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Efficient Design and Decoding of Polar Codes

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Short summary: Polar codes are shown to be instances of both **generalized concatenated codes** and **multilevel codes**. It is shown that the performance of a polar code can be improved by representing it as a multilevel code and applying the multistage decoding algorithm with maximum likelihood decoding of outer codes. Additional performance improvement is obtained by replacing polar outer codes with other ones with better error correction performance. In some cases this also results in complexity reduction. It is shown that Gaussian approximation for density evolution enables one to accurately predict the performance of polar codes and concatenated codes based on them.

I. Introduction

The practical performance of polar codes under the successive cancellation (SC) decoding reported up to now turns out to be worse than that of LDPC and Turbo codes.

This paper demonstrates

1) Polar codes can be efficiently constructed using Gaussian approximation for density evolution.

2) It is shown that polar codes can be treated in the framework of multilevel coding. This enables one to improve the performance of polar codes by considering them as multilevel or, equivalently, generalized concatenated (GCC) ones, and using block-wise near-maximum-likelihood decoding of outer codes. In some cases this results also in reduced decoding complexity.

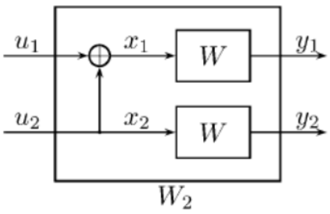
3) A simple algorithm for construction of GCC with inner polar codes.

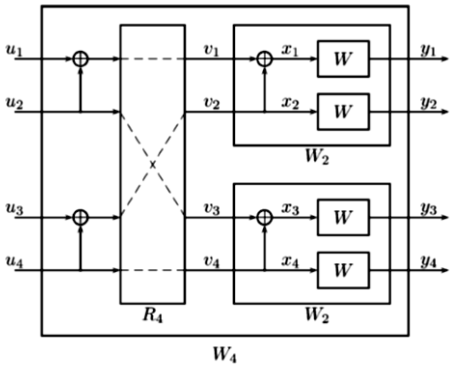
II. Background

A. Polar codes

Consider a binary input output symmetric memoryless channel with output probability density function $W(y|x)$, $y \in Y$, $x \in \mathbb{F}_2$. It can be transformed into a vector channel given by $W_n(y_1^n | u_1^n) = W^n(y_1^n | u_1^n G_n)$, where $W^n(y_1^n | x_1^n) = \prod_{i=1}^{n} W(y_i | x_i)$, $G_n = B_s F^{\otimes s}$, $n = 2^s$, $F = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $\otimes s$ denotes s-times Kronecker product of a matrix with itself, and B_s is a $2^s \times 2^s$ bit reversal permutation matrix.

For example)





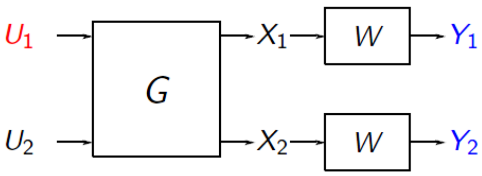
$W_2(y_1^2 | u_1^2) = W^2(y_1^2 | u_1^2 G_2)$ $\quad\quad$ $W_4(y_1^4 | u_1^4) = W_2(y_1^2 | u_1 \oplus u_2, u_3 \oplus u_4)\, W_2(y_3^4 | u_2, u_4)$

The vector channel can be further decomposed into equivalent subchannels

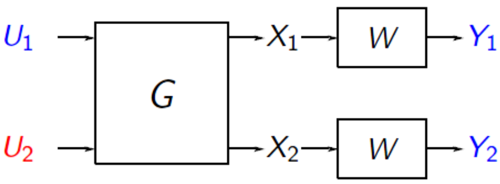
$$W_N^{(i)}(y_1^N, u_1^{i-1} | u_i) = \sum_{u_{i+1}^N} \frac{1}{2^{N-1}} W_N(y_1^N | u_1^N).$$

For example)

According to this, we can write $(W, W) \mapsto (W_2^{(1)}, W_2^{(2)})$ for any given B-DMC W.



