Multiuser Detection of Sparsely Spread CDMA

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I. Introduction

This paper has discussed about design and analysis of multiuser detection (MUD) using sparsely spread CDMA systems. The objective of the MUD problem is how to detect multiple user signals simultaneously at the low computational cost. The main obstacle is multiple-access interference (MAI). These multiple user signals are interference for each user detection one another. The MAI problem arise in most CDMA systems, and optimal detection in such systems requires exponentially growing computation as the number of user increases. This paper investigates a suboptimal MUD detection using sparse CDMA systems. The key idea of the proposed system is to encode the transmitted waveforms using sparsely spread CDMA codes and detect the signal using a linear-complexity belief propagation (BP) algorithm. We summarize the contributions of this work is following:

1) Description of the sparse CDMA system
2) Ensemble of the sparsely spread CDMA codes
3) Design of the BP algorithm for the MUD problem
4) Asymptotic analysis of performance of the BP algorithm based MUD detection

In this report, we aim to sketch the key point of each contribution of this paper.

II. Description of the sparse CDMA system

We consider a fully-synchronous CDMA system which is able to simultaneously transmit K user signals. As shown in Fig.1, symbols X_k from the k-th user is multiplied by the spreading code $\{S_{lk}\}_{l=1}^{L}$ having code length L, being transmitted to the receiver with gain $\frac{A_k}{\sqrt{\Lambda_k}}$ for the transmit power regulation. Then, the receive observes the L channel outputs per a symbol transmission from K users, given by

$$Y_l = \sum_{k=1}^{K} S_{lk} \frac{A_k}{\sqrt{\Lambda_k}} X_k + N_l \quad \text{for } l = 1 \text{ to } L, \tag{1}$$

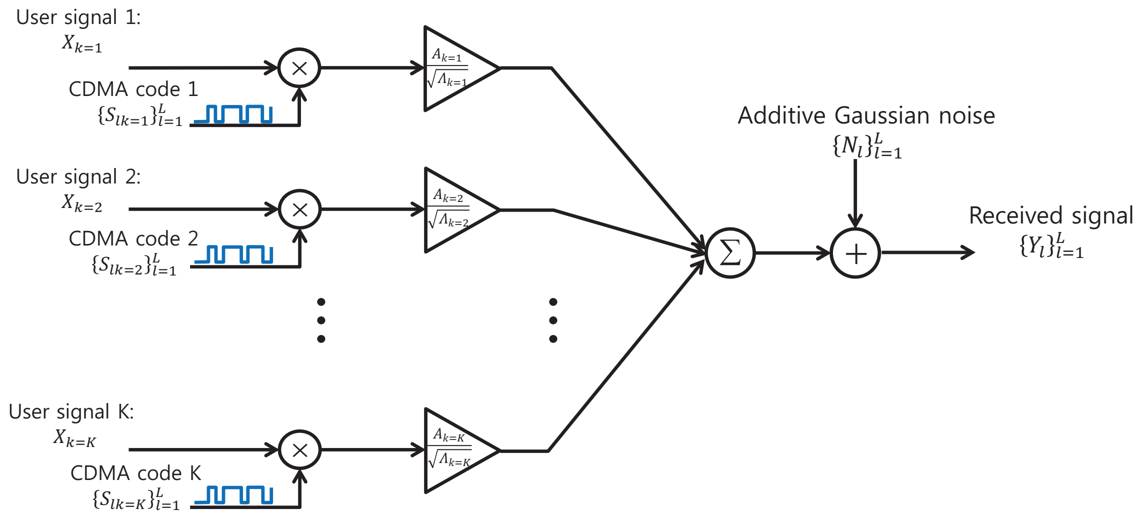

Fig. 1. System model

where we consider additive noise following the zero-mean and unit variance Gaussian distribution, i.e., $N_l \sim \mathcal{N}(0,1)$. In vector form, the expression in (1) can be represented as

$$\underline{Y} = \mathbf{S}\mathbf{A}\underline{X} + \underline{N}, \tag{2}$$

where $\underline{Y} = [Y_1, ..., Y_L] \in \mathbb{R}^L$ denotes the channel output vector, $\underline{X} = [X_1, ..., X_K] \in \mathcal{X}^K \subset \mathbb{R}^K$ is the input symbol vector, $\mathbf{S} \in \mathbb{R}^{L \times K}$ is the sparse spreading matrix, and $\mathbf{A} = diag(\frac{A_1}{\sqrt{\Lambda_1}}, \frac{A_2}{\sqrt{\Lambda_2}}, ..., \frac{A_K}{\sqrt{\Lambda_K}})$ is the gain matrix which has a diagonal form. In the system model, we additionally assume that input symbols X_k, elements of the spreading codes S_{lk} and the transmit gain A_k are i.i.d. drawn from P_X,P_S,P_A respectively. In the receiver side, the goal of the multiuser detector is to estimate the input vector $\underline{X}$ from the channel output vector $\underline{Y}$ given $\mathbf{S}, \mathbf{A}, P_X$.

