[14] R. Nabar, D. Gore, and A. Paulraj, "Optimal selection and use of transmit antennas in wireless systems," in *Proc. Int. Conf. Telecommunications (ICT'00)*, Acapulco, Mexico, 2000.
[15] D. A. Gore and A. Paulraj, "Space-time block coding with optimal antenna selection," in *Proc. Int. Conf. Acoustics, Speech, and Signal Processing*, 2001, pp. 2441–2444.
[16] ——, "Statistical MIMO antenna sub-set selection with space-time coding," in *Proc. Int. Conf. Communications (ICC'2002)*, 2002, pp. 641–645.
[17] H. El Gamal and M. O. Damen, "An algebraic number theoric framework for space-time coding," in *Proc. IEEE Int. Symp. Information Theory*, Lausanne, Switzerland, June 2002, p. 132.
[18] S. Galliou and J.-C. Belfiore, "A new family of full rate, fully diverse space-time codes based on Galois theory," in *Proc. IEEE Int. Symp. Information Theory*, Lausanne, Switzerland, June 2002, p. 419.
[19] X. Ma and G. B. Giannakis, "Layered space-time complex field coding: Full diversity with full-rate, and tradeoffs," in *Proc. Sensor Array and Multichannel Signal Processing Workshop*, 2002, pp. 442–446.
[20] S. Baro, G. Bauch, and A. Hansmann, "Improved codes for space-time trellis codes," *IEEE Commun. Lett.*, vol. 41, pp. 20–22, Jan. 2000.

On Performance Limits of Space–Time Codes: A Sphere-Packing Bound Approach

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***Abstract*—A sphere-packing bound (SPB) on average word-error probability (WEP) is derived to determine the performance limits of space–time codes on Rayleigh block-fading channels under delay and maximum energy constraints. Two other explicit bounds, a looser bound and a tight approximate bound, are also derived to provide more intuition on how the system parameters affect the performance limits. Moreover, it is shown that as the block length grows to infinity, the SPB converges to the outage probability, and the asymptotic behavior of performance limits is determined by the outage probability.**

***Index Terms*—Fading channels, multiple antennas, outage probability, performance limits, space–time codes.**

I. INTRODUCTION

In this correspondence, a sphere-packing bound (SPB) is derived on the average word-error probability (WEP) of codes over Rayleigh block-fading multiple-input–multiple-output (MIMO) channels. The SPB defines a bound on the performance (in an average WEP sense) of all possible codes satisfying the design constraints. Thus, it can be used to specify the merit of a code in a sense that how far the code performs from the performance limits. Moreover, it can be used to obtain some intuition on how system parameters affect the performance limits. Because the SPB does not have a closed form, inequalities are used to obtain two other bounds with explicit expressions, namely, a looser bound and a tight approximate bound.

The results presented here show that for data rates less than the *ergodic capacity* [1], the performance limits improve significantly if a code spans a larger number of fading blocks. On the other hand, increasing the *block length* (length of portion of the code within one block) improves the performance limits marginally. In fact, as block length grows to infinity, the SPB converges to the *outage probability* (see [2] and [3] for definition); and the asymptotic behavior of performance limits of space–time codes are determined by the outage probability. Hence, for large block lengths, the SPB determines the *outage probability*.

To show how effective the SPB is in determining the performance, it is shown that for a 2×2 MIMO system, the 64-state space–time trellis-coded modulation developed in [4], performs 2.5 and 1.4 dB away from the performance limits for data rates of 3 and 2 bits/s/Hz, respectively.

In Section II, the system model is introduced. In Section III, it is shown that for sufficiently large dimension, the received signal space is bounded within a hyperellipsoid with an arbitrarily high probability. The derivation of the three sphere-packing lower bounds is given in Section IV. In Section V, it is shown that outage probability determines the asymptotic behavior of the performance limits. Finally, we conclude in Section VI.

II. SYSTEM MODEL

We assume a wireless communication system employing n transmit and m receive antennas. The channel is assumed to be memoryless, MIMO block fading. The channel gains remain constant throughout the duration of each fading block, but they change independently from one block to another. The fading is assumed to be Rayleigh, where the channel gains are independent and identically distributed (i.i.d.) complex normal with zero mean and unit variance.

It is assumed that the delay constraint is equal to the duration of K fading blocks (fading intervals), and the codewords (words) span K fading blocks. In each fading block, the channel is used L times, which is called the *block length* of the code. At each channel use, all antennas are used simultaneously, and n complex symbols are transmitted through n transmit antennas.

Let X_k be an $n \times L$ matrix denoting the nL symbols that are transmitted over the kth fading block. The channel output is described as

$$Y_k = H_k X_k + Z_k, \qquad k = 0, \dots, K-1 \tag{1}$$

where $\{H_k \sim \mathcal{CN}(0_{m\times n}, I_{mn})\}_{k=0}^{K-1}$ is a sequence of multivariate i.i.d. complex-normal $m \times n$ channel matrices over K fading blocks. The noise $\{Z_k \sim \mathcal{CN}(0_{m\times L}, \sigma^2 I_{mL})\}_{k=0}^{K-1}$ is a sequence of multivariate i.i.d. complex-normal $m \times L$ noise matrices.

To simplify the notation of the channel equation, we define

$$\boldsymbol{X} \triangleq \begin{bmatrix} X_0 \\ X_1 \\ \vdots \\ X_{K-1} \end{bmatrix}, \; \boldsymbol{Z} \triangleq \begin{bmatrix} Z_0 \\ Z_1 \\ \vdots \\ Z_{K-1} \end{bmatrix}, \; \boldsymbol{Y} \triangleq \begin{bmatrix} Y_0 \\ Y_1 \\ \vdots \\ Y_{K-1} \end{bmatrix}$$

$$\boldsymbol{H} \triangleq \begin{bmatrix} H_0 & 0 & \cdots & 0 \\ 0 & H_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & H_{K-1} \end{bmatrix}.$$

Thus, (1) can be rewritten as

$$\boldsymbol{Y} = \boldsymbol{H}\boldsymbol{X} + \boldsymbol{Z} \tag{2}$$

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where $\boldsymbol{X}$ is an $nK \times L$ matrix of symbols denoting the transmitted word. The dimension of the code is $2nLK$ and the data rate is R bits per dimension; thus, the code $\mathcal{X}_s$ consists of $M = \lceil 2^{2nLKR} \rceil$ equiprobable codewords $\boldsymbol{X}^0, \boldsymbol{X}^1, \ldots, \boldsymbol{X}^{M-1}$. The codewords are constrained to have a maximum energy of LKE_s, where E_s denotes the average transmit energy per channel use. In other words, the code is considered to be a set of M points contained in an (nLK)-hypersphere in $\mathbb{C}^{nLK}$ which is centered at the origin with a radius of $\sqrt{LKE_s}$, denoted by $B_{nLK}(0, \sqrt{LKE_s})$.