$$W_2^{(1)}(y_1^2 | u_1) \triangleq \sum_{u_2} \frac{1}{2} W_2(y_1^2 | u_1^2)$$
$$= \sum_{u_2} \frac{1}{2} W(y_1 | u_1 \oplus u_2) W(y_2 | u_2)$$



$$W_2^{(2)}(y_1^2, u_1 | u_2) \triangleq \frac{1}{2} W_2(y_1^2 | u_1^2)$$
$$= \frac{1}{2} W(y_1 | u_1 \oplus u_2) W(y_2 | u_2)$$

Given y_1^n and estimates $\hat{u}_1^{i-1}$ of u_1^{i-1}, the SC decoding algorithm attempts to estimate u_i. This can be implemented by computing the following log-likelihood ratios $L_n^{(i)}\left(y_1^n, \hat{u}_1^{i-1}\right) = \log \frac{W_n^{(i)}\left(y_1^n, \hat{u}_1^{i-1} \mid u_i = 0\right)}{W_n^{(i)}\left(y_1^n, \hat{u}_1^{i-1} \mid u_i = 1\right)}$:

$$L_n^{(2i-1)}\left(y_1^n, \hat{u}_1^{2i-2}\right) = 2\tanh^{-1}\left(\tanh\left(L_{n/2}^{(i)}\left(y_1^{n/2}, \hat{u}_{1,e}^{2i-2} \oplus \hat{u}_{1,o}^{2i-2}\right)/2\right)\tanh\left(L_{n/2}^{(i)}\left(y_{n/2+1}^n, \hat{u}_{1,e}^{2i-2}\right)/2\right)\right), \quad (1)$$

$$L_n^{(2i)}\left(y_1^n, \hat{u}_1^{2i-1}\right) = L_{n/2}^{(i)}\left(y_{n/2+1}^n, \hat{u}_{1,e}^{2i-2}\right) + (-1)^{\hat{u}_{2i-1}} L_{n/2}^{(i)}\left(y_1^{n/2}, \hat{u}_{1,e}^{2i-2} \oplus \hat{u}_{1,o}^{2i-2}\right) \quad (2)$$

where $\hat{u}_{1,e}^{i}, \hat{u}_{1,o}^{i}$ are subvectors of $\hat{u}_1^{i}$ with even and odd indices, respectively, and $L_1^{(i)}(y_i) = \log \frac{W(y_i \mid 0)}{W(y_i \mid 1)}$.

III. Design of Polar Codes Based on Gaussian Approximation

The main drawback of the polar code construction method based on density evolution is its high computational complexity. The most practically important case corresponds to the AWGN channel. In this scenario, $L_1^{(i)}(y_i) \sim N\left(\frac{2}{\sigma^2}, \frac{4}{\sigma^4}\right)$, provided that the all-zero codeword is transmitted.

The value given by (1)-(2) can be considered as Gaussian random variables with $\boldsymbol{D}\left[L_n^{(i)}\right] = 2\boldsymbol{E}\left[L_n^{(i)}\right]$, where $\boldsymbol{E}$ and $\boldsymbol{D}$ are the mean and variance, respectively. This enable one to compute only the expected value of $L_n^{(i)}$, drastically reducing thus the complexity. In the case of polar codes this approach reduces to

$$\boldsymbol{E}\left[L_n^{(2i-1)}\right] = \phi^{-1}\left(1 - \left(1 - \phi\left(\boldsymbol{E}\left[L_{n/2}^{(i)}\right]\right)\right)^2\right), \quad (3)$$

$$\boldsymbol{E}\left[L_n^{(2i)}\right] = 2\boldsymbol{E}\left[L_{n/2}^{(i)}\right] \quad (4)$$

where

$$\phi(x)=\begin{cases}1-\frac{1}{\sqrt{4\pi x}}\int_{-\infty}^{\infty}\tanh\frac{u}{2}e^{-\frac{(u-x)^2}{4x}}dx, & x>0\\ 1 & , \quad x=0.\end{cases}$$

