{1, 2, \ldots, N\}$ and $j \in \{1, 2, \ldots, M\}$, assuming that all the channel qualities in networks are equal, and ii) *heterogeneous connectivity* in which two types of outage probabilities exist, i.e., $\delta_1 = \delta_{U_i,BS} = \delta_{U_i,R_j}$ and $\delta_2 = \delta_{R_j,BS}$, with each outage probability assumed to be merely a function of transmit power. Note that each element of a random matrix follows the probability distributions defined in (6) and (7).

Let S_k denote the probability, $S_k := \Pr\{\sum_{i=1}^{k} \beta_{ji} = 0\}$, for the sum of the first k random elements, $k \in \{1, 2, \ldots, N\}$, in the jth row of a combination matrix $\mathbf{P}$. For the homogeneous case, Lemma 5 provides this probability S_k.

Lemma 5: For the homogeneous connectivity with the distributions (6) and (7), the probability S_k is given by

$$S_k = \delta + (1-\delta)\left(q^{-1} + (1-q^{-1})\left(1 - \frac{1-\delta}{1-q^{-1}}\right)^k\right). \quad (15)$$

Proof: See Appendix A. ∎

Before attempting to derive $\mathbb{E}[L(\mathbf{A})]$ of a random matrix $\mathbf{A}$ from Lemma 5, recall that $L(\mathsf{A})$ is the number of all nonzero vectors $\mathbf{c}$ satisfying the linear dependency in (10). The following Proposition 6 gives $\mathbb{E}[L(\mathbf{A})]$ for the homogeneous (N, M) wireless cooperative network.

Proposition 6: Given an (N, M) cooperative network with the homogeneous connectivity based on some outage probability δ, $\mathbb{E}[L(\mathbf{A})]$ of a $(N + M) \times N$ random transmission matrix $\mathbf{A}$ over the finite field $\mathbb{F}_q$ is

$$\mathbb{E}[L(\mathbf{A})] = \sum_{k=1}^{N} \binom{N}{k} (q-1)^k \delta^k \Bigg[\delta + (1-\delta) \times \left(q^{-1} + (1-q^{-1})\left(1 - \frac{1-\delta}{1-q^{-1}}\right)^k\right)\Bigg]^M. \quad (16)$$

Proof: See Appendix B. ∎

Consider $\mathbb{E}[L(\mathbf{A})]$ under the heterogeneous case; i.e., $\delta_1 = \delta_{U_i,BS} = \delta_{U_i,R_j}$ and $\delta_2 = \delta_{R_j,BS}$. In the following section, we aim to study the manner in which the outage probabilities δ_1 and δ_2 affect the recovery performance, assuming that these outage probabilities rely on the transmit power at the sources and relays.

Proposition 7: Given the heterogeneous (N, M) cooperative network defined by the two outage probabilities δ_1 and δ_2, $\mathbb{E}[L(\mathbf{A})]$ of a $(N + M) \times N$ random transmission matrix $\mathbf{A}$ over finite fields $\mathbb{F}_q$ is given by

$$\mathbb{E}[L(\mathbf{A})] = \sum_{k=1}^{N} \binom{N}{k} (q-1)^k \delta_1^k \Bigg[\delta_2 + (1-\delta_2) \times \left(q^{-1} + (1-q^{-1})\left(1 - \frac{1-\delta_1}{1-q^{-1}}\right)^k\right)\Bigg]^M. \quad (17)$$

The proof is omitted. However, it can be proved by following the formalism given in Proposition 6, using two outage probabilities, δ_1 and δ_2, instead of single outage probability as in Proposition 6.

B. General Connectivity

Thus far, we have obtained $\mathbb{E}[L(\mathbf{A})]$ of a random transmission matrix $\mathbf{A}$ for homogeneous and heterogeneous cases. In this subsection, we extend it to a more general case where δ_{U_i,R_j}, $\delta_{R_j,BS}$, and $\delta_{U_i,BS}$ are used as defined in Section III. Outage probabilities for the wireless links are obtained using (1) and (2), which are functions of transmit power and variance of the channel gain. After obtaining the outage probability for the each link, we determine the probability distributions for all elements in $\mathbf{A}$. We call this the *general connectivity* case. Since it involves an exhaustive search of combinations of column vectors in a random matrix, the general connectivity requires a more complicated computation than that of the homogeneous and heterogeneous connectivity to derive $\mathbb{E}[L(\mathbf{A})]$ where the proposed approach using the upper bound on the dimension of nullspace will be most effective.

Proposition 8: Given an (N, M) cooperative network with the general connectivity based on the outage probabilities defined in (1) and (2), $\mathbb{E}[L(\mathbf{A})]$ of a $(N + M) \times N$ random transmission matrix $\mathbf{A}$ over finite fields $\mathbb{F}_q$ is

$$\mathbb{E}[L(\mathbf{A})] = \sum_{k=1}^{N} (q-1)^k Q_k, \quad (18)$$

where $Q_k := \sum_{l=1}^{|\mathcal{L}_k|} Q_{k,l}$, $l \in \{1, 2, \ldots, |\mathcal{L}_k|\}$, $|\mathcal{L}_k| := \binom{N}{k}$, and $\mathcal{L}_{k,l}$ is the lth entry of a set $\mathcal{L}_k$. Let $\mathcal{L}_k$ denote the collection of the sets of k distinct indices among $[N] := \{1, 2, \ldots, N\}$, i.e., $\mathcal{L}_k := \{\{\lambda_1, \lambda_2, \ldots, \lambda_k\} : \lambda_i \in \{1, 2, \ldots, N\}, \lambda_i \neq \lambda_j, i \neq j\}$. $Q_{k,l} := \Pr\{\sum_{i \in \mathcal{L}_{k,l}} c_i A_i = 0\}$.

Proof: See Appendix C. ∎

We use Proposition 8 to obtain $\mathbb{E}[L(\mathbf{A})]$ for $q = 2$ in a (2, 1) cooperative wireless network as follows.