III. Maximum-Likelihood Decoder and the Bounding Region

Under the assumption that the channel realization matrix is known at the receiver and the code has M equiprobable codewords, the maximum-likelihood decoder (MLD) is described as follows. Suppose that codeword $\boldsymbol{X}^i \in \mathcal{X}_s$ is transmitted. The probability density function (pdf) of the received signal $\boldsymbol{Y}$, conditioned on $\boldsymbol{H}$ and $\boldsymbol{X}^i$, is

$$p(\boldsymbol{Y}|\boldsymbol{X}^i, \boldsymbol{H}) = \frac{1}{\pi^{mLK}\sigma^{2mLK}} e^{-\frac{\|\boldsymbol{Y}-\boldsymbol{H}\boldsymbol{X}^i\|_F^2}{\sigma^2}} \tag{3}$$

where $\|A\|_F^2 \triangleq \mathrm{tr}(A'A)$ (the Frobenius norm). The MLD decodes the received signal $\boldsymbol{Y}$ to $\boldsymbol{X}^i$ if

$$\forall j \in \{0, \ldots, M-1\} - \{i\}, \qquad \log \frac{p(\boldsymbol{Y}|\boldsymbol{X}^i, \boldsymbol{H})}{p(\boldsymbol{Y}|\boldsymbol{X}^j, \boldsymbol{H})} \geq 0. \tag{4}$$

The set of all received signals that satisfy (4) form the *Voronoi region* of codeword $\boldsymbol{X}^i$, which is denoted by $\Lambda_{i,\mathcal{X}_s}(\boldsymbol{H}) \in \mathbb{C}^{mLK}$. By computing the Voronoi regions for all of the codewords, the MLD partitions the received signal space $\mathbb{C}^{mLK}$ into M disjoint regions $\{\Lambda_{i,\mathcal{X}_s}(\boldsymbol{H})\}_{i=0}^{M-1}$. Because $\mathbb{C}^{mLK}$ is decomposed into a finite number of disjoint regions, the Voronoi regions of some of the codewords are unbounded. To obtain the SPB, the Voronoi regions have to be bounded. This requires the received signal space to be within a bounded region. In the following, it is shown that for any $0 < \epsilon < 1$ and any $\delta > 0$ there exists a bounding region $S_\delta(\boldsymbol{H})$ such that for sufficiently large dimension, the received signals are contained in this region with an arbitrarily high probability $(> 1-\epsilon)$. As a result, all Voronoi regions are considered to be bounded by their intersections with $\lim_{\delta \to 0} S_\delta(\boldsymbol{H})$. For the sake of completeness, two results on derivation of the bounding region are stated. These results are direct extensions of the results for single-input–single-output (SISO) additive white Gaussian noise (AWGN) channels, proved in [5].

Lemma 1: For any $\delta > 0$ and any $0 < \epsilon < 1$, there exists an N such that for all $mLK \geq N$

$$P\left(\left|\frac{\mathrm{tr}(\boldsymbol{Z}\boldsymbol{Z}')}{mLK} - \sigma^2\right| < \delta\right) > 1 - \epsilon.$$

Proof: The proof is virtually identical to the one given in [5, Ch. 5.5]. □

Lemma 1 states that if the dimension of $\mathcal{X}_s$ is large enough, then with an arbitrarily high probability $(>1-\epsilon)$, the normalized norm of the noise lies in a δ-distance from σ^2. Now, using Lemma 1, the following theorem is stated and proven.

Theorem 1: For any $0 < \epsilon < 1$ and any $\delta > 0$ there exists an N such that for all $mLK \geq N$, $P(\boldsymbol{Y} \in S_\delta(\boldsymbol{H})) > 1 - \epsilon$, where $S_\delta(\boldsymbol{H})$ is defined by

$$S_\delta(\boldsymbol{H}) \triangleq \left\{\boldsymbol{Y} \in \mathbb{C}^{mLK} \middle| \mathrm{tr}\left(\boldsymbol{Y}'\left(mLK(\sigma^2+\delta)I_{mK} + \frac{mLKE_s}{n}\boldsymbol{H}\boldsymbol{H}'\right)^{-1}\boldsymbol{Y}\right) \leq 1\right\}. \tag{5}$$

Proof: See Appendix, Subsection A. □

This theorem states that if the dimension of $\mathcal{X}_s$ is large enough, then with an arbitrarily high probability, the received signals lie inside an (mLK)-hyperellipsoid $S_\delta(\boldsymbol{H}) \in \mathbb{C}^{mLK}$. Similar to the discussions in [5, Ch. 5.5], we take $\delta \to 0$ and define $S(\boldsymbol{H}) \triangleq \lim_{\delta \to 0} S_\delta(\boldsymbol{H})$. The volume of $S(\boldsymbol{H})$ is [6]

$$\mathrm{Vol}(S(\boldsymbol{H})) = \frac{\pi^{mLK}}{(mLK)!}(mLK\sigma^2)^{mLK} \cdot \prod_{k=0}^{K-1} \det\left(I_m + \frac{E_s}{n\sigma^2}H_kH_k'\right)^L. \tag{6}$$

From now on, the received signal space is considered to be contained in $S(\boldsymbol{H})$, and all of the Voronoi regions are modified and bounded by their intersection with $S(\boldsymbol{H})$. Therefore, the summation of their volumes is equal to the volume of $S(\boldsymbol{H})$.

IV. Derivation of the SPB

For a given code $\mathcal{X}_s$, the average WEP is determined by the ensemble average of the conditional WEP given $\boldsymbol{H}$, i.e.,

$$P_{\mathcal{X}_s}(\varepsilon) = \mathrm{E}\left[P_{\mathcal{X}_s}(\varepsilon|\boldsymbol{H})\right] \tag{7}$$

where $P_{\mathcal{X}_s}(\varepsilon|\boldsymbol{H})$ denotes the conditional WEP for code $\mathcal{X}_s$ given $\boldsymbol{H}$. Taking the minimum of both sides of (7) over $\mathcal{X}_s$ we clearly obtain

$$\min_{\mathcal{X}_s} P_{\mathcal{X}_s}(\varepsilon) \geq \mathrm{E}\left[\min_{\mathcal{X}_s} P_{\mathcal{X}_s}(\varepsilon|\boldsymbol{H})\right]. \tag{8}$$

This means that the minimum of the average WEP is lower-bounded by the expectation of the minimum conditional WEP, where the minimization is applied over all possible codes satisfying the design constraints. Assuming all the codewords in $\mathcal{X}_s$ are equiprobable, $P_{\mathcal{X}_s}(\varepsilon|\boldsymbol{H})$ is described as

$$P_{\mathcal{X}_s}(\varepsilon|\boldsymbol{H}) = \frac{1}{M}\sum_{i=0}^{M-1} P_{\mathcal{X}_s}(\varepsilon|\boldsymbol{X}^i, \boldsymbol{H}) \tag{9}$$

where $P_{\mathcal{X}_s}(\varepsilon|\boldsymbol{X}^i, \boldsymbol{H})$ denotes the conditional WEP given $\boldsymbol{X}^i$ and $\boldsymbol{H}$. An error occurs when $\boldsymbol{X}^i$ is transmitted and the received signal $\boldsymbol{Y}$ lies outside the Voronoi region of $\boldsymbol{X}^i$, $\Lambda_{i,\mathcal{X}_s}(\boldsymbol{H})$. Hence, $P_{\mathcal{X}_s}(\varepsilon|\boldsymbol{X}^i, \boldsymbol{H})$ is written as

$$P_{\mathcal{X}_s}(\varepsilon|\boldsymbol{X}^i, \boldsymbol{H}) = \int_{\boldsymbol{Y} \notin \Lambda_{i,\mathcal{X}_s}(\boldsymbol{H})} p(\boldsymbol{Y}|\boldsymbol{X}^i, \boldsymbol{H})\, d\boldsymbol{Y}. \tag{10}$$

