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| Signal Recovery from Random Measurements  Via Orthogonal Matching Pursuit |

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**Short summary**: We discuss from this paper how to derive the theoretical probability of recovery for the orthogonal matching pursuit (OMP) algorithm. It is known that, till date, there is no theoretical phase transition analysis present for the OMP algorithms. The analysis presented in this paper could be used as a starting point to derive the phase transitions for the OMP algorithm. We discuss a key theorem and its proof in this talk.

# System Model and Background

* Consider a model,  with and  is a *K*-sparse signal.
* Let be the i*th* column of A and assume that.
* The goal of OMP is to estimate the support of  iteratively.
* At each iteration, OMP selects a column of *A* that is most correlated with the current residual. OMP then updates the residual by projecting  onto a linear space spanned by the selected columns. The algorithm iterates until certain stopping rule is satisfied.

A. The OMP algorithm: Description

Input:

* An measurement matrix 
* A measurement vector of size 
* A *K* sparse signal vector of size 

Output after *K* iterations

* An estimate of 
* An index set such that  and the elements of 
* An *K* dimensional approximation for the measurement vector
* An *K* dimensional residualfor the measurement vector

Procedure:

1. Initialization :

* Initialize the residual, the index set, an empty set. Let  be an empty matrix.
* Set the counter value.

1. Finding the index and updating the index set

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* Update the index set 

1. Estimate the signal using the obtained index set

* Form a matrix 
* Estimate the signal by solving the least squares problem: 

1. Calculate new approximation of the measurement vector and new residual

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1. Increment *p* and return to step 2 if . Halt if 

# Analysis Using Random Ensembles

In this section, we study how to derive the probability of recovery by OMP with Gaussian matrices.

Admissible matrices

An admissible matrix for *K*-sparse signals is an matrix with the following four properties.

(M0) Independence: The columns of are statistically independent

(M1) Normalization:. This assumption is for mathematical convenience. Signal recovery problem does not depend on the scale of the measurement matrix

(M2) Joint Correlation: Let be a sequence of *K* vectors whose L2 norms does not exceed 1. Let be the column of  that is independent from this sequence. Then



(M3) Smallest singular value: For a given sub-matrix from , the *K*th smallest singular value 

Theorem 6: (OMP with admissible measurement matrices): Suppose that is an arbitrary *K*-sparse signal in , and draw a random admissible measurement matrix independent from the signal. Given the data , the OMP can reconstruct the signal with the success probability given by

Result:



Proof:

Without loss of generality, assume that the first *K* entries of the original signal are nonzero, while the remaining *N - K* entries equal zero. Therefore, the data vector  is a linear combination of the first *K* columns from the matrix. Partition the matrix as  so that *B* has first *K* columns and *C* has the *N*-*K* columns. Note that the vector is statistically independent from the random matrix *C*.

Consider the event where the algorithm correctly identifies the signal after *K* iterations. We only decrease the probability of success if we impose the additional requirement that the smallest singular values of *B* meets a lower bound. To that end, define the event



Note that the event  implies that *B* has full column rank (and thus, guarantees unique solution). Now can then be written by using the conditional probability as



We aim to calculate. To prove that occurs conditioned on , it suffices to check that OMP correctly identifies the columns of . For this purpose, we define the *greedy selection ratio.* For a vector, the greedy selection ratio is given by



Ifis the residual vector, then from Step 2 of the OMP, we find that the OMP choose a column from if . If , the OMP has no provision to choose either a column from and.

Observation: Greedy selection ratio is based on the assumption that we know the support and hence the matrices and.

Imaginary experiment: Suppose that we execute *K* iterations of OMP with the input signal and the restricted measurement matrix to obtain a sequence of residuals and a sequence of column indices. The algorithm is deterministic, so these sequences are both functions of and. In particular, the residuals are statistically independent from. It is also evident that each residual lies in the column span of.

Actual experiment: Execute OMP with the input signal and the full matrix to obtain the actual sequence of residuals and the actual sequence of column indices . Conditional on , OMP succeeds in reconstructing after *K* iterations if and only if the algorithm selects the *K* columns of in some order. Using induction, it is possible to prove that this situation occurs when for each .

The success probability is and conditioned on, the success probability is



where is a sequence of *K* random vectors that fall in the column space of , and these vectors are statistically independent of . We need to derive a lower bound for .

Assume that occurs. For each index the greedy ratio is



Now we aim to calculate the ratio by first approximating it. We first make use of in the denominator to get



The denominator is a scalar and it can be moved inside the inner product form the vector.

Now we know that the ratio and thus  which implies that . Thus, for each 



What we need is , that is,



Every column of is independent of and also from. Using the joint correlation property (M2), the above product of terms can be written as



The property M3 says . Therefore,





Reference

1. D. Donoho and X. Huo, ``Uncertainty principles and ideal atomic decomposition," IEEE Trans. Info. Theory, vol. 47, no. 7, pp. 2845-2862, 2001.
2. A. B. Heim, Y. Eldar, and M. Elad, ``Coherence-based performance guarantees for estimating a sparse vector under random noise," IEEE Trans. On Sig. Process. vol. 58, no.10, pp. 5030-5042, Oct. 2010.
3. J. Tropp, ``Greed is good: Algorithmic results for sparse approximation,"IEEE Trans. Info. Theory, vol. 50, no. 10, pp. 2231-2242, 2004.
4. Q. Mo and Y. Shen, ``A remark on the restricted isometry property in orthogonal matching pursuit," IEEE Trans. Inf. Theory, vol. 58, no. 6, pp. 3654-3656, 2012.
5. J. Wang and B. Shim, ``On the recovery limit of sparse signals using orthogonal matching pursuit," IEEE Trans. Signal Process., vol. 60, no. 9, pp. 4973-4976, Sept. 2012.
6. T. Cai and L. Wang, ``Orthogonal matching pursuit for sparse signal recovery with noise," IEEE Trans. Inf. Theory, vol. 57, no. 7, pp.4680–4688, 2011.
7. Y. Shen and S. Li, ``Sparse signals recovery from noisy measurements by orthogonal matching pursuit," Arxiv Preprint arXiv: 1105.6177, 2011.