Example 2: Let us consider a (2, 1) cooperative wireless network for $q = 2$, $N = 2$, and $M = 1$. There are three nonzero vectors $\mathbf{c}$ in $\mathbb{F}_2^2$: (10), (01), and (11). For each nonzero vector, we obtain the probability $Q_{k,l}$ as follows. First, the probability $Q_{1,1}$ is

$$\begin{aligned} Q_{1,1} &= \Pr\{c_1 A_1 = 0\} \\ &= \Pr\{\alpha_{11} = 0\} \Pr\{\beta_{11} = 0\} \\ &= \delta_{U_1,BS} \left(\delta_{R_1,BS} + (1 - \delta_{R_1,BS})\, \delta_{U_1,R_1}\right). \end{aligned} \quad (19)$$

The probability $Q_{1,2}$ is

$$\begin{aligned} Q_{1,2} &= \Pr\{c_2 A_2 = 0\} \\ &= \Pr\{\alpha_{22} = 0\} \Pr\{\beta_{12} = 0\} \\ &= \delta_{U_2,BS} \left(\delta_{R_1,BS} + (1 - \delta_{R_1,BS})\, \delta_{U_2,R_1}\right). \end{aligned} \quad (20)$$

The probability $Q_{2,1}$ is

$$\begin{aligned} Q_{2,1} &= \Pr\{c_1 A_1 + c_2 A_2 = 0\} \\ &= \Pr\{\alpha_{11} = 0\} \Pr\{\alpha_{22} = 0\} \Pr\{\beta_{11} + \beta_{12} = 0\} \\ &= \delta_{U_1,BS} \delta_{U_2,BS} \big(\delta_{R_1,BS} \\ &\quad + (1 - \delta_{R_1,BS}) \Pr\{\beta_{11} + \beta_{12} = 0 | \bar{\mathcal{E}}_1\}\big). \end{aligned} \quad (21)$$

In this example, $\mathbb{E}[L(\mathbf{A})]$ is then given by

$$\mathbb{E}\left[L(\mathbf{A})\right] = Q_{1,1} + Q_{1,2} + Q_{2,1}. \tag{22}$$

■

In addition, it would be intriguing to determine if the proposed evaluation framework developed thus far can be extended to cases where the outages between different links are not independent, but correlated. Such cases may occur when the channels between two nodes are not perfectly orthogonal. Then, such a case becomes an interesting problem to show how the proposed evaluation framework can be utilized to compute the decoding failure probability in the correlated link outages cases. In general, this is a difficult task and would require an additional research paper to effectively provide all details. These details will be provided in a future work. In this paper, we aim to show that the framework is extendible to correlated link outage cases.

For this purpose, we again utilize Proposition 8 to compute $\mathbb{E}[L(\mathbf{A})]$ and extend Example 2 for correlated cases. The outage probabilities are not independent from each other. This can be addressed by considering a joint probability distribution for the random matrix. Using the joint probability distribution, we can again compute the last line of (44) in Appendix, instead of the product of probabilities. This is the main change that allows correlated cases to extend the proposed evaluation framework. In Example 3, given the joint probability distributions, we compute $Q_{1,1}$, $Q_{1,2}$, and $Q_{2,1}$ as shown in Example 2.

Example 3: Consider a (2, 1) cooperative wireless network for $q = 2$, $N = 2$, and $M = 1$. There are two sets of channel correlations that are assumed in this example. The first set of correlated channels is between $U_1 - R_1$ and $U_2 - R_1$; the second set of correlated channels is between U_1-BS and U_2-BS. We assume that all other combinations of channels are mutually independent. Note that both sets of correlations occur in the broadcasting phase. A pair of two outage events, U_1-BS and U_2-BS, create a joint probability as $\Pr\{\alpha_{11} = \theta_1, \alpha_{22} = \theta_2\} = \Theta_{\theta_1,\theta_2}$ for each $(\theta_1, \theta_2) \in \mathbb{F}_2^2$, where $\sum_{\theta_1,\theta_2} \Theta_{\theta_1,\theta_2} = 1$. For example, when both channels are simultaneously successful during the broadcasting phase, we can set the particular probability as $\Pr\{\alpha_{11} = 1, \alpha_{22} = 1\} = \Theta_{1,1}$. Similarly, other probabilities can be defined according to the conditions of the two outage events, U_1-BS and U_2-BS. Note that the conditional joint probability is set as $\Pr\{\beta_{11} = 0, \beta_{12} = 0|\mathcal{E}_1\} = 1$ since the two elements, β_{11} and β_{12}, are zero when the channel outage between R_1 and the BS occurs. In addition, a set of the two outage events, $U_1 - R_1$ and $U_2 - R_1$, can determine the values of the two random elements β_{11} and β_{12} once the channel outage between R_1 and BS does not occur, i.e., when $\bar{\mathcal{E}}_1$ is true. In this case, let the conditional joint probability distribution be known acknowledged and given as $\Pr\{\beta_{11} = \gamma_1, \beta_{12} = \gamma_2|\bar{\mathcal{E}}_1\} = \Gamma_{\gamma_1,\gamma_2}$ for each $(\gamma_1, \gamma_2) \in \mathbb{F}_2^2$, where $\sum_{\gamma_1,\gamma_2} \Gamma_{\gamma_1,\gamma_2} = 1$. For $q = 2$, Table III summarizes this conditional joint probability distribution according to the conditions of the two outage events $U_1 - R_1$ and $U_2 - R_1$.

For three nonzero vectors $\mathbf{c}$ in $\mathbb{F}_2^2$, we can again compute $Q_{1,1}$, $Q_{1,2}$, and $Q_{2,1}$. The computation of $Q_{1,1}$ and $Q_{1,2}$ is straightforward because of our assumption that the two sets of channel correlations are independent. The results are

TABLE III
DETERMINATION OF THE CONDITIONAL JOINT PROBABILITY FOR THE TWO ELEMENTS β_{11} AND β_{12}, WHERE "O" INDICATES NO OUTAGE AND "×" INDICATES AN OUTAGE

(U_1-R_1, U_2-R_1)	(β_{11},β_{12})	$\Pr\{\beta_{11} = \gamma_1, \beta_{12} = \gamma_2 \lvert \bar{\mathcal{E}}_1\}$
(O, O)	(1, 1)	$\Gamma_{1,1}$
(O, ×)	(1, 0)	$\Gamma_{1,0}$
(×, O)	(0, 1)	$\Gamma_{0,1}$
(×, ×)	(0, 0)	$\Gamma_{0,0}$

$$Q_{1,1} = (\Theta_{0,0} + \Theta_{0,1})\left(\delta_{R_1,BS} + (1 - \delta_{R_1,BS})(\Gamma_{0,0} + \Gamma_{0,1})\right) \tag{23}$$

and

$$Q_{1,1} = (\Theta_{0,0} + \Theta_{1,0})\left(\delta_{R_1,BS} + (1 - \delta_{R_1,BS})(\Gamma_{0,0} + \Gamma_{1,0})\right) \tag{24}$$