For most codes, $\mathcal{X}_s$, a closed-form expression for (10), is unattainable. However, since $p(\boldsymbol{Y}|\boldsymbol{X}^i, \boldsymbol{H})$ is monotonically decreasing with respect to $\|\boldsymbol{Y} - \boldsymbol{H}\boldsymbol{X}^i\|_F$, and it has spherical symmetry around $\boldsymbol{H}\boldsymbol{X}^i$, we find a simple lower bound on (10) as justified in [7].

Lemma 2: Let $B_{mLK}(\boldsymbol{H}\boldsymbol{X}^i, r_{i,\mathcal{X}_s}(\boldsymbol{H}))$ be an (mLK)-hypersphere centered at $\boldsymbol{H}\boldsymbol{X}^i$ with a radius of $r_{i,\mathcal{X}_s}(\boldsymbol{H})$, where $r_{i,\mathcal{X}_s}(\boldsymbol{H})$, the *equivalent radius* of $\Lambda_{i,\mathcal{X}_s}(\boldsymbol{H})$, is selected such that $\Lambda_{i,\mathcal{X}_s}(\boldsymbol{H})$ and $B_{mLK}(\boldsymbol{H}\boldsymbol{X}^i, r_{i,\mathcal{X}_s}(\boldsymbol{H}))$ have the same volume. By substituting $\Lambda_{i,\mathcal{X}_s}(H)$ with $B_{mLK}(\boldsymbol{H}\boldsymbol{X}^i, r_{i,\mathcal{X}_s}(\boldsymbol{H}))$, $P_{\mathcal{X}_s}(\varepsilon|\boldsymbol{X}^i, \boldsymbol{H})$ is lower-bounded by

$$P_{\mathcal{X}_s}(\varepsilon|\boldsymbol{X}^i, \boldsymbol{H}) \geq e^{-\frac{r_{i,\mathcal{X}_s}^2(\boldsymbol{H})}{\sigma^2}} \sum_{k=0}^{mLK-1} \frac{1}{k!}\left(\frac{r_{i,\mathcal{X}_s}^2(\boldsymbol{H})}{\sigma^2}\right)^k. \tag{11}$$

Proof: See Appendix, Subsection B. □

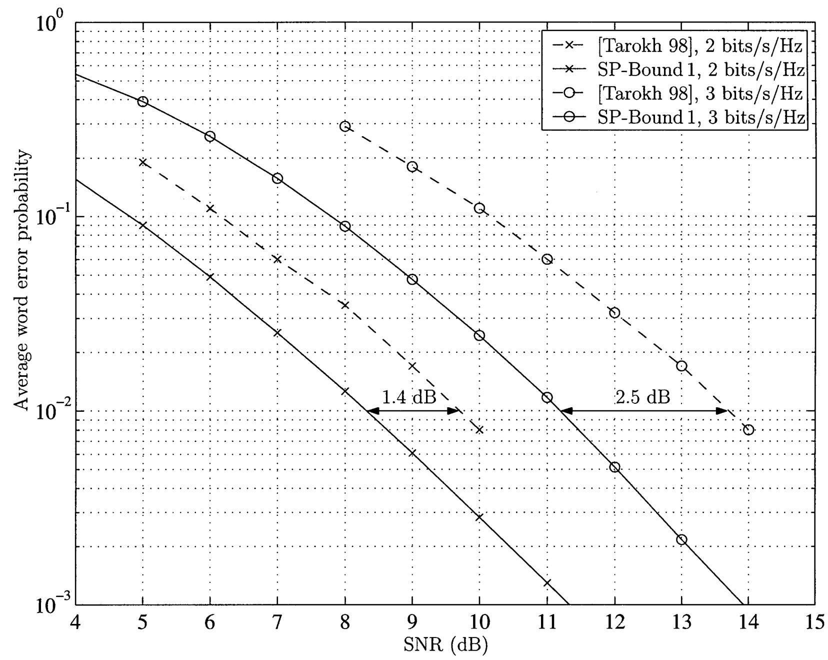

Fig. 1. The average WEP of a 64-state TCM developed in [4] compared with SP-Bound 1 (16). The results are shown for a system with $n = 2$ for two data rates of 2 and 3 bits/s/Hz.

Applying Lemma 2 to (9) and taking the minimum over all codes $\mathcal{X}_s \subset B_{nLK}(0, \sqrt{LKE_s})$, we obtain

$$\min_{\mathcal{X}_s} P_{\mathcal{X}_s}(\varepsilon|\boldsymbol{H}) \geq \min_{\mathcal{X}_s} \frac{1}{M} \sum_{i=0}^{M-1} e^{-\frac{r^2_{i,\mathcal{X}_s}(\boldsymbol{H})}{\sigma^2}} \cdot \sum_{k=0}^{mLK-1} \frac{1}{k!} \left(\frac{r^2_{i,\mathcal{X}_s}(\boldsymbol{H})}{\sigma^2}\right)^k. \quad (12)$$

Because the Voronoi regions are disjoint and bounded inside $S(\boldsymbol{H})$, the minimization of the RHS of (12) is subject to a volume constraint as follows.

Theorem 2: The minimum of

$$\frac{1}{M} \sum_{i=0}^{M-1} e^{-\frac{r^2_{i,\mathcal{X}_s}(\boldsymbol{H})}{\sigma^2}} \sum_{k=0}^{mLK-1} \frac{1}{k!} \left(\frac{r^2_{i,\mathcal{X}_s}(\boldsymbol{H})}{\sigma^2}\right)^k$$

with respect to the equivalent radiuses $\{r_{i,\mathcal{X}_s}(\boldsymbol{H})\}_{i=0}^{M-1}$, subject to the volume constraint

$$\sum_{i=0}^{M-1} \mathrm{Vol}(B_{mLK}(\boldsymbol{H}\boldsymbol{X}^i, r_{i,\mathcal{X}_s}(\boldsymbol{H}))) = \mathrm{Vol}(S(\boldsymbol{H})) \quad (13)$$

occurs when all $r_{i,\mathcal{X}_s}(\boldsymbol{H})$'s are equal; i.e., all of the Voronoi regions have the same volume, equal to $\frac{1}{M}$ of the volume of the bounding region $S(\boldsymbol{H})$. The optimum (minimizing) equivalent square radius is

$$r_o^2(\boldsymbol{H}) = mLK\sigma^2 2^{-\frac{2n}{m}R} \left[\prod_{k=0}^{K-1} \det\left(I_m + \frac{E_s}{n\sigma^2} H_k H_k'\right)\right]^{\frac{1}{mK}}. \quad (14)$$