III. Ensemble of the sparsely spread CDMA codes

Let $\mathbf{H} \in \{0,1\}^{L \times K}$ denote an incidence matrix of the spreading codes $\mathbf{S}$ which indicates nonzero position of the matrix $\mathbf{S}$. The authors also defined that two notation from the incidence matrix, which are

$$\text{The k-th symbol degree: } \Lambda_k = \sum_{l=1}^{L} H_{lk} \tag{3}$$

$$\text{The l-th chip degree: } \Gamma_k = \sum_{k=1}^{K} H_{lk} \tag{4}$$

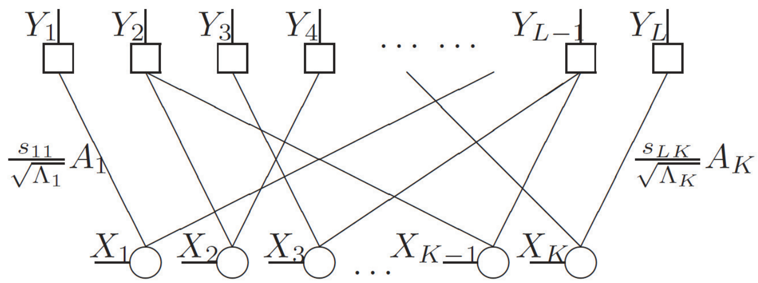

Fig. 2. Factor graphical representation of the CDMA system

Similarly, the average of the symbol and chip degree is defined as $\overline{\Lambda} = \frac{1}{K}\sum_{k=1}^{K}\Lambda_k$ and, $\overline{\Gamma} = \frac{1}{L}\sum_{l=1}^{L}\Gamma_l$, respectively. Then, the factor graph representation of the CDMA system is given in Fig.2. The authors of this paper have tried to analyze the performance of this CDMA system by assuming the following:

1) *Large-system limit*: The system size is very large, i.e., $K, L \to \infty$, and its system load remains a constant, i.e., $\beta \to K/L$.
2) *No-short-cycle*: Under the large-system-limit, the factor graph of the CDMA system does not include cycles shorter than the number of the BP iterations denoted by t.
3) *Chip-semi-regular*: Under the large-system-limit, the chip degree concentrate around their average, i.e., for every l and very small constant $\epsilon > 0$, $\lim_{K,L\to\infty} \Pr\{|\Gamma_l - \mathbf{E}\overline{\Gamma}| > \varepsilon\mathbf{E}\overline{\Gamma}\} = 0$.

Throughout this report, such CDMA system satisfying above assumption is referred to *large-sparse-system* (LSS).

IV. Design of the BP algorithm for the MUD problem

Before discussing the BP detection algorithm for the MUD system, let us summarize the important known facts the BP algorithms

1) BP basically aims to find marginal posterior PDF of each element X_k.
2) In order to reduce the complexity, BP removes the duplicated calculation with message exchanging over the graph connection.
3) *Optimality of BP*: BP provides exact inference (optimal) of the marginal PDFs if the corresponding factor graph is perfectly tree-structured.
4) *Loopy BP*: BP is well applied to graphs with cycles and provides good approximation of the marginal PDFs in practice even through the performance is suboptimal .

For the description of the BP algorithm, we define two notation for the message: *symbol-to-chip* (StC) messages denoted by $V_{k\to l}^{(t)}(x)$ and *chip-to-symbol* (CtS) messages denoted by $U_{l\to k}^{(t)}(x)$ where t is the number of iterations. In addition, $V_k(x)$ denotes the marginal PDF of X_k. For convenience, we define a set of edge representing statistical connection over the factor graph as $\mathcal{E} := \{(l,k)|S_{lk} \neq 0\}$. Also we define ∂l (resp. ∂k) as the subset of symbols (resp. ships) which have the statistical connection to chip l (resp. symbol k), called its neighborhood. Then, the iterative BP algorithm for computing the marginal PDF of all symbols is shown in Algo.1. This iterative BP algorithm performs exact marginalization of each symbol X_k given the entire observation $\underline{Y}$ if the factor graph is cycle-free. In practical CDMA systems, however, the average node degree is always greater than 2 such that cycles are inevitable. Thus, the BP algorithm performs approximate inference by assuming that all nodes, $\{X_k\}$ and $\{Y_l\}$, are i.i.d. each other.