The computation of $Q_{2,1}$ is given as follows:

$$\begin{aligned} Q_{2,1} &= \Pr\{c_1 A_1 + c_2 A_2 = 0\} \\ &= \Pr\{\alpha_{11} = 0, \alpha_{22} = 0, \beta_{11} + \beta_{12} = 0\} \\ &= \Pr\{\alpha_{11} = 0, \alpha_{22} = 0, \beta_{11} + \beta_{12} = 0|\mathcal{E}_1\} \Pr\{\mathcal{E}_1\} \\ &\quad + \Pr\{\alpha_{11} = 0, \alpha_{22} = 0, \beta_{11} + \beta_{12} = 0|\bar{\mathcal{E}}_1\} \Pr\{\bar{\mathcal{E}}_1\} \\ &\overset{(a)}{=} \Pr\{\alpha_{11} = 0, \alpha_{22} = 0\} \\ &\quad \times \left(\Pr\{\beta_{11} + \beta_{12} = 0|\mathcal{E}_1\} \Pr\{\mathcal{E}_1\}\right. \\ &\quad\quad \left. + \Pr\{\beta_{11} + \beta_{12} = 0|\bar{\mathcal{E}}_1\} \Pr\{\bar{\mathcal{E}}_1\}\right) \\ &= \Theta_{0,0}\left(\delta_{R_1,BS} + (\Gamma_{0,0} + \Gamma_{1,1})(1 - \delta_{R_1,BS})\right), \end{aligned} \tag{25}$$

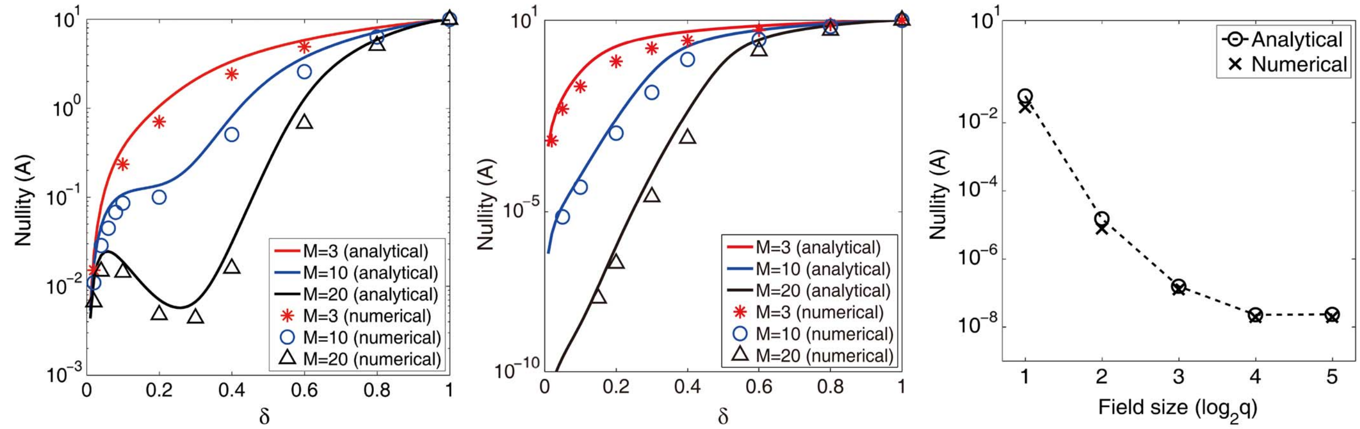
where equality (a) is based on the fact that the relation between the two sets, $(\alpha_{11}, \alpha_{22})$ and (β_{11}, β_{12}), is independent. We finally obtain $\mathbb{E}[L(\mathbf{A})] = Q_{1,1} + Q_{1,2} + Q_{2,1}$ for correlated cases by using the proposed evaluation framework. ■

C. Asymptotic Nullity

In practice, the computation of (18) requires a significant amount of time, because all combinations of column vectors must be collected as the number of sources and relays increases. In larger networks, this process is complicated; therefore, it will be beneficial to reduce the resources required for this computation. In this subsection, we aim to obtain an asymptotic form of (18) for utilization in large-scale networks.

As previously mentioned, the homogeneous connectivity scheme is a specific case among general connectivity schemes. We can exhibit a simple form of $\mathbb{E}[L(\mathbf{A})]$ in terms of Q_k for the homogeneous topology of cooperative networks. Based on this approach, we can obtain an asymptotic result of (18) in general connectivity schemes. Let us consider $\mathbb{E}[L(\mathbf{A})]$ for $q = 2$ in the homogeneous connectivity. Thus, $\mathbb{E}[L(\mathbf{A})] = \sum_{k=1}^{N} Q_k$ in (18). Using (44), Q_1 is given by

$$Q_1 = N\delta S_1. \tag{26}$$



(a) (b) (c)

Fig. 3. The nullity of random matrix $\mathbf{A}$ for a homogeneous (10, M) cooperative wireless network with $N = 10$ and $M = 3, 10,$ and 20. Solid lines indicate the upper bounds on $\mathbb{E}[\mathsf{nullity}(\mathbf{A})]$ using Proposition 6, and markers indicate numerically simulated results of $\mathbb{E}[\mathsf{nullity}(\mathbf{A})]$, respectively: (a) $q = 2$ and (b) $q = 4$. Fig. 3(c) shows $\mathbb{E}[\mathsf{nullity}(\mathbf{A})]$ with different field sizes from $q = 2$ to 32 under fixed $N = 10$, $M = 10$, and $\delta = 0.05$.