The error probability for each subchannel is given by

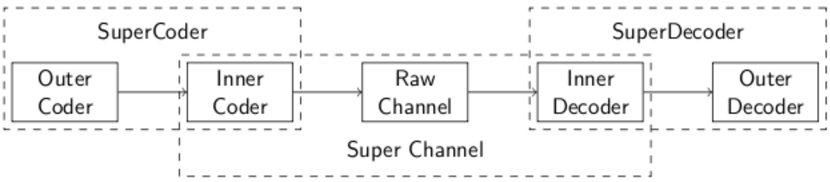
$$\pi_i\approx Q\left(\sqrt{\boldsymbol{E}\left[L_n^{(i)}\right]/2}\right),\ 1\le i\le n.$$

IV. Decomposition of Polar Codes

The overall performance of a polar code is dominated by the performance of the worst subchannel. The proposed approach avoids this problem by performing joint decoding over a number of subchannels.

A. Generalized concatenated polar codes

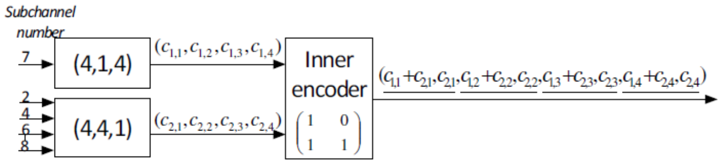
The recursive structure of polar codes enables one to consider them as GCC. Namely, the generator matrix of a polar code can be represented as $G=AF^{\otimes s}=A\left(F^{\otimes(s-l)}\otimes F^{\otimes l}\right)$, where $F=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$ and A is a full-rank matrix with at most one non-zero element in each column.



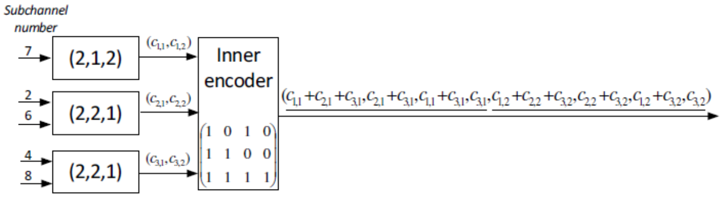
Inner code encoding: Inner codes $\mathbb{C}_i$ of length $n=2^l$ is generated by rows $i,\ldots,2^l$ of matrix $B_lF^{\otimes l}$.

Outer code encoding: The generator matrix of the $\left(1+R(i,l)\right)$-th outer code C_i is obtained by taking rows $1+R(j,s-l)$ of $F^{\otimes(s-l)}$, such that row $1+R\left(i2^{s-l}+j,s\right)$ of $F^{\otimes s}$ is included into the generator matrix of the original polar code, where $0\le i<2^l$, $0\le j<2^{s-l}$, and

$$R\left(\sum_{j=0}^{m-1}2^j i_j,m\right)=\sum_{j=0}^{m-1}2^j i_{m-1-j},\quad i_j\in\{0,1\}.$$



(a) $l = 1$

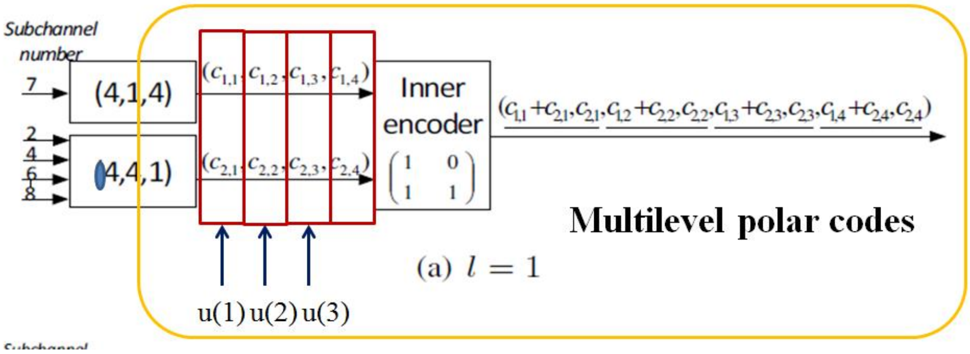


(b) $l = 2$

Fig. 3. Representation of $(8, 5, 2)$ polar code as GCC.