Proof: See Appendix, Subsection C. □

Applying Theorem 2 to (12) with some straightforward manipulations results in

$$\min_{\mathcal{X}_s} P_{\mathcal{X}_s}(\varepsilon|\boldsymbol{H}) \geq e^{-\frac{r_o^2(\boldsymbol{H})}{\sigma^2}} \sum_{k=0}^{mLK-1} \frac{1}{k!} \left(\frac{r_o^2(\boldsymbol{H})}{\sigma^2}\right)^k \quad (15)$$

where $r_o^2(\boldsymbol{H})$ is given by (14). The right-hand side (RHS) of (15) expresses the conditional SPB. For future reference, we denote the conditional SPB by $P_{\mathrm{sp}}(mLK, \frac{r_o^2(\boldsymbol{H})}{\sigma^2})$.

SP-Bound 1: Applying (15) to (8) results in the derivation of the SPB on average WEP

$$\min_{\mathcal{X}_s} P_{\mathcal{X}_s}(\varepsilon) \geq \mathrm{E}\left[P_{\mathrm{sp}}\left(mLK, \frac{r_o^2(\boldsymbol{H})}{\sigma^2}\right)\right], \quad (16)$$

where the RHS of (16) is the SPB, called SP-Bound 1 and denoted by $\overline{P}_{\mathrm{sp}}^{(1)}(n, m, L, K, R)$. SP-Bound 1 defines a lower bound on the average WEP of all the codes satisfying the design constraints. The expectation in the RHS of (16) can be numerically computed and used to determine the performance limits of codes.

Example: Fig. 1 shows SP-Bound 1 in comparison with the average WEP of the 64-state trellis-coded modulation (TCM) that was developed in [4]. SP-Bound 1 is computed using a Monte Carlo simulation with 10^5 repetitions. The graphs are shown for a system with $n = 2$, $K = 1$, and $L = 30$ for data rates of 2 and 3 bits/s/Hz. It is observed that for an average WEP of 10^{-2} at a data rate of 2 bits/s/Hz, the mentioned code is about 1.4 dB away from SP-Bound 1, and at a data rate of 3 bits/s/Hz, it is 2.5 dB away from SP-Bound 1.

This example shows that SPB is useful in determining how far a code performs from the limits. However, since (16) does not have a closed-form expression, it is difficult to understand how the system parameters affect the performance limits. Therefore, it is of interest to obtain simplified closed-form expressions for (16).

If $n \geq m$ (respectively, $n < m$) the matrices $\{H_k H_k'\}_{k=0}^{K-1}$ (respectively, $\{H_k' H_k\}_{k=0}^{K-1}$) have Wishart distributions [8]. The moment-generating function of the Wishart distribution motivates the approximation of the determinant functions in (16) with trace functions to obtain closed-form expressions for (16). Therefore, two explicit bounds are found, namely SP-Bound 2 and SP-Bound 3.

SP-Bound 2: Because $P_{\mathrm{sp}}(mLK, \frac{r_o^2(\boldsymbol{H})}{\sigma^2})$ is monotonically decreasing with respect to $r_o^2(\boldsymbol{H})$, the following inequality is used to obtain an explicit lower bound for SP-Bound 1.

Proposition 1: Based on the inequality between the geometric average and the arithmetic average, the following inequality holds

$$\left[\prod_{k=0}^{K-1} \det\left(I_m + \frac{E_s}{n\sigma^2} H_k H_k'\right)\right]^{\frac{1}{mK}} \le 1 + \frac{1}{mK}\left[\sum_{k=0}^{K-1} \mathrm{tr}\left(\frac{E_s}{n\sigma^2} H_k H_k'\right)\right]. \quad (17)$$

Applying Proposition 1 to (16) results in derivation of a looser lower bound for average WEP, which is described by (see Appendix, Subsection D)

$$\min_{\mathcal{X}_s} P_{\mathcal{X}_s}(\varepsilon) \ge \overline{P}_{\mathrm{sp}}^{(2)}(n, m, L, K, R)$$

where

$$\begin{aligned}\overline{P}_{\mathrm{sp}}^{(2)}&(n, m, L, K, R)\\ &\triangleq \frac{e^{-mLK2^{-\frac{2n}{m}R}}}{\left(1+\frac{2^{-\frac{2n}{m}R}LE_s}{n\sigma^2}\right)^{nmK}} \sum_{k=0}^{mLK-1}\left(mLK2^{-\frac{2n}{m}R}\right)^k\\ &\cdot \sum_{i=0}^{k}\left(\frac{\frac{2^{-\frac{2n}{m}R}LE_s}{n\sigma^2}}{\left(1+\frac{2^{-\frac{2n}{m}R}LE_s}{n\sigma^2}\right)\left(mLK2^{-\frac{2n}{m}R}\right)}\right)^i\\ &\cdot \frac{(nmK+i-1)!}{i!(k-i)!(nmK-1)!} \end{aligned} \quad (18)$$

denotes SP-Bound 2. For the case that

$$R \gg \frac{m}{2n}\log_2(mLK) \quad \text{and} \quad \frac{2^{-\frac{2n}{m}R}LE_s}{n\sigma^2} \gg 1$$

(18) is simplified to

$$\overline{P}_{\mathrm{sp}}^{(2)}(n, m, L, K, R) \approx \frac{e^{-mLK2^{-\frac{2n}{m}R}}}{\left(1+\frac{2^{-\frac{2n}{m}R}LE_s}{n\sigma^2}\right)^{nmK}}\binom{nmK+mLK-1}{mLK-1}. \quad (19)$$

SP-Bound 2 is looser than SP-Bound 1 in all situations, but its advantage is its explicit expression, which is helpful to understand the effect of system parameters on performance limits.

SP-Bound 3: A tight approximate lower bound is obtained by using the following proposition.

Proposition 2: Let $r = \min(n, m)$. Then, one can apply the approximate inequality

$$\left[\prod_{k=0}^{K-1} \det\left(I_m + \frac{E_s}{n\sigma^2} H_k H_k'\right)\right]^{\frac{1}{mK}} \lessapprox \frac{1}{rK}\left[\sum_{k=0}^{K-1} \mathrm{tr}\left(\frac{E_s}{n\sigma^2} H_k H_k'\right)\right] \quad (20)$$

to (16) and obtain a tight approximate lower bound for SP-Bound 1 in high signal-to-noise ratio (SNR).