For $k = 2$, we have $Q_2 = \binom{N}{2}\delta^2 S_2$. We further obtain $Q_3 = \binom{N}{3}\delta^3 S_3$ for $k = 3$. The general expression of Q_k is given by

$$Q_k = \binom{N}{k}\delta^k S_k. \tag{27}$$

In this case, $\mathbb{E}[L(\mathbf{A})]$ in (18) is

$$\mathbb{E}\left[L(\mathbf{A})\right] = \sum_{k=1}^{N}\binom{N}{k}\delta^k S_k. \tag{28}$$

In high SNR regions, assuming δ is small, an approximation form of (28) is obtained as

$$\begin{aligned}\mathbb{E}\left[L(\mathbf{A})\right] &= \sum_{k=1}^{N} Q_k \\ &\overset{(a)}{\approx} \binom{N}{1}\delta S_1 + \binom{N}{2}\delta^2 S_2,\end{aligned} \tag{29}$$

where (a) is based on the fact that the order of Q_k for $k \geq 3$ is greater than two with respect to δ. This approximation indicates that for the computation of $\mathbb{E}[L(\mathbf{A})]$, two terms Q_1 and Q_2 are sufficient in high SNR regions. Therefore, in the high SNR regions, $\mathbb{E}[L(\mathbf{A})]$ converges to the second order of the transmit SNR. For any finite field and the general connectivity, this approximation is satisfied.

Corollary 9: Given an (N, M) cooperative network with general connectivity having the distributions (6) and (7), $\mathbb{E}[L(\mathbf{A})]$ is simplified in the high SNR regime

$$\mathbb{E}\left[L(\mathbf{A})\right] \approx (q-1)Q_1 + (q-1)^2 Q_2. \tag{30}$$

Remark 3: Proposition 8 provides a closed-form solution to the expectation of the number of nonzero vectors in the nullspace of the random transmission matrix $\mathbf{A}$. This enables us to evaluate the performance of a general network with randomly deployed nodes, without separately processing each count of transmission matrix count. Corollary 9 is a closed-form approximation of (18) that is useful for performance evaluations of large size networks.

VI. Numerical and Simulation Results

In this section, the performance of source message reconstruction at the BS is investigated by utilizing the proposed evaluation framework, i.e., $\mathbb{E}[\mathsf{nullity}(\mathbf{A})]$ and P_{fail}. For the homogeneous connectivity scheme, we employ Proposition 6 to analytically derive the upper bound on $\mathbb{E}[\mathsf{nullity}(\mathbf{A})]$ defined in Section V as a function of the outage probability of the single channel link. We compare the upper bounds with the numerically simulated results of $\mathbb{E}[\mathsf{nullity}(\mathbf{A})]$ as well as P_{fail}. Subsequently, we employ Proposition 8 to investigate the upper bounds of a general cooperative network in which sources and relays are deployed in a 2D space. Furthermore, we show the results of the upper bound on $\mathbb{E}[\mathsf{nullity}(\mathbf{A})]$ and P_{fail} for a given transmission matrix to investigate the impact of the number of relays and the field size of network coding.

Fig. 3 shows analytically obtained upper bounds and numerically averaged results of $\mathbb{E}[\mathsf{nullity}(\mathbf{A})]$ for a random transmission matrix in a (10, M) cooperative wireless network given the homogeneous connectivity scheme, where $N = 10$ and $M = 3$, 10, and 20 for $q = 2$ in Fig. 3(a), and 4 in Fig. 3(b). We observe that $\mathbb{E}[\mathsf{nullity}(\mathbf{A})]$ increases as the outage probability slightly increases. Based on Fig. 3(b), it is evident that a nonbinary network coding scheme provides superior reconstruction performance for source messages at the BS when compared to binary coding; moreover, increasing the field size of network coding also improves recovery performance. As the outage probability is reduced to zero, $\mathbb{E}[\mathsf{nullity}(\mathbf{A})]$ approaches zero for all field sizes.

Fig. 4 shows the analytically derived upper bounds using Proposition 7 for a heterogeneously connected (20, 20) cooperative wireless network. Regardless of the value of δ_2, when the outage probability δ_1 approaches one, $\mathbb{E}[\mathsf{nullity}(\mathbf{A})]$ barely reaches 20 for both field sizes, i.e., $q = 2$ and 4. This indicates that all channel links undergo outage events, causing all elements of the transmission matrix to become zero. In Fig. 4(a), for $q = 2$, there is an oscillation in the proximity of $\delta_1 = 0.3$ such that $\mathbb{E}[\mathsf{nullity}(\mathbf{A})]$ decreases as δ_1 increases to 0.3, and beyond this point $\mathbb{E}[\mathsf{nullity}(\mathbf{A})]$ increases. This oscillation also appears in Fig. 3(a). This behavior results from the fact that the rows of $\mathbf{P}$ tend to be identical as the outage probabilities

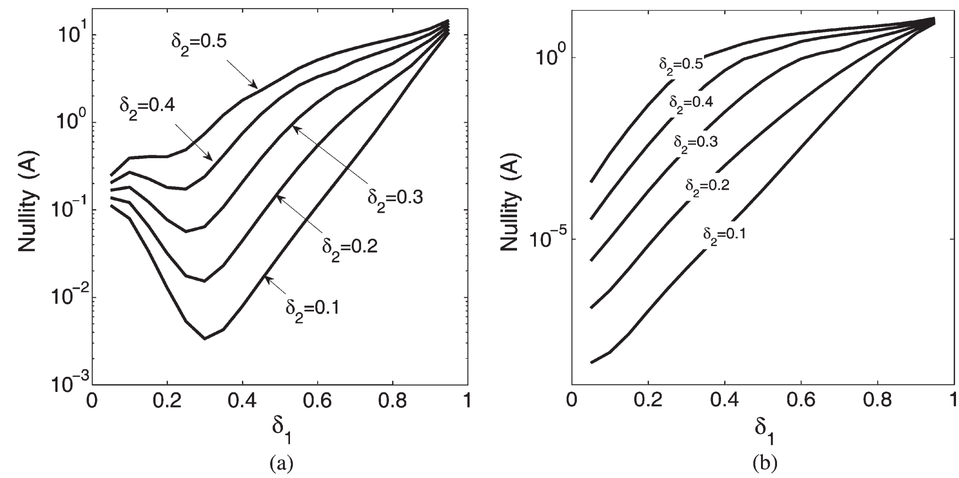


Fig. 4. Upper bounds on $\mathbb{E}[\mathsf{nullity}(\mathbf{A})]$ using Proposition 3 and 7 in a heterogeneous (20, 20) cooperative wireless network with two outage probabilities δ_1 and δ_2 for (a) $q = 2$, and (b) $q = 4$.