B. Multilevel polar codes

In the context of polar codes, signal constellation A is given by 2^n binary n-vectors $a(u)$, which can be obtained as $a(u)=uB_lF^{\otimes l}$, $u\in GF(2)^n$, where $n=2^l$. This constellation is recursively partitioned into subsets $A(u_1^i)$ by fixing the values of $u_1,\dots,u_i$. The elements of u are obtained as codeword symbols of outer codes C_i of length $N=2^{s-l}$. That is, one can construct N vectors $u^{(j)}=(c_{1,j},\dots,c_{n,j})$, $1\le j\le N$, where $(c_{i,1},\dots,c_{i,N})\in C_i$, $1\le i\le n$ and obtain a multilevel codeword $\left(u^{(1)}B_lF^{\otimes l},\dots,u^{(N)}B_lF^{\otimes l}\right)$.



The multilevel polar codes can be decoded by multistage decoding algorithm.

V. Concatenated Codes Based on Polar Codes

The performance of a polar code under the multistage decoding with block-wise maximum-likelihood decoding of outer codes can be improved by changing the set of frozen bits. Furthermore, if the algorithm used to perform block-wise decoding of outer codes does not take into account their structure, one can use any linear block code with suitable parameters, not necessary polar, as C_i. This enables one to employ outer codes with better error correction performance.

A. Capacity rule

The rate R_i of C_i should be chosen equal to the capacity C_i of the *i*-th subchannel of the multilevel code, which is induced by matrix $B_l F^{\otimes l}$. According to [10], one obtains

$$C_i = I\left(y_1^n; u_i \mid u_1^{i-1}\right) = E_{u_1^{i-1}}\left[C\left(A\left(u_1^{i-1}\right)\right)\right] - E_{u_1^i}\left[C\left(A\left(u_1^i\right)\right)\right]$$

where

$$C(B) = \int_{R^n} \sum_{a \in B} \frac{W^n\left(y_1^n \mid a\right)}{|B|} \log_2\left(\frac{|B| W^n\left(y_1^n \mid a\right)}{\sum_{b \in B} W^n\left(y_1^n \mid b\right)}\right) dy_1^n$$

is the capacity when using the subset B of $\mathbb{F}_2^n$ for transmission over the vector channel $W^n\left(y_1^n \mid x_1^n\right)$. In the case of binary input memoryless output symmetric channels, one can drop the expectation operator to obtain $C_i = C\left(A^{(i-1)}\right) - C\left(A^{(i)}\right)$, where $A^{(i)} = A\underbrace{(0,\ldots,0)}_{i\text{ times}}$.

It can be seen that the latter set is a linear block code C_i generated by $l-i$ last rows of $B_l F^{\otimes l}$. The expression can be further simplified to

$$C\left(A^{(i)}\right) = \int_{R^N} \prod_{j=1}^{N} W\left(y_j \mid 0\right) \log_2\left(\frac{|C_i| \prod_{j=1}^{N} W\left(y_j \mid 0\right)}{\sum_{b \in B} \prod_{j=1}^{N} W\left(y_j \mid b_j\right)}\right) dy_1^N .$$

Hence, the capacity of the *i*-th subchannel of the multilevel polar code can be computed as

$$C_i = \int_{R^N} \prod_{j=1}^{N} W\left(y_j \mid 0\right) \log_2\left(\frac{2 \sum_{b \in C_{i+1}} \prod_{j=1}^{N} W\left(y_j \mid b_j\right)}{\sum_{b \in C_i} \prod_{j=1}^{N} W\left(y_j \mid b_j\right)}\right) dy_1^N . \qquad (5)$$

Obviously, employing this rule results in a capacity achieving concatenated code, provided that the outer codes can achieve the capacity too. However, evaluating (5) seems to be a difficult task.

