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| On Some Multiple Integrals Involving Determinants |

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| Authors: | N. G. de Bruijn |
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| Speaker: | Oliver |

**Short summary**: This paper introduces a way to evaluate a (ordered) multiple integration whose integrand is a determinant, i.e.,



where



An example for is the Vandermonde matrix such as the one shown below with th element 



Main Result:



where is a Pfaffian of a skew-symmetric matrix *A* () , whose entries are calculated from the diagonal entries of the matrix .

# Background

1. Alternating function: A function of *N* variables is said to be *alternating* if [interchang](http://en.wiktionary.org/wiki/interchange" \o "interchange)ing any two variables changes the [sign](http://en.wiktionary.org/wiki/sign) of the function.

Ex. 1.  ; 

Ex. 2.  is an alternating function

1. Symmetric function: A symmetric function of *N* variables is one whose value at any of the *N*!permutation is the same.

Ex. 

1. Result from Integration theory: Ordered integral can be converted into an unordered integral provided the integrand is a symmetric function.



1. Determinant expansion: Let  be the determinant of an matrix . The expansion of the determinant contains *N*! terms.

Ex. Consider a matrix



Its determinant expansion along the first row is given by



* + - The above expansion contains 3! = 6 terms.
    - Each term contains all the 3 functions 
    - Unordered integration of each term gives the same value 

1. Signature function: Let be a function defined as
2.  if 
3.  is alternating in .

In other words,



There are *N* variables and hence *N*! permutations are possible.

The function is called signature (sign) of the permutation.

When is not a permutation, then 

Ex. Let us take *N* = 3 and 

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|  |  |

Now consider 

; ;

* + The values of the *E*- functions do not depend on the actual values of 
  + *E*- function of *N* variables can be written in terms of two variables.

1. General definition: The signature function of *N* variables can be written in terms of *two* variables as (for even *N* i.e., *N* = 2*m*)



Ex. Let us take *N* = 2. (*m* = 1)



We can view terms  in the above summation as a matrix



In terms of the matrix  we can write  as



Let us take *N* = 4 (*m* = 2)



For a general *N*, we have 







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| --- |
| If *A* is an matrix with entries then |

Ex.



1. The signature function for odd number of variables can be extended from the even number of variables. (Please refer 1).

# Evaluation of the multiple integral

Let us consider the following *N*-dimensional integration:



1. Multiplying the above integrand by  does not affect its value.



1. The integrand now becomes symmetric in the variables , so that it can be written as



1. The determinant in the above integral contains *N*! terms and each term contribute the same value to the integral, thus the integral can be written in terms of only one term as



1. Now expand by using





 

The element of the skew-symmetric matrix is written as



For skew-symmetric matrices and.

References

1. N. G. De Bruijn, “On some multiple integrals involving determinants,” Journal of Indian Mathematical Society, vol. 19, pp. 133-151, 1955.