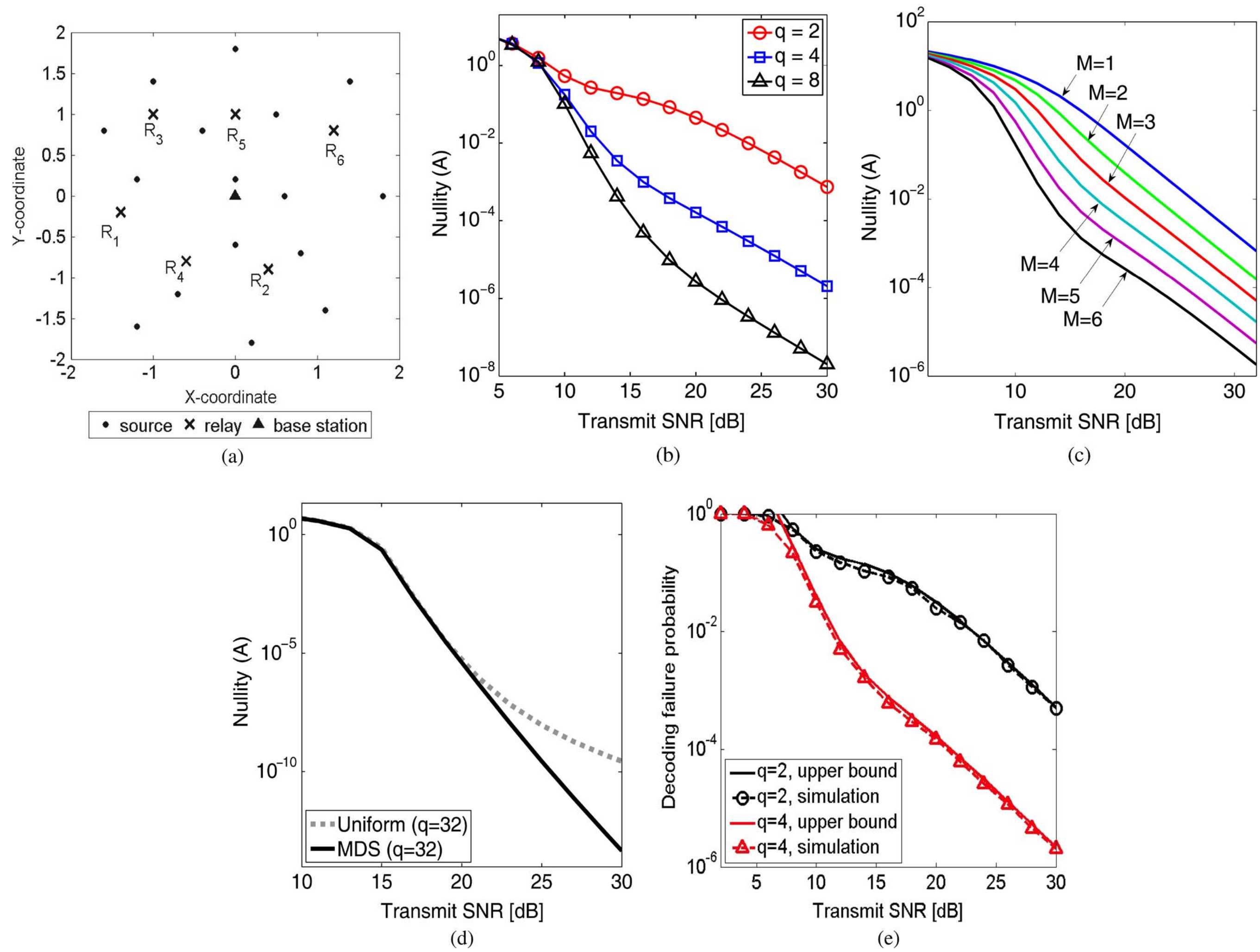
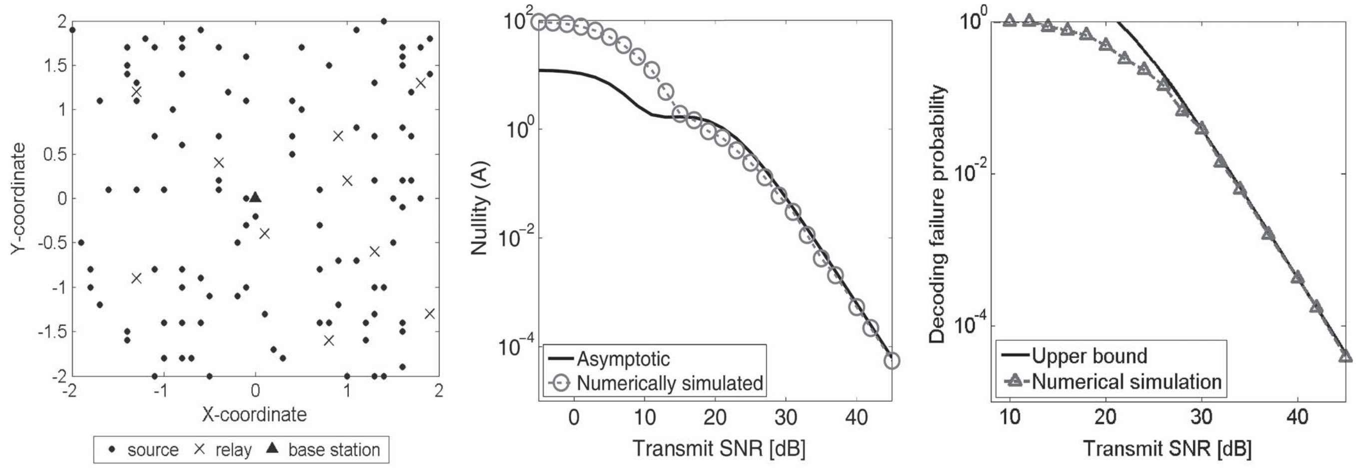


Fig. 5. (a) Location of 16 sources and 6 relays in 2D space for an (16, 6) cooperative wireless network. (b) Results of upper bounds on $\mathbb{E}[\mathsf{nullity}(\mathbf{A})]$ with differing network coding field sizes $q = 2, 4$, and 8, (c) varying the number of relays at $q = 4$. (d) Comparison of upper bounds on $\mathbb{E}[\mathsf{nullity}(\mathbf{A})]$ for the uniform and MDS distributions. (e) Comparison of the decoding failure probabilities with the numerical simulation and the upper bound using Proposition 8 for $q = 2$ and 4.