B. Equal error probability rule

The probability of incorrect decoding of a binary linear block code C can be obtained as

$$p_e \le \sum_{j=d}^{N} A_j Q\left(\sqrt{\frac{\boldsymbol{E}[L_i]}{2} j}\right)$$

where A_i are weight spectrum coefficients of code C, and d is its minimum distance. Since it is in general difficult to obtain code weight spectrum, and union bound is known to be not tight in the low-SNR region, one can use simulations to obtain a performance curve for the case of AWGN channel and some fixed (probably, non-ML) decoding algorithm, and use least squares fitting to find suitable α and δ, so that the decoding error probability is given by

$$p_e(m) \approx \alpha Q\left(\sqrt{\frac{\boldsymbol{E}[L_i]}{2} \delta}\right).$$

Assume now that the outer codes C_i are selected from some family of error-correcting codes (not necessary polar) of length N. Let K_t, D_t and $P_t(m)$ be the dimension, minimum distance and decoding error probability function for the t-th code, respectively, where m is the expected value of LLR.

Figure 4 presents a simple algorithm for construction of a generalized concatenated (multilevel) code of rate R according to the equal error probability rule. The algorithm employs the bisection method to approximately solve the equation $\sum_{i=1}^{2^l} K(i,P) = RN2^l$, where $K(i,P)$ is the maximum dimension of a code capable of achieving error probability P at the i-th subchannel. The parameter ε is a sufficiently small constant, which affects the precision of the obtained estimate for P. The code is optimized for the case of AWGN channel with noise variance σ^2. The algorithm returns the dimensions of optimal codes for each level, as well as an estimate for the decoding error probability for each code.

$\textsc{CodeOptimization}(\sigma, R, N, l)$

$\mathbf{E}[L_1^{(1)}] \leftarrow 2/\sigma^2$

Compute $m_i = \mathbf{E}[L_{2^l}^{(i)}], 1 \le i \le 2^l$ via (5)–(6)

$P' \leftarrow 1; P'' \leftarrow 0$

while $P' - P'' > \epsilon P'$

do $\tilde{P} \leftarrow (P' + P'')/2$

  $t_i \leftarrow \arg\max_{t: P_t(m_i) \le \tilde{P}} K_t, 1 \le i \le 2^l$

  $K \leftarrow \sum_{i=1}^{2^l} K_{t_i}$

  if $K < RN2^l$

    then $P'' \leftarrow \tilde{P}$

    else $P' = \tilde{P}$

return $(K_{t_1}, \ldots, K_{t_{2^l}}), \tilde{P}$

The SC/multistage decoder produces an error if decoding of any of the component codes is incorrect. Therefore, the overall error probability of the GCC can be computed as

$$\begin{aligned} P &= 1 - P\{C_1, \ldots, C_n\} \\ &= 1 - P\{C_1\} P\{C_2 \mid C_1\} \cdots P\{C_n \mid C_1, \ldots, C_{n-1}\} \\ &\approx 1 - \prod_{i=1}^{n} \left(1 - P_{t_i}(m_i)\right) \approx 1 - (1-P)^n \end{aligned}$$

where C_i denotes the event of correct decoding of the outer code at the i-th level, P is the quantity computed by the above algorithm, and t_i is the index of the code selected for the i-th subchannel. This expression enables semi-analytic prediction of the performance of the concatenated code, based on the available performance results for component outer codes.

C. Decoding complexity

One can use any suitable algorithm to implement soft-decision decoding of outer codes in the GCC obtained either by decomposing a polar code, or constructed explicitly using the algorithm in Figure 4. Box-and-match algorithm is one of the most efficient methods to perform near maximum likelihood decoding of short linear block codes [20]. Its worstcase complexity for the case of (N, K) code with order t reprocessing is given by $O((N-K)K^t) = O(N^{t+1})$, although in practice it turns out to be much more efficient. Decoding of a concatenated code of length $v = Nn$ involves decoding of N inner codes using the SC algorithm, and decoding of n outer codes. Therefore the overall complexity is given by $O(N^{t+1} n C_b + Nn \log n C_s)$, where C_b and C_s are some factors which reflect the cost of elementary operations performed by these algorithms.

