# Signal Recovery From Random Measurements Via Orthogonal Matching Pursuit

J. A. Tropp and A. C. Gilbert

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**Presenter : Sangjun Park**

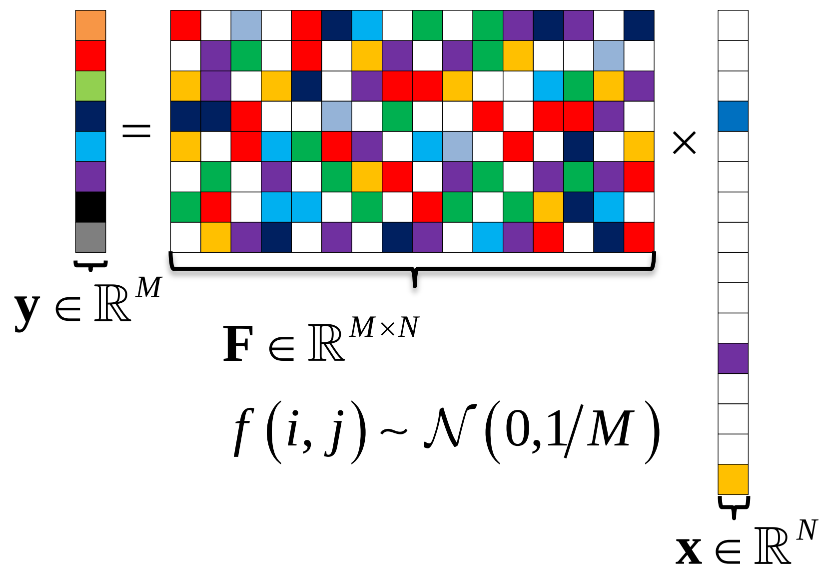
GIST, Dept. of Information and Communication, INFONET Lab.

Gwangju Institute of Science and Technology

# Questions and System Model

- Let us suppose that we aim to find the support set of a sparse vector by using OMP.

- Then, what is a sufficient condition for successful OMP?



# Orthogonal Matching Pursuit

- OMP finds one index at a time for approximating the solution of

$$\min_{\mathbf{x}} \|\mathbf{x}\|_0 \text{ subject to } \|\mathbf{y}-\mathbf{A}\mathbf{x}\|_2^2 \le \varepsilon^2$$

**Initialization**

$i=0, \mathbf{x}^0=\mathbf{0}_N$

$\mathbf{r}^0=\mathbf{y}, \mathcal{Q}=\{\ \}$

$i=i+1$

**Main Iteration**

1. **Choose** $t=\arg\max_{j=1,\cdots,N} \left|\left\langle \mathbf{r}^{i-1}, \mathbf{a}_j \right\rangle\right|$
2. **Update** $\mathcal{Q}=\mathcal{Q}\cup\{t\}$
3. **Least Square** $\mathbf{x}^i=\min_{\mathbf{x}} \|\mathbf{y}-\mathbf{A}\mathbf{x}\|_2^2 \text{ s.t. } \mathrm{Supp}(\mathbf{x})=\mathcal{Q}$
4. **Update** $\mathbf{r}^i=\mathbf{y}-\mathbf{A}\mathbf{x}^i$

$\|\mathbf{r}\|_2^2 \le \varepsilon^2$

No

Yes

# Sufficient conditions for successful OMP

- There are many papers that report sufficient conditions for successful OMP.

| Year | A sufficient condition | Types |
|---|---|---|
| 2004 | $\mu < 1/(2K-1)$ | Deterministic |
| 2010 | $\delta_{K+1} < 1/(3\sqrt{K})$ | Deterministic |
| 2012 | $\delta_{K+1} < 1/(\sqrt{K}+1)$ | Deterministic |
| This paper | $M = \Omega(K\log(N))$ | Probabilistic |

2007: J. Tropp, "Greed is good: Algorithmic results for sparse approximation," IEEE Trans. On. Inform. Theory
2010: M. A. Davenport, M. B. Wakin, "Analysis of Orthogonal Matching Pursuit Using the Restricted Isometry Property", IEEE Trans. On. Inform. Theory
2012: J. Wang and B. Shim, "On the recovery Limit of Sparse Signals Using Orthogonal Matching Pursuit", IEEE Trans. Signal Processing Letter

# The short overview of the paper [2012]

- To derive their sufficient condition, the authors considered the event that OMP correctly selects index $j$ at the $i^{\text{th}}$ iteration.
- The event occurs if $\min_{t\in\mathcal{I}} \left\| \langle \mathbf{a}_t, \mathbf{y} \rangle \right\|_2 > \max_{t\notin\mathcal{I}} \left\| \langle \mathbf{a}_t, \mathbf{y} \rangle \right\|_2$.
- They have shown that the left term is lower bounded by

$$\min_{t\in\mathcal{I}} \left\| \langle \mathbf{a}_t, \mathbf{y} \rangle \right\|_2 \geq \frac{1}{\sqrt{K}} \left(1-\delta_K\right) \left\| \mathbf{x}_{\mathcal{I}} \right\|_2 .$$

- Also, they have shown that the right term is upper bounded by

$$\max_{t\notin\mathcal{I}} \left\| \langle \mathbf{a}_t, \mathbf{y} \rangle \right\|_2 \leq \left(1-\delta_{K+1}\right) \left\| \mathbf{x}_{\mathcal{I}} \right\|_2 .$$

- Then, they have derived their sufficient condition from the two bounds.

# The