Proof: See Appendix, Subsection E. □

By applying Proposition 2 to (16), we obtain the following approximate lower bound for average WEP in high-SNR regimes (see Appendix, Subsection F):

$$\min_{\mathcal{X}_s} P_{\mathcal{X}_s}(\varepsilon) \gtrapprox \overline{P}_{\mathrm{sp}}^{(3)}(n, m, L, K, R)$$

where

$$\begin{aligned}\overline{P}_{\mathrm{sp}}^{(3)}(n, m, L, K, R) &\triangleq \frac{1}{\left(1+\frac{2^{-\frac{2n}{m}R}mLE_s}{rn\sigma^2}\right)^{nmK}}\\ &\cdot\sum_{k=0}^{mLK-1}\frac{(k+nmK-1)!}{k!(nmK-1)!}\left(\frac{\frac{2^{-\frac{2n}{m}R}mLE_s}{rn\sigma^2}}{1+\frac{2^{-\frac{2n}{m}R}mLE_s}{rn\sigma^2}}\right)^k \end{aligned} \quad (21)$$

denotes SP-Bound 3, which is tighter than SP-Bound 2 (18) in all situations. In the case of $\frac{2^{-\frac{2n}{m}R}mLE_s}{rn\sigma^2} \gg 1$, SP-Bound 3 is simplified to

$$\overline{P}_{\mathrm{sp}}^{(3)}(n, m, L, K, R) \approx \frac{1}{\left(1+\frac{2^{-\frac{2n}{m}R}mLE_s}{rn\sigma^2}\right)^{nmK}}\binom{nmK+mLK-1}{mLK-1}. \quad (22)$$

Comparing (19) and (22) shows that SP-Bound 2 and SP-Bound 3 have similar asymptotic behaviors.

V. Asymptotic Performance Limits of Space–Time Codes

It was shown in the preceding section that the minimum conditional WEP is lower-bounded by the conditional SPB, where the equivalent square radius of all the Voronoi regions are equal to $r_o^2(\boldsymbol{H})$ (Theorem 2). Lemma 1 states that for sufficiently large dimension, the additive noise lies around the surface of an (mLK)-hypersphere with radius $\sqrt{mLK\sigma^2}$ with an arbitrarily high probability. Using these two statements, similar to [5, Ch. 5.5], we state the following proposition.

Proposition 3: If $r_o^2(\boldsymbol{H}) < mLK\sigma^2$, the conditional SPB converges to one as the dimension of the code increases, and an error will occur with high probability. On the other hand, if $r_o^2(\boldsymbol{H}) \ge mLK\sigma^2$, the conditional SPB converges to zero uniformly and the probability of error is negligible.

To obtain more intuition on this issue, let us rewrite $r_o^2(\boldsymbol{H})$ using two new variables, namely,

$$C(H_k) \triangleq \frac{1}{2n}\log_2\det\left(I_n + \frac{E_s}{n\sigma^2} H_k' H_k\right) \quad \text{bits/dim} \quad (23)$$

which is the *instantaneous channel capacity* of kth fading block [1], and

$$C(\boldsymbol{H}) \triangleq \frac{1}{K}\sum_{k=0}^{K-1} C(H_k) \quad \text{bits/dim} \quad (24)$$

which denotes the average of the $C(H_k)$'s over K consecutive fading blocks. Noting the fact that

$$\det\left(I_m + \frac{E_s}{n\sigma^2} H_k H_k'\right) = \det\left(I_n + \frac{E_s}{n\sigma^2} H_k' H_k\right)$$

we rewrite (14) using (24). Hence,

$$r_o^2(\boldsymbol{H}) = mLK\sigma^2 2^{\frac{2n}{m}(C(\boldsymbol{H})-R)}. \quad (25)$$

Because $C(\boldsymbol{H})$ is a random variable, we partition the range of $C(\boldsymbol{H})$ into two parts using the test function $C(\boldsymbol{H}) \gtreqless R$. This imposes the following decomposition on SPB (16):

$$\begin{aligned}\overline{P}_{\mathrm{sp}}^{(1)}&(n, m, L, K, R)\\ &= P(C(\boldsymbol{H}) \ge R)\mathrm{E}\left[P_{\mathrm{sp}}\left(mLK, \frac{r_o^2(\boldsymbol{H})}{\sigma^2}\right)\middle| C(\boldsymbol{H}) \ge R\right]\\ &\quad + P(C(\boldsymbol{H}) < R)\mathrm{E}\left[P_{\mathrm{sp}}\left(mLK, \frac{r_o^2(\boldsymbol{H})}{\sigma^2}\right)\middle| C(\boldsymbol{H}) < R\right] \end{aligned} \quad (26)$$

where $P(C(\boldsymbol{H}) < R)$ is the *outage probability* [2], [3]. Suppose n, m, and K are fixed. From Proposition 3 and (25), it follows that as the block length L grows to infinity, the conditional SPB,

$P_{\rm sp}(mLK, \frac{r_o^2(\boldsymbol{H})}{\sigma^2})$, converges to one (respectively, to zero) if $C(\boldsymbol{H}) < R$ (respectively, $C(\boldsymbol{H}) \geq R$). Hence, the average of $P_{\rm sp}(mLK, \frac{r_o^2(\boldsymbol{H})}{\sigma^2})$ over all $\boldsymbol{H}$ with $C(\boldsymbol{H}) < R$ (respectively, $C(\boldsymbol{H}) \geq R$) goes to one (respectively, to zero). Therefore, as L grows to infinity, the first term of (26) converges to zero and the second term converges to the outage probability. Hence, we have

$$\lim_{L\to\infty} \overline{P}_{\rm sp}^{(1)}(n, m, L, K, R) = P(C(\boldsymbol{H}) < R). \tag{27}$$

In other words, as the block length increases, the SPB converges to the outage probability. Because L does not incorporate in the expression of the outage probability, it is not possible to obtain an arbitrarily small average WEP by increasing L. This is also observable from (21) which consists of two multiplicative terms, where the first term is of order $O(L^{-nmK})$, and the second term is of order $O(L^{nmK})$. Thus, (21) is of order $O(1)$ with respect to L.

To improve the performance, we have to change a parameter that reduces $P(C(\boldsymbol{H}) < R)$. Because the pdf of $C(\boldsymbol{H})$ is not explicitly known, the probability of outage does not have an explicit expression; although it can be computed numerically. However, we can obtain helpful intuition by studying the mean and variance of $C(\boldsymbol{H})$. Recalling that the channel realization matrices are i.i.d. in all fading blocks and all subchannels, we have

$$\mathrm{E}[C(\boldsymbol{H})] = \mathrm{E}\left[\frac{1}{2n}\log_2 \det\left(I_n + \frac{E_s}{n\sigma^2} H'H\right)\right] \tag{28}$$

which is called the *ergodic capacity* [1], [3]. It is clearly seen that the parameter K does not contribute in the expression of the mean value of the capacity. However, it plays an important role in the variance of the capacity as follows:

$$\mathrm{Var}[C(\boldsymbol{H})] = \frac{1}{K}\,\mathrm{Var}\left[\frac{1}{2n}\log_2 \det\left(I_n + \frac{E_s}{n\sigma^2} H'H\right)\right]. \tag{29}$$

It is seen that increasing K does not affect $\mathrm{E}[C(\boldsymbol{H})]$, but it decreases $\mathrm{Var}[C(\boldsymbol{H})]$ and concentrates $C(\boldsymbol{H})$ around its mean. Because the variance of $C(\boldsymbol{H})$ goes to zero with increasing K, if $R \leq \mathrm{E}[C(\boldsymbol{H})]$, then $\lim_{K\to\infty} P(C(\boldsymbol{H}) < R) = 0$. This indicates that if the data rate is smaller than the ergodic capacity, the outage probability goes to zero as K increases, which enhances the performance limits of space–time codes dramatically.