Algorithm 1 Iterative BP

Inputs: Channel output $\underline{Y}$, Spreading matrix $\mathbf{\Phi}$, Gain matrix $\mathbf{A}$, Prior knowledge $p_X(x)$

Outputs: Marginal PDFs $V_k(x)$ for every k

1)Initialization:

set $U^0_{l\to k}(x) = 1 \ \forall x \in \mathcal{X}$ for every $(l,k) \in \mathcal{E}$

2)Iterations:

for $t = 1$ **to** T **do**

set $V^{(t)}_{k\to l}(x) \propto p_X(x) \times \prod_{j\in\partial k\backslash l} U^{(t-1)}_{j\to k}(x)$ for every $(l,k) \in \mathcal{E}$

set $U^{(t)}_{l\to k}(x) \propto \mathbf{E}\left\{p_{Y_l|\underline{X}}(y|\underline{X})|X_k = x, V^{(t)}_{k\to l}\right\}$

$$:= \sum_{(x_i)_{\partial l\backslash k}} \exp\left[-\frac{1}{2}\left(y_l - \frac{s_{lk}a_k}{\sqrt{\Lambda_k}}x - \sum_{i\in\partial l\backslash k}\frac{s_{li}a_i}{\sqrt{\Lambda_i}}x_i\right)^2\right]$$

$$\times \prod_{i\in\partial l\backslash k} V^{(t)}_{i\to l}(x_i) \text{ for every } (l,k) \in \mathcal{E}$$

end for

3)Marginal PDFs calculation:

set $V_k(x) \propto p_X(x) \prod_{j\in\partial k} U^{(T)}_{j\to k}(x)$ for every k

The LLR form of BP algorithm is simply obtained by fixing a reference point $x_0 \in \mathcal{X}$ and then defining LLR messages as

$$\text{LLR CtS message: } L^{(t)}_{l\to k}(x) := \log\frac{\mathbf{E}\left\{p_{Y_l|\underline{X}}(y|\underline{X})|X_k = x, L^{(t)}_{k\to l}\right\}}{\mathbf{E}\left\{p_{Y_l|\underline{X}}(y|\underline{X})|X_k = x_0, L^{(t)}_{k\to l}\right\}} \tag{5}$$

$$\text{LLR StC message: } L_{k\to l}^{(t)}(x) := \log p_X(x) + \sum_{j\in\partial k\backslash l} L_{j\to k}^{(t-1)}(x) \tag{6}$$

V. Asymptotic analysis of performance of BP

The key result of this paper states that

The marginal posterior computed for each symbol X_k using BP after t itertions essentially converges to the marginal posterior of a scalar Gaussian channel as the system size increases.

Now, we provide mathematical support for the statement above by stage. Let $P_{X_k}^{bp}(\cdot|\underline{Y}_k^{(t)}, \mathbf{S}, \mathbf{A})$ denote the output CDF from BP, which is approximate posterior of X_k given $\underline{Y}_k^{(t)}$. Here, $\underline{Y}_k^{(t)}$ is all observations within distance $2t-1$ to X_k on the factor graph. If X_k and Y_l is directly connected, the distance will be 1. In addition, let us introduce the canonical scalar Gaussian channel, given as

$$Z = \sqrt{g}X + N, \tag{7}$$

where $X \sim P_X$ and $N \sim \mathcal{N}(0,1)$ are independent, and g denotes the channel gain. For remainder derivation, we use $P_{X|Z;g}(\cdot|z;g)$ to denote the CDF of the posterior distribution of X given Z, according to the Gaussian channel model in (7).

***Theorem** 1 (Gaussian convergence of Marginal posterior):* Given fixed iterations t, the marginal posterior of X_k converges to that of the Gaussian channel, i.e. for every k

$$P_{X_k}^{bp}(x|\underline{Y}_k^{(t)}, \mathbf{S}, \mathbf{A}) \to P_{X|Z;g}(x|h(\underline{Y}_k^{(t)}, \mathbf{S}, \mathbf{A}); \eta^{(t)}A_k^2), \tag{8}$$

in probability under the LSS setup, where the Gaussian channel output is given as $Z = h(\underline{Y}_k^{(t)}, \mathbf{S}, \mathbf{A}) \sim \mathcal{N}(\sqrt{\eta^{(t)}}ax, 1)$, the channel gain $A_k\sqrt{\eta^{(t)}}$ is determined by the following recursion:

$$\frac{1}{\eta^{(t)}} = 1 + \beta\overline{\text{var}}\left\{AX|\sqrt{\eta^{(t-1)}}AX + N\right\}, \tag{9}$$

and

$$\overline{\text{var}}\left\{U|V\right\} := \mathbf{E}\left\{(U - \mathbf{E}\left\{U|V\right\})^2\right\}. \tag{10}$$