δ_1 and δ_2 approach zero. For $q = 4$, however, this behavior disappears owing to the extension of the field size from binary to quaternary.

Now, let us consider the (16, 6) cooperative wireless network shown in Fig. 5(a), in which there are 16 sources and 6 relays: R_1 through R_6. We randomly deploy these relays in a 2D space. We assume that all the transmit powers of sources and relays in both transmission phases are equal. Fig. 5(b) shows the upper bound on $\mathbb{E}[\mathsf{nullity}(\mathbf{A})]$ of the transmission matrix for $q = 2, 4,$ and 8; the benefit of increasing the field size of



(a) (b) (c)

Fig. 6. (a) Locations of 100 sources and 10 relays in a 2D space for an (100, 10) cooperative wireless network, (b) Comparison of $\mathbb{E}[\text{nullity}(\mathbf{A})]$ with numerically simulated result and the upper bound using Corollary 9 with $q = 2$ and the uniform distribution. (c) Comparison of the decoding failure probability with the numerical simulation and the upper bound using Corollary 9.

network coding appears in this scheme. Fig. 5(c) shows the upper bound on $\mathbb{E}[\text{nullity}(\mathbf{A})]$ with respect to the number of relays. When $M = 1$, R_1 is used, while R_1 and R_2 are used as relays for $M = 2$, and relays R_1, R_2, and R_3 are used for $M = 3$. For $M = 4, 5,$ and 6, we deploy one relay in order. We investigate the impact of the number of relays, as shown in Fig. 5(c), where increasing the number of relays contributes to the increasingly high likelihood of deriving random transmission matrices of full rank.

Other intriguing results indicate that the value of $\mathbb{E}[\text{nullity}(\mathbf{A})]$ differs slightly for the uniform and MDS distributions of the combination matrix defined in Section IV-B; furthermore, the recovery performance obtained using the MDS distribution is superior to that of the uniform distribution in high SNR regions. Comparative results of $\mathbb{E}[\text{nullity}(\mathbf{A})]$ for the two cooperative networks are shown in Fig. 5(d). We observe that there are minimal differences between the uniform and MDS distributions for the recovery performance in the low SNR regions. In particular, the benefit of using the systematic generator of MDS codes appears only in the high SNR regions.

To validate the usefulness of our asymptotic nullity, we consider an (100, 10) cooperative wireless network as a large-scale network in which 100 sources and 10 relays are deployed in the 2D space shown in Fig. 6(a). For Corollary 9, we show that in the high SNR regions, the asymptotic nullity of (30) is similar to the numerical results that were obtained from randomly generated transmission matrices. A comparison of those results is shown in Fig. 6(b). Using the asymptotic nullity, the complexity of (18) can be significantly reduced. In addition, the nullity of the random transmission matrix can be obtained efficiently. Our proposed framework provides the ability to evaluate reconstruction performance in large-scale networks.

Fig. 5(e) shows the comparison of numerically simulated decoding failure probabilities and upper bounds using (14) for an (16, 6) cooperative wireless network with $q = 2$ and 4. The gap between the results is evident in small SNR regions. However, the upper bound on the decoding failure probability is tight in high SNR regions. This behavior is shown in Fig. 6(c) in which the upper bound is obtained from the approximation form of $\mathbb{E}[\text{nullity}(\mathbf{A})]$ in (30). Based on those results, we show that predicting the performance of source message reconstruction for an (N, M) cooperative wireless network is straightforwardly possible in large-scale networks.

VII. Conclusion

In this paper, we considered a cooperative wireless network where N sources are assisted with M relays in two phase transmissions. Our main goal was to propose a new performance analysis framework for evaluating the reconstruction performance of source messages at the BS. To handle dynamic network topologies, we modeled the elements of the transmission matrix as random variables. This enabled us to develop a systematic approach, to avoid the exhaustive evaluations used in DNC and GDNC schemes [5]–[8]. To complete the performance evaluation, we derived two tight upper bounds on the expected dimension of the nullspace of the random transmission matrix, as well as the decoding failure probability. The result is a framework that is more effective than the rank-based method proposed in the previous literature.

Three types of connectivity schemes are considered in this paper, as they make the framework to be general and scalable. They enabled us to show the reconstruction performance of our proposed framework using multiple sources and multiple relays randomly deployed in a 2D space. In addition, it enables us to investigate the impact of the number of relays and the field size of network coding on the system performance; an example is shown in Figs. 5 and 6 for example. In particular, the ability to generate a precise prediction of network coding performance for a network with a large number of sources and relays is a significant benefit. We can formulate challenging scenarios and generate accurate response in an efficient manner, without resorting to extensive computer simulations. For example, we can determine the advantage that using an MDS code, rather than random code, provides when designing the transmission matrices, as relays are added and field sizes are increased; we can also determine how the position of relays and sources, with respect to the base station locations, affect the performance of cooperative communications. These questions are important engineering inquires in terms of wireless networks design. These questions, which could not be readily answered in the past, but can now be answered precisely using the proposed framework describe in this paper. In addition, we demonstrate that the proposed framework can be extended to channel correlation cases; however, it is necessary to generalize for any categorization of cooperative wireless network coding schemes.