VI. Conclusion

The performance limits of space–time codes over Rayleigh MIMO channels were addressed using an SPB approach. Three sphere packing lower bounds were derived on average WEP of space–time codes. The results show that as codes span a larger number of fading blocks, the performance limits improve dramatically. Moreover, it was shown that the performance limits improve marginally as L grows. In fact, as L grows to infinity, the SPB converges to the outage probability, and the asymptotic behavior of the performance limits is determined by outage probability.

Appendix

A. Proof of Theorem 1

Under the linear transformation $\boldsymbol{H}$, an (nLK)-hypersphere

$$B_{nLK}(0, \sqrt{LKE_s}) \in \mathbb{C}^{nLK}$$

is transformed into an (mLK)-hyperellipsoid centered at origin and characterized by $\frac{mKLE_s}{n}\boldsymbol{H}\boldsymbol{H}'$ [6], which is described by the set

$$\left\{\boldsymbol{Y} \in \mathbb{C}^{mLK} \,\middle|\, \mathrm{tr}\left(\boldsymbol{Y}'\left(\frac{mLKE_s}{n}\boldsymbol{H}\boldsymbol{H}'\right)^{-1}\boldsymbol{Y}\right) \leq 1\right\}. \tag{30}$$

For the affine transform $\boldsymbol{Y} = \boldsymbol{H}\boldsymbol{X} + \boldsymbol{Z}$, the received signal space is the (mLK)-hyperellipsoid (30), where its center is translated by $\boldsymbol{Z}$. Therefore, to find the bounding region $S_\delta(\boldsymbol{H})$ we find the addition of two sets. One set is described by the (mLK)-hyperellipsoid (30) and the other one is

$$\left\{\boldsymbol{Z} \in \mathbb{C}^{mLK} \,\middle|\, \left|\frac{\mathrm{tr}(\boldsymbol{Z}\boldsymbol{Z}')}{mLK} - \sigma^2\right| \leq \delta\right\}.$$

As the result, we obtain the following (mLK)-hyperellipsoid:

$$\left\{\boldsymbol{Y} \in \mathbb{C}^{mLK} \,\middle|\, \mathrm{tr}\left(\boldsymbol{Y}'\left(mLK(\sigma^2+\delta)\boldsymbol{I}_{mK} + \frac{mLKE_s}{n}\boldsymbol{H}\boldsymbol{H}'\right)^{-1}\boldsymbol{Y}\right) \leq 1\right\},$$

as the bounding region $S_\delta(\boldsymbol{H})$, where, with an arbitrarily high probability ($>1-\epsilon$), the received signals lie in $S_\delta(\boldsymbol{H})$, if the dimension is sufficiently large. This concludes the proof of Theorem 1. □

B. Proof of Lemma 2

As justified in [7], if we substitute $\Lambda_{i,\mathcal{X}_s}(\boldsymbol{H})$ by $B_{mLK}(\boldsymbol{H}\boldsymbol{X}^i, r_{i,\mathcal{X}_s}(\boldsymbol{H}))$, we obtain

$$P_{\mathcal{X}_s}(\varepsilon|\boldsymbol{X}^i, \boldsymbol{H}) \geq \int_{\boldsymbol{Y}\notin B_{mLK}(\boldsymbol{H}\boldsymbol{X}^i,\, r_{i,\mathcal{X}_s}(\boldsymbol{H}))} p(\boldsymbol{Y}|\boldsymbol{X}^i, \boldsymbol{H})\,d\boldsymbol{Y}.$$

Now, using the expression for the surface area of an (mLK)-hyperspheres [6], we obtain

$$\begin{aligned}P_{\mathcal{X}_s}(\varepsilon|\boldsymbol{X}^i, \boldsymbol{H}) &\geq \int_{r_{i,\mathcal{X}_s}(\boldsymbol{H})}^{\infty} \frac{2\pi^{mLK} r^{2mLK-1}}{(mLK-1)!}\,\frac{e^{-\frac{r^2}{\sigma^2}}}{\pi^{mLK}\sigma^{2mLK}}\,dr\\ &= e^{-\frac{r_{i,\mathcal{X}_s}^2(\boldsymbol{H})}{\sigma^2}}\sum_{k=0}^{mLK-1}\frac{1}{k!}\left(\frac{r_{i,\mathcal{X}_s}^2(\boldsymbol{H})}{\sigma^2}\right)^k.\end{aligned}$$

This concludes the proof of Lemma 2. □

C. Proof of Theorem 2

Recalling that the volume of $S(\boldsymbol{H})$ is described by (6), we divide both sides of (13) by $\mathrm{Vol}(B_{mLK}(0, \sigma))$. Thus, we obtain

$$\sum_{i=0}^{M-1}\left(\frac{r_{i,\mathcal{X}_s}(\boldsymbol{H})}{\sigma}\right)^{2mLK} = \det\left(mLK\boldsymbol{I}_{mK} + \frac{mLKE_s}{n\sigma^2}\boldsymbol{H}\boldsymbol{H}'\right)^L.$$

For the sake of simplicity, let

$$x_i \triangleq \left(\frac{r_{i,\mathcal{X}_s}(\boldsymbol{H})}{\sigma}\right)^{2mLK}$$

and

$$g(x_i) = e^{-x_i^{\frac{1}{mLK}}}\sum_{k=0}^{mLK-1}\frac{1}{k!}x_i^{\frac{k}{mLK}}.$$

We reconfigure the optimization problem as follows:

$$\min_{x_0,\ldots,x_{M-1}} \frac{1}{M}\sum_{i=0}^{M-1} g(x_i) \quad \text{subject to} \quad \sum_{i=0}^{M-1} x_i = \det\left(mLK\boldsymbol{I}_{mK} + \frac{mLKE_s}{n\sigma^2}\boldsymbol{H}\boldsymbol{H}'\right)^L.$$

Taking the first two derivatives of $g(x_i)$ with respect to x_i, it can be inspected that $g_{nLK}(x_i)$ is a convex monotonically decreasing function for $x \geq 0$. Thus, applying Jensen's inequality, we obtain

$$\begin{aligned}\frac{1}{M}\sum_{i=0}^{M-1} g(x_i) &\geq g\left(\frac{1}{M}\sum_{i=0}^{M-1} x_i\right)\\ &= g\left(\frac{1}{M}\det\left(mLK\boldsymbol{I}_{mK} + \frac{mLKE_s}{n\sigma^2}\boldsymbol{H}\boldsymbol{H}'\right)^L\right).\end{aligned}$$