Proof: The authors proved Theorem 1 by considering messages of the LLR form given in (5) and (6). Proving Theorem 1 is equivalent to showing the LLR StC message is Gaussian distributed with $X_k \sim \mathcal{N}(A_k^2\eta^{(t)}X_k, 1)$. We summarize this proof in four steps.

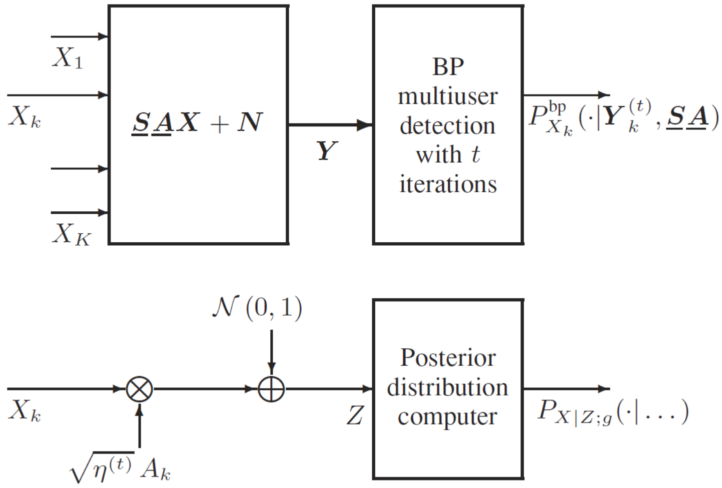

Fig. 3. Upper diagram: Multiuser channel and BP detection. Lower diagram: The asymptotically equivalent scalar Gaussian channel

Step I: The StC message is Gaussian RV by the central limit theorem (CLT): Under the no-short-cycle assumption, all CtS messages $L_{l\to k}^{(t)}$ are i.i.d. conditioned on $X_k = x_k$. From (6), by CLT, the message is a Gaussian random vector.

Step II: LLR obtained from a scalar Gaussian channel is also Gaussian distributed. Namely, for $Y = \sqrt{\gamma}X + N$, its LLR is a Gaussian RV, i.e.,

$$\log\frac{p_{Y|X}(Y|x_1)}{p_{Y|X}(Y|x_0)} = \sqrt{\gamma}(x_1 - x_0)Y - \gamma(x_1^2 - x_0^2)/2 \tag{11}$$

Step III: Calculation of mean and covariance of the StC messages given as

$$\mathbf{E}[L_{k\to l}^{(t)}(x)] = \log p_X(x) + \sum_{j\in\partial k\setminus l}\mathbf{E}[L_{j\to k}^{(t-1)}(x)]. \tag{12}$$

We first consider the mean of the CtS messages. To this end, we have

$$\begin{aligned} f(y,x) &:= \mathbf{E}\left\{p_{Y_l|\underline{X}}(y|\underline{X})|X_k = x, L_{k\to l}^{(t)}\right\} \\ &= \sum_{(x_i)_{\partial l\setminus k}}\left(\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}(y - \sum_{i\in\partial l\setminus k}\frac{s_{li}A_i}{\sqrt{\Lambda_i}}x_i - c_k x)^2 + \underbrace{\sum_{i\in\partial l\setminus k}L_{i\to l}^{(t)}(x_i)}_{=B_{x_k}}\right]\right), \end{aligned} \tag{13}$$

where we use $c_k = \frac{s_{lk}a_k}{\sqrt{\Lambda_k}}$. Then, we apply the 2nd order Taylor approximation with respect to $x = 0$ as

$$\begin{aligned} f(y,x) &\approx f(y,x=0) + f\prime(y,x=0)x + \frac{1}{2}f\prime\prime(y,x=0)x^2 \\ &= g_0(y) + g_1(y)c_k x + \frac{1}{2}g_2(y)c_k^2 x^2, \end{aligned} \tag{14}$$