APPENDIX A
PROOF OF LEMMA 5

For the homogeneous connectivity, we use $\delta = \delta_{U_i,BS} = \delta_{U_i,R_j} = \delta_{R_j,BS}$. The conditional probability $\Pr\{\beta_{ji} = \theta|\bar{\mathcal{E}}_j\}$ of each element β_{ji} is defined using (7), and the other conditional probability can be set as $\Pr\{\beta_{ji} = 0|\mathcal{E}_j\} = 1$. Using the total probability theorem, the probability $\Pr\{\sum_{i=1}^{k} \beta_{ji} = 0\}$ can be decomposed by the condition of the outage event $\mathcal{E}_j$, and then given as follows:

$$\begin{aligned} S_k &= \Pr\{\mathcal{E}_j\} \Pr\left\{ \sum_{i=1}^{k} \beta_{ji} = 0 \middle| \mathcal{E}_j \right\} \\ &\quad + \Pr\{\bar{\mathcal{E}}_j\} \Pr\left\{ \sum_{i=1}^{k} \beta_{ji} = 0 \middle| \bar{\mathcal{E}}_j \right\} \\ &\overset{(a)}{=} \delta + (1-\delta) \Pr\left\{ \sum_{i=1}^{k} \beta_{ji} = 0 \middle| \bar{\mathcal{E}}_j \right\}, \end{aligned} \tag{31}$$

where (a) follows from the fact that $\Pr\{\sum_{i=1}^{k} \beta_{ji} = 0|\mathcal{E}_j\} = 1$, as $\Pr\{\beta_{ji} = 0|\mathcal{E}_j\} = 1$ (note that the conditional probability $\Pr\{\beta_{ji} = \theta|\bar{\mathcal{E}}_j\}$ is independent of this). Let f_k be the probability, i.e., $f_k := \Pr\{\sum_{i=1}^{k} \beta_{ji} = 0|\bar{\mathcal{E}}_j\}$. Given the conditional probability defined in (7) and $f_0 = 1$, the probability f_k can be rewritten by [43]

$$\begin{aligned} f_k &= \Pr\left\{ \sum_{i=1}^{k-1} \beta_{ji} = 0 \middle| \bar{\mathcal{E}}_j \right\} \Pr\{\beta_{jk} = 0|\bar{\mathcal{E}}_j\} \\ &\quad + \sum_{\theta \in \mathbb{F}_q \setminus \{0\}} \Pr\left\{ \sum_{i=1}^{k-1} \beta_{ji} = \theta \middle| \bar{\mathcal{E}}_j \right\} \Pr\{\beta_{jk} = -\theta|\bar{\mathcal{E}}_j\} \\ &= f_{k-1}\delta + (1 - f_{k-1}) \frac{1-\delta}{q-1}. \end{aligned} \tag{32}$$

Let $g_k := f_k - q^{-1}$. By rewriting (32) as a function of g_k, we have a simple closed form:

$$g_k = g_{k-1} \left(1 - \frac{1-\delta}{1-q^{-1}} \right). \tag{33}$$

Applying a geometric series to (33), we obtain f_k as

$$f_k = q^{-1} + (1-q^{-1}) \left(1 - \frac{1-\delta}{1-q^{-1}} \right)^k. \tag{34}$$

Finally, the probability S_k can be obtained by substituting (34) into (31) as:

$$S_k = \delta + (1-\delta) \left(q^{-1} + (1-q^{-1}) \left(1 - \frac{1-\delta}{1-q^{-1}} \right)^k \right). \tag{35}$$

APPENDIX B
PROOF OF PROPOSITION 6

Let us consider a vector $\mathbf{c} = (c_1, \ldots, c_N) \in \mathbb{F}_q^N$ in which the first k entries (and only the first k entries) are nonzero, i.e., $\mathbf{c} = (c_1, \ldots, c_k, 0, \ldots, 0)$. Let P_k be the probability that the sum of the first k column vectors is zero, i.e., $P_k := \Pr\{\sum_{i=1}^{k} c_i A_i = 0\}$. As (13), $\mathbb{E}[L(\mathbf{A})]$ is given by

$$\begin{aligned} \mathbb{E}[L(\mathbf{A})] &= \sum_{\mathbf{c} \in \mathbb{F}_q^N \setminus \{0^T\}} \Pr\{\mathbf{Ac} = 0^T\} \\ &= \sum_{k=1}^{N} \binom{N}{k} (q-1)^k P_k. \end{aligned} \tag{36}$$

Since all links in the wireless network are assumed to be spatially and temporally independent, the rows of the transmission matrix are also independent. Thus, P_k is given by

$$\begin{aligned} P_k &= \Pr\left\{ \sum_{i=1}^{k} c_i A_i = 0 \right\} \\ &= \prod_{i=1}^{k} \Pr\{c_i \alpha_{ii} = 0\} \prod_{j=1}^{M} \Pr\left\{ \sum_{i=1}^{k} c_i \beta_{ji} = 0 \right\}. \end{aligned} \tag{37}$$

Let H_k be the probability as $H_k := \Pr\{\sum_{i=1}^{k} c_i \beta_{ji} = 0\}$. For $k = 1$, it is easy to show $\Pr\{c_1 \beta_{j1} = 0\} = \Pr\{\beta_{j1} = 0\}$ for $c_1 \in \mathbb{F}_q \setminus \{0\}$ because of the multiplication property in finite fields. Next, we prove that $H_k = S_k$ for $k \geq 2$ where $c_1, c_2, \ldots, c_k \in \mathbb{F}_q \setminus \{0\}$ denote the k nonzero elements. The probability H_k is represented by

$$\begin{aligned} H_k &= \sum_{\theta \in \mathbb{F}_q} \Pr\left\{ \sum_{i=1}^{k-1} c_i \beta_{ji} = \theta, c_k \beta_{jk} = -\theta \right\} \\ &= \Pr\left\{ \sum_{i=1}^{k-1} c_i \beta_{ji} = 0, c_k \beta_{jk} = 0 \right\} \\ &\quad + \sum_{\theta \in \mathbb{F}_q \setminus \{0\}} \Pr\left\{ \sum_{i=1}^{k-1} c_i \beta_{ji} = \theta, c_k \beta_{jk} = -\theta \right\}. \end{aligned} \tag{38}$$

Decomposing the outage event $\mathcal{E}_j$, (38) can be rewritten by

$$\begin{aligned} H_k &= \Pr\left\{ \sum_{i=1}^{k-1} c_i \beta_{ji} = 0, c_k \beta_{jk} = 0 \middle| \bar{\mathcal{E}}_j \right\} \Pr\{\bar{\mathcal{E}}_j\} \\ &\quad + \Pr\left\{ \sum_{i=1}^{k-1} c_i \beta_{ji} = 0, c_k \beta_{jk} = 0 \middle| \mathcal{E}_j \right\} \Pr\{\mathcal{E}_j\} \\ &\quad + \sum_{\theta \in \mathbb{F}_q \setminus \{0\}} \left(\Pr\left\{ \sum_{i=1}^{k-1} c_i \beta_{ji} = \theta, c_k \beta_{jk} = -\theta \middle| \bar{\mathcal{E}}_j \right\} \Pr\{\bar{\mathcal{E}}_j\} \right. \\ &\quad \left. + \Pr\left\{ \sum_{i=1}^{k-1} c_i \beta_{ji} = \theta, c_k \beta_{jk} = -\theta \middle| \mathcal{E}_j \right\} \Pr\{\mathcal{E}_j\} \right). \end{aligned} \tag{39}$$