Therefore, to minimize the object function, all x_i's should be equal to

$$x_o \triangleq \frac{1}{M}\det\left(mLK\boldsymbol{I}_{mK} + \frac{mLKE_s}{n\sigma^2}\boldsymbol{H}\boldsymbol{H}'\right)^L.$$

Substituting 2^{2nLKR} for M (for simplicity we ignore the $\lceil\cdot\rceil$ function), the optimum (minimizing) equivalent square radius is

$$r_o^2(\boldsymbol{H}) = mLK\sigma^2 2^{-\frac{2n}{m}R}\left[\prod_{k=0}^{K-1}\det\left(I_m + \frac{E_s}{n\sigma^2}H_kH_k'\right)\right]^{\frac{1}{mK}}.$$

This concludes the proof of Theorem 2. □

D. Proof of (18) and (19)

We rewrite the conditional SPB (15) as

$$\begin{aligned}P_{\rm sp}\left(mLK, \frac{r_o^2(\boldsymbol{H})}{\sigma^2}\right) &= \sum_{k=0}^{mLK-1}\frac{1}{k!}\frac{d^k}{ds^k}\\ &\cdot e^{-(1-s)mLK2^{-\frac{2n}{m}R}\left[\prod_{i=0}^{K-1}\det\left(I_m+\frac{E_s}{n\sigma^2}H_iH_i'\right)\right]^{\frac{1}{mK}}}\Bigg|_{s=0}.\end{aligned}$$

Now, by applying Proposition 1 to this expression, we obtain

$$\begin{aligned}\min_{\mathcal{X}_s} P_{\mathcal{X}_s}(\varepsilon) \geq& \sum_{k=0}^{mLK-1}\frac{1}{k!}\frac{d^k}{ds^k}e^{-(1-s)mLK2^{-\frac{2n}{m}R}}\\ &\prod_{i=0}^{K-1}\mathrm{E}\left[e^{-(1-s)\frac{2^{-\frac{2n}{m}R}LE_s}{n\sigma^2}\mathrm{tr}(H_iH_i')}\right]\Bigg|_{s=0}.\end{aligned}$$

If $n \geq m$ (respectively, $n < m$), then H_iH_i''s (respectively, $H_i'H_i$'s) have Wishart distribution [8]. Without loss of generality, let $n \geq m$. Using the moment-generating function of Wishart distribution [8], we gain

$$\begin{aligned}\min_{\mathcal{X}_s} P_{\mathcal{X}_s}(\varepsilon) \geq& \sum_{k=0}^{mLK-1}\frac{1}{k!}\frac{d^k}{ds^k}\\ &\frac{e^{-(1-s)mLK2^{-\frac{2n}{m}R}}}{\det\left(I_m + (1-s)\frac{2^{-\frac{2n}{m}R}LE_s}{n\sigma^2}I_m\right)^{nK}}\Bigg|_{s=0}\end{aligned}$$

which is simplified to

$$\begin{aligned}\min_{\mathcal{X}_s} P_{\mathcal{X}_s}(\varepsilon) \geq& \frac{e^{-mLK2^{-\frac{2n}{m}R}}}{\left(1+\frac{2^{-\frac{2n}{m}R}LE_s}{n\sigma^2}\right)^{nmK}}\sum_{k=0}^{mLK-1}\left(mLK2^{-\frac{2n}{m}R}\right)^k\\ &\cdot\sum_{i=0}^{k}\left(\frac{\frac{2^{-\frac{2n}{m}R}LE_s}{n\sigma^2}}{\left(1+\frac{2^{-\frac{2n}{m}R}LE_s}{n\sigma^2}\right)\left(mLK2^{-\frac{2n}{m}R}\right)}\right)^i\\ &\cdot\frac{(nmK+i-1)!}{i!(k-i)!(nmK-1)!}.\end{aligned}$$

One can easily verify that the same result is obtained if $n < m$. Recalling that

$$\binom{n+i}{i} = \sum_{k=0}^{i}\binom{n-1+k}{k}$$

for the case that $R \gg \frac{m}{2n}\log_2(mLK)$ and $\frac{2^{-\frac{2n}{m}R}LE_s}{n\sigma^2} \gg 1$, we have

$$\min_{\mathcal{X}_s} P_{\mathcal{X}_s}(\varepsilon) \geq \frac{e^{-mLK2^{-\frac{2n}{m}R}}}{\left(1+\frac{2^{-\frac{2n}{m}R}LE_s}{n\sigma^2}\right)^{nmK}}\binom{nmK+mLK-1}{nLK-1}.$$

This concludes the proof of (18) and (19). □

E. Proof of Proposition 2

We prove the assertion for $K = 1$ and the proof of the general case is trivial by extension. Hence, we drop indexing H by k. Let $r = \min(m, n)$, and let $\mathcal{H} = \{H \in \mathbb{C}^{nm} | \mathrm{rank}(HH') = r\}$. We show that the approximate inequality is valid if $H \in \mathcal{H}$. Because the set $\mathcal{H}^c = \mathbb{C}^{nm} - \mathcal{H}$ is a closed subvariety in $\mathbb{C}^{nm}$ with zero measure [9], we conclude that the approximate inequality can be used to obtain a tight approximate lower bound for $\overline{P}_{\rm sp}^{(1)}(n, m, L, 1, R)$.

Let $H \in \mathcal{H}$. If $m \leq n$, then for high SNR ($\frac{E_s}{n\sigma^2} \gg 1$)

$$\begin{aligned}\det\left(I_m + \frac{E_s}{n\sigma^2}HH'\right)^{\frac{1}{m}} &\approx \det\left(\frac{E_s}{n\sigma^2}HH'\right)^{\frac{1}{m}}\\ &\leq \frac{1}{m}\mathrm{tr}\left(\frac{E_s}{n\sigma^2}HH'\right).\end{aligned}$$

On the other hand, if $n < m$, then for high SNR

$$\begin{aligned}\det\left(I_m + \frac{E_s}{n\sigma^2}HH'\right)^{\frac{1}{m}} &\approx \left(\det\left(\frac{E_s}{n\sigma^2}H'H\right)^{\frac{1}{n}}\right)^{\frac{n}{m}}\\ &\leq \det\left(\frac{E_s}{n\sigma^2}H'H\right)^{\frac{1}{n}} \leq \frac{1}{n}\mathrm{tr}\left(\frac{E_s}{n\sigma^2}H'H\right).\end{aligned}$$

In summary, if $\mathrm{rank}(HH') = r$, then for high SNR

$$\det\left(I_m + \frac{E_s}{n\sigma^2}HH'\right)^{\frac{1}{m}} \lessapprox \frac{1}{r}\mathrm{tr}\left(\frac{E_s}{n\sigma^2}HH'\right). \tag{31}$$