where we define

$$g_0(y) := \frac{1}{\sqrt{2\pi}} \sum_{(x_i)_{\partial l \setminus k}} \exp\left[-\frac{1}{2}(y - \sum_{i\in\partial l\setminus k} \frac{s_{li}A_i}{\sqrt{\Lambda_i}}x_i)^2 + B_{x_k}\right] \tag{15}$$

$$g_1(y) := \frac{1}{\sqrt{2\pi}} \sum_{(x_i)_{\partial l \setminus k}} (y - \sum_{i\in\partial l\setminus k} \frac{s_{li}A_i}{\sqrt{\Lambda_i}}x_i) \exp\left[-\frac{1}{2}(y - \sum_{i\in\partial l\setminus k} \frac{s_{li}A_i}{\sqrt{\Lambda_i}}x_i)^2 + B_{x_k}\right] \tag{16}$$

$$g_2(y) := \frac{1}{\sqrt{2\pi}} \sum_{(x_i)_{\partial l \setminus k}} ((y - \sum_{i\in\partial l\setminus k} \frac{s_{li}A_i}{\sqrt{\Lambda_i}}x_i)^2 - 1) \exp\left[-\frac{1}{2}(y - \sum_{i\in\partial l\setminus k} \frac{s_{li}A_i}{\sqrt{\Lambda_i}}x_i)^2 + B_{x_k}\right]. \tag{17}$$

Then, from (5), the CtS LLR message is given as

$$\begin{aligned} L_{l\to k}^{(t)} &= \log \frac{g_0(y) + g_1(y)c_k x + \frac{1}{2}g_2(y)c_k^2 x^2}{g_0(y) + g_1(y)c_k x_0 + \frac{1}{2}g_2(y)c_k^2 x_0^2} \\ &\approx \frac{g_1(y)}{g_0(y)}c_k(x - x_0) + \frac{g_2(y)}{g_0(y)}c_k^2(x^2 - x_0^2) - \frac{1}{2}\frac{g_1^2(y)}{g_0^2(y)}c_k^2(x^2 - x_0^2), \end{aligned} \tag{18}$$

where we further apply the 2nd order Taylor approximation of $\log(x)$. The mean of the CtS message $\mathbf{E}[L_{j\to k}^{(t)}(x)]$ can be obtained by taking integration to (18) with respect to y. Then, using (12), the mean of the LLR StC message is obtained as

$$\mathbf{E}\left[L_{k\to l}^{(t)}(x)\right] = \Theta(x_k(x - x_0) - (x^2 - x_0^2)/2) \tag{19}$$

where

$$\Theta = A_k^2 \int \frac{g_1^2(y)}{g_0(y)} dy \frac{\sum_{j\in\partial k\setminus l} S_{jk}^2}{\Lambda_k}. \tag{20}$$

In (20), by law of large number, $\frac{\sum_{j\in\partial k\setminus l} S_{jk}^2}{\Lambda_k} \to 1$. Here, importantly note that the result in (19) is exactly equivalent to mean of LLR in a scalar Gaussian channel

$$\begin{aligned} X_k &= \sqrt{\Theta}X + N \\ &= A_k \sqrt{\int \frac{g_1^2(y)}{g_0(y)} dy} X + N \end{aligned} \tag{21}$$

where X_k here is a symbol obtained from the BP iteration given the observation $\underline{Y}$ (which is equivalent to Z in Th.1) and $N \sim \mathcal{N}(0,1)$ is additive noise with unit variance. Although the derivation of covariance is omitted here, they are also equivalent.

Now, we summarize the proof as following

1) LLR of StC message is asymptotically a Gaussian RV by CLT from Step I.
2) LLR of a scalar Gaussian channel is a Gaussian RV from Step II.
3) The mean and covariance of LLR StC message have the exactly same form as the LLR of the scalar Gaussian channel in (21) from Step III.
4) From (12), each individual symbol X_k via BP is also Gaussian distributed with $N \sim \mathcal{N}(\sqrt{\Theta}, 1)$.

The last piece of the proof of Theorem 1 is to quantify the corresponding SNR Θ with respect to the number of BP iterations t by showing that

$$\lim_{\substack{L,K\to\infty \\ \overline{\Gamma}\to\infty}} \int \frac{g_1^2(y)}{g_0(y)} dy = \eta^{(t)}. \tag{22}$$

But, the proof of (22) was not well explained in the paper. One thing is that one can derive the recursion in (9) by showing (22).

REFERENCES

[1] D. Guo and C. C. Wang, "Multiuser detection of sparsely spread CDMA," *IEEE J. Sel. Areas Comm.*, vol. 26, no. 3, pp. 421-431, Mar. 2008.