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| The Exact Support Recovery of Sparse Signals with Noise via Orthogonal Matching Pursuit |

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**Short summary**: This letter derives sufficient conditions for the OMP to recover the support set of a sparse vector from noise corrupted measurements. In particular, the conditions are given in terms of the minimum absolute values of the signal amplitudes. That is, if the minimum values of the non-zero coefficient of the signal satisfy certain bound then OMP guarantees exact support recovery.

# System Model and Background

* Consider a model,  with and  is a *K*-sparse signal.
* Let be the i*th* column of A and assume that.
* Let  and .
* The goal of OMP is to estimate the support of  iteratively.
* At each iteration, OMP selects a column of A that is most correlated with the current residual. OMP then updates the residual by projecting  onto a linear space spanned by the selected columns. The algorithm iterates until certain stopping rule is satisfied.

A. The OMP algorithm

Notations: For two sets and, let  and. Let  denotes a sub-matrix whose column indices are elements of the set and  denotes the elements of  whose indices are specified by  and  represents the pseudo-inverse of A.

1. Initialize: Given A and, set the initial residual vector (that is), the initial index set as empty, and the iteration counter t=1.
2. Find the index  and update the support set estimate 
3. Estimate: and update the residual 
4. Halt if some stopping rule is satisfied. Otherwise, set t=t+1 and return to step 2.

Stopping rule design for the OMP depends on noise. In noiseless case, (when) the stopping rule can simply be. This letter considers two types of bounded noises, namely, bounded noise, and  bounded noise,. The stopping rules for these two noises in terms of residuals are and, respectively. This paper also considers the case when  follows.

# RIP and a Few associated Lemmas

Two features of a sensing matrix are often used to analyze and derive the recovery performance guarantee of OMP. One is the Mutual Incoherence Property (MIP) [1] defined as. And, the other one is restricted isometry property (RIP).

* A matrix A satisfies RIP of order *K* with parameter  if it is the smallest constant such that



holds for any *K*-sparse vector 

* ***Lemma 1:*** Suppose that a matrix *A* satisfies RIP of order *K*. Let be an index set with. Then all singular values of sub-matrix, which are denoted by, satisfy



* ***Remark 1:*** For any given matrix. Let, then



* ***Lemma 2:*** Suppose that a matrix *A* satisfies RIP of order *K*. Let be an index set with. Then all eigenvalues of matrix, which are denoted by, satisfy



* ***Lemma 3***: Suppose that a matrix *A* satisfies RIP of order *K*. Let and  be two disjoint sets with. Then for any vector  with , it holds that



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| Recovery conditions of OMP algorithm | | | |
| MIP | | RIP | |
| Noiseless case | Noisy case | Noiseless case | Noisy case |
| [3] | [2] | [4, 5] | and  [6]  and  [7] |

# Exact support set recovery of sparse signals

Condition

Let be an original support set of the signal. Let  is the residual at the t*th* iteration,. The condition for OMP to select a correct index at t*th* iteration is



*A.  bounded noise*

***Theorem 1:*** Suppose that  and the matrix *A* satisfies condition. Then OMP with stopping rule  will exactly recover the support  of *K*-sparse signal, if the minimum magnitude of nonzero elements of  satisfies



*Proof*

* Suppose that OMP selects only correct indexes at the first t-1 iterations, then  and the support of the solution  obtained at t-1*th* iteration is  and .
* We can write the residual as



* Our goal is to find the RHS and LHS of the condition 
* Let us start with the LHS, that is, 



* Now from Lemma 3 it holds for any 



* Also, since, we have



* Now, the LHS becomes



* Let us find the RHS, that is,. Let us recall that the residual  is orthogonal to the columns of, that is,. Then



* Thus, has only  non-zero elements. By using the relation , we have



* Now,



*  (Consequence of RIP)
* 
* Therefore, the RHS is lower bounded by



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* Using Eqns. , , and , we can find that for the condition to be satisfied , the following inequality must hold true



which is stated in Theorem 1.

* After all the *K* indexes in ** have been identified, we find a new estimator via. Then, the residual  obeys



* Therefore, OMP stops after *K* iterations during which the stopping rule is satisfied.

B. bounded noise:

***Theorem 2:*** Suppose that  and the matrix A satisfies condition . Then OMP with stopping rule will exactly recover the support  of K-sparse signal, if the minimum magnitude of nonzero elements of  satisfies



C. Gaussian noise case

It is well known that when the noise in the model  follows iid Gaussian distribution with zero-mean and variance, then



***Theorem 3:*** Suppose that each element of the noise vector follows Gaussian with zero mean and variance  and the matrix A satisfies condition . Then, OMP with stopping rule will exactly recover the support  of *K*-sparse signal with probability at least 1-(1/m), if the minimum magnitude of nonzero elements of  satisfies



*Remarks:*

* In bounded noise case, the minimum magnitude of the *K*-sparse signal needs to be in the same order of the noise level.
* In bounded noise case, the minimum magnitude needs to be about times the noise level
* In the Gaussian case, the minimum magnitude depends on the size “*m*” of the matrix. Thus, *m* must be chosen to satisfy  (for example say), then the minimum magnitude need to be about *K* times of.

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