VI. Numerical Results

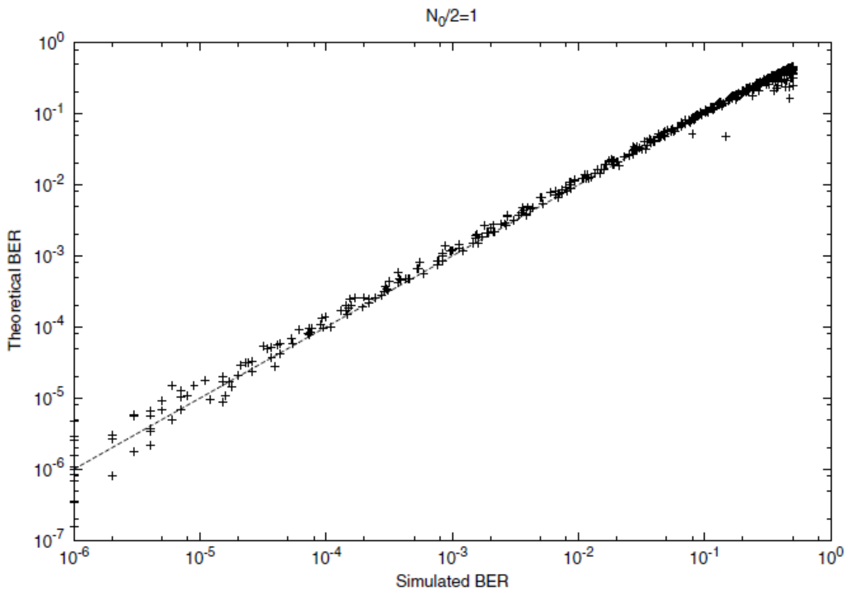


Fig. 5. Accuracy of Gaussian approximation.

Figure 5 presents simulation results illustrating the accuracy of bit error rate analysis based on the Gaussian approximation.

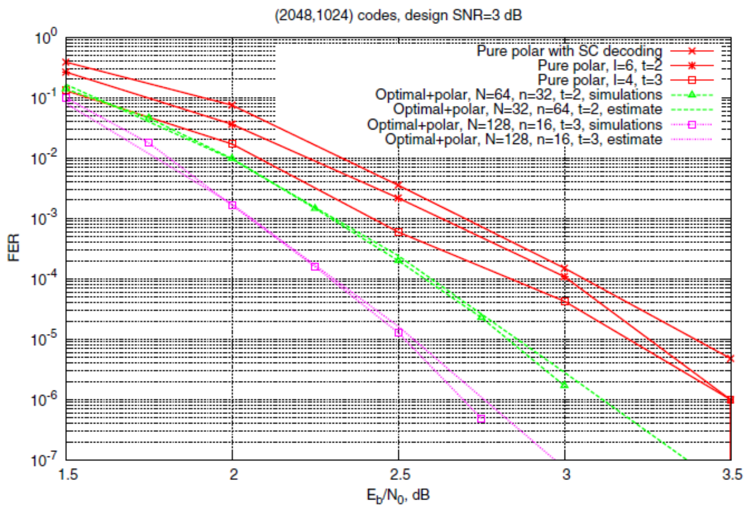


Fig. 6. Performance of polar and concatenated codes.

Figure 6 presents the performance of polar codes of length 2048 designed using the Gaussian approximation method for the case of AWGN channel with Eb/N0 = 3 dB. For multistage decoding, degree *l* decomposition of the original polar code was performed, and box-and-match algorithm with order *t* reprocessing was used for decoding of outer polar codes.

It can be seen that block-wise decoding of outer codes provides up to 0.25 dB performance gain compared to SC decoding. Higher values of N do not provide any noticeable performance improvement. The figure presents also the performance of GCC based on inner polar codes and outer optimal linear block codes with multistage decoding. It can be seen that increasing the length of outer codes provides additional 0.5 dB performance gain. This is due to much higher minimum distance of optimal codes compared to polar codes of the same length, obtained by decomposing the polar code of length Nn.