Since $\Pr\{\beta_{ji} = 0|\mathcal{E}_j\} = 1$, (39) can be represented by

$$\begin{aligned} H_k &= \Pr\left\{ \sum_{i=1}^{k-1} c_i \beta_{ji} = 0, c_k \beta_{jk} = 0 \middle| \bar{\mathcal{E}}_j \right\} \Pr\{\bar{\mathcal{E}}_j\} + \Pr\{\mathcal{E}_j\} \\ &\quad + \sum_{\theta \in \mathbb{F}_q \setminus \{0\}} \Pr\left\{ \sum_{i=1}^{k-1} c_i \beta_{ji} = \theta, c_k \beta_{jk} = -\theta \middle| \bar{\mathcal{E}}_j \right\} \Pr\{\bar{\mathcal{E}}_j\}. \end{aligned} \tag{40}$$

Noting that wireless channels are independent from each other under the condition of $\bar{\mathcal{E}}_j$, (40) can be decomposed by

$$\begin{aligned} H_k = &\Pr\left\{\sum_{i=1}^{k-1} c_i\beta_{ji} = 0 \middle| \bar{\mathcal{E}}_j\right\} \Pr\left\{c_k\beta_{jk} = 0|\bar{\mathcal{E}}_j\right\} \Pr\{\bar{\mathcal{E}}_j\} \\ &+ \Pr\{\mathcal{E}_j\} + \sum_{\theta\in\mathbb{F}_q\setminus\{0\}} \Pr\left\{\sum_{i=1}^{k-1} c_i\beta_{ji} = \theta \middle| \bar{\mathcal{E}}_j\right\} \\ &\times \Pr\left\{c_k\beta_{jk} = -\theta|\bar{\mathcal{E}}_j\right\} \Pr\{\bar{\mathcal{E}}_j\}. \end{aligned} \tag{41}$$

Using recursion, the probability H_k is given by

$$\begin{aligned} H_k &= \Pr\left\{\sum_{i=1}^{k-1} \beta_{ji} = 0 \middle| \bar{\mathcal{E}}_j\right\} \Pr\{\beta_{jk} = 0|\bar{\mathcal{E}}_j\} \Pr\{\bar{\mathcal{E}}_j\} \\ &\quad + \Pr\{\mathcal{E}_j\} + \sum_{\theta\in\mathbb{F}_q\setminus\{0\}} \Pr\left\{\sum_{i=1}^{k-1} \beta_{ji} = \theta \middle| \bar{\mathcal{E}}_j\right\} \\ &\quad \times \Pr\{\beta_{jk} = -\theta|\bar{\mathcal{E}}_j\} \Pr\{\bar{\mathcal{E}}_j\} \\ &= \sum_{\theta\in\mathbb{F}_q} \Pr\left\{\sum_{i=1}^{k-1} \beta_{ji} = \theta, \beta_{jk} = -\theta\right\} \\ &= \Pr\left\{\sum_{i=1}^{k} \beta_{ji} = 0\right\}. \end{aligned} \tag{42}$$

Thus, we simply rewrite P_k as follows

$$P_k = \delta^k S_k^M. \tag{43}$$

The proof of Proposition 6 is complete.

Appendix C
Proof of Proposition 8

For general connectivity, each element of $\mathbf{A}$ have a different the probability distribution. This result in different probabilities $Q_{k,l}$ for that any k column vectors of $\mathbf{A}$ that are linearly dependent. The total number of $Q_{k,l}$ is $|\mathcal{L}_k| := \binom{N}{k}$. We have to consider all different probabilities $Q_{k,l}$ with respect to all sets $\mathcal{L}_{k,l}$. The probability Q_k should be summed over all different probabilities $Q_{k,l}$, i.e., $Q_k := \sum_{l=1}^{|\mathcal{L}_k|} Q_{k,l}$ for $l \in \{1, 2, \ldots, |\mathcal{L}_k|\}$ and $k \in \{1, 2, \ldots, N\}$. Thus, all $Q_{k,l}$ are enumerated and collected to obtain the probability Q_k, which is derived as follows:

$$\begin{aligned} Q_k &= \sum_{l=1}^{|\mathcal{L}_k|} Q_{k,l} = \sum_{l=1}^{|\mathcal{L}_k|} \Pr\left\{\sum_{i\in\mathcal{L}_{k,l}} c_i A_i = 0\right\} \\ &= \sum_{l=1}^{|\mathcal{L}_k|} \prod_{m=1}^{k} \Pr\{\alpha_{l_m l_m} = 0\} \prod_{j=1}^{M} \Pr\left\{\sum_{m=1}^{k} \beta_{jl_m} = 0\right\}, \end{aligned} \tag{44}$$

where l_m is the mth entry of the set $\mathcal{L}_{k,l}$, $m \in \{1, 2, \ldots, k\}$. As similarly obtained in (31), (44) can be rewritten as

$$\begin{aligned} Q_k = \sum_{l=1}^{|\mathcal{L}_k|} \prod_{m=1}^{k} \Pr\{\alpha_{l_m l_m} = 0\} \prod_{j=1}^{M} \Bigg(&\delta_{R_j,BS} \\ &+ \left(1 - \delta_{R_j,BS}\right) \Pr\left\{\sum_{m=1}^{k} \beta_{jl_m} = 0 \middle| \bar{\mathcal{E}}_j\right\}\Bigg). \end{aligned} \tag{45}$$

In order to determine $\mathbb{E}[L(\mathbf{A})]$, we count the number of vectors $\mathbf{c}$ having the first k nonzero elements, i.e., $(q-1)^k$. Finally, we perform the summation over all k, and obtain $\mathbb{E}[L(\mathbf{A})]$ as follows,

$$\mathbb{E}[L(\mathbf{A})] = \sum_{k=1}^{N} (q-1)^k Q_k. \tag{46}$$

The proof of Proposition 8 is complete.

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