Now, we write $\overline{P}_{\rm sp}^{(1)}(n, m, L, 1, R)$ as

$$\begin{aligned}\overline{P}_{\rm sp}^{(1)}(n, m, L, 1, R) =& P(\mathcal{H})\mathrm{E}\left[P_{\rm sp}\left(mL, \frac{r_o^2(H)}{\sigma^2}\right)\Bigg|\mathcal{H}\right]\\ &+ P(\mathcal{H}^c)\mathrm{E}\left[P_{\rm sp}\left(mL, \frac{r_o^2(H)}{\sigma^2}\right)\Bigg|\mathcal{H}^c\right].\end{aligned}$$

Because $\mathcal{H}^c$ has zero measure [9], $P(\mathcal{H}^c) = 0$ and $P(\mathcal{H}) = 1$. Hence,

$$\overline{P}_{\rm sp}^{(1)}(n, m, L, 1, R) = \mathrm{E}\left[P_{\rm sp}\left(mL, \frac{r_o^2(H)}{\sigma^2}\right)\Bigg|\mathcal{H}\right]. \tag{32}$$

Thus, applying (31) to (32), we obtain a tight approximate lower bound for $\overline{P}_{\rm sp}^{(1)}(n, m, L, 1, R)$ in high SNR. This concludes the proof of Proposition 2. □

F. Proof of (21)

We follow a process similar to that of Subsection D of this appendix. Here, we use Proposition 2 to obtain

$$\begin{aligned}\min_{\mathcal{X}_s} P_{\mathcal{X}_s}(\varepsilon) \gtrapprox& \sum_{k=0}^{mLK-1}\frac{1}{k!}\frac{d^k}{ds^k}\\ &\prod_{i=0}^{K-1}\mathrm{E}\left[e^{-(1-s)\frac{2^{-\frac{2n}{m}R}mLE_s}{rn\sigma^2}\mathrm{tr}(H_iH_i')}\right]\Bigg|_{s=0}.\end{aligned}$$

Without loss of generality, let assume $n \geq m$ which means that $H_i H_i'$'s have Wishart distribution [8]. Using the moment-generating function of Wishart distribution, we obtain

$$\min_{\mathcal{X}_s} P_{\mathcal{X}_s}(\varepsilon)$$

$$\gtrapprox \sum_{k=0}^{mLK-1} \frac{1}{k!} \frac{d^k}{ds^k} \left. \frac{1}{\det\left(I_m + (1-s)\frac{2^{-\frac{2n}{m}R}mLE_s}{rn\sigma^2} I_m\right)^{nK}} \right|_{s=0}$$

$$= \frac{1}{\left(1 + \frac{2^{-\frac{2n}{m}R}mLE_s}{rn\sigma^2}\right)^{nmK}} \cdot \sum_{k=0}^{mLK-1} \frac{(k+nmK-1)!}{k!(nmK-1)!}$$

$$\cdot \left(\frac{\frac{2^{-\frac{2n}{m}R}mLE_s}{rn\sigma^2}}{1 + \frac{2^{-\frac{2n}{m}R}mLE_s}{rn\sigma^2}} \right)^k.$$

Note that the same result is obtained if $n < m$. This concludes the proof of (21). □

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REFERENCES

[1] E. Teletar, "Capacity of multi-antenna Gaussian channels," Lucent Technology, Bell Labs., Tech. Rep., June 1995.
[2] E. Biglieri, J. Proakis, and S. Shamai (Shitz), "Fading channels: Information-theoretic and communications aspects," *IEEE Trans. Inform. Theory*, vol. 44, pp. 2619–2692, Oct. 1998.
[3] E. Biglieri, G. Caire, and G. Taricco, "Limiting performance of block-fading channels with multiple antennas," *IEEE Trans. Inform. Theory*, vol. 47, pp. 1273–1289, May 2001.
[4] V. Tarokh, N. Seshadri, and A. R. Calderbank, "Space-time codes for high data rate wireless communication: Performance criterion and code construction," *IEEE Trans. Inform. Theory*, vol. 44, pp. 744–765, Mar. 1998.
[5] J. M. Wozencraft and I. M. Jacobs, *Principle of Communication Engineering*. New York: Wiley, 1965.
[6] A. M. Mathai, "An introduction to geometrical probability," in *Statistical Distributions and Models with Applications*. New York: Gordon and Breach, 1999.
[7] G. M. Poscetti, "An upper bound for probability of error related to a given decision region in n-dimensional signal set," *IEEE Trans. Inform. Theory*, vol. IT-17, pp. 203–206, Mar. 1971.
[8] A. Edelman, "Eigenvalues and condition numbers of random matrices," Ph.D. dissertation, MIT, Cambridge, MA, May 1989.
[9] J. Harris, *Algebraic Geometry: A First Course*. New York: Springer-Verlag, 1992, vol. 133–GTM.

Receive Antenna Selection for MIMO Flat-Fading Channels: Theory and Algorithms

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***Abstract*—This correspondence discusses the problem of the receive antenna subset selection in multiple-element antenna (MEA) transmission systems. The antennas are selected so as to maximize the channel capacity. A set of near-optimal selection algorithms is presented. The first algorithm in particular allows statistical analysis of selection gains. We present tight analytic lower bounds on the outage capacity achievable through antenna selection. Extensive simulations validating analysis and illustrating performance of the selection algorithms are also presented.**

***Index Terms*—Antenna selection, diversity, multiple-input multiple-output (MIMO), spatial multiplexing.**

I. INTRODUCTION

Multiple-antenna technology significantly improves wireless link performance. The extra degrees of freedom afforded by the multiple antennas may be used either for increasing reliability through space–time diversity techniques [1]–[3] or for increased data rate through spatial multiplexing techniques [4]–[7]. However, a major limiting factor in the deployment of multiple-input multiple-outpu (MIMO) systems is the cost of multiple analog chains (amplifiers, analog-to-digital converters, etc.). Antenna subset selection, where transmission and/or reception is performed through a selection of the total available antennas is a powerful solution that reduces the need for multiple analog chains yet retains many diversity benefits. The core idea of antenna selection is to use a limited number of analog chains that is adaptively switched to a subset of available antennas. An appropriate subset of antennas can be identified, e.g., within the training phase, by subsequently probing all receive antennas with the available set of receive chains. This general approach provides certain diversity benefits at a low additional cost that is mainly determined by low-cost radio-frequency (RF) switches rather than by expensive analog chains.

Early work on antenna selection focused on selection in multiple-input single output/single-input multiple-output (MISO/SIMO) systems. This included the hybrid selection/maximal ratio combining approach in [8]. Recently, there has been increasing interest [9]–[15] in applying antenna subset selection techniques to MIMO links. In [9], the authors present a criterion for selecting antenna subsets that maximize the channel capacity. As shown in [10], antenna selection techniques applied to low-rank channels can increase capacity. A fast selection algorithm based on "water-pouring" type ideas is presented in [11]. In [13], Heath *et al.*, discuss antenna subset selection for spatial multiplexing systems with practical receivers. Antenna selection algorithms/analysis for space–time codes based on exact and statistical channel knowledge may be found in [15]. Recently, Molisch *et al.* presented an algorithm that maximizes an upper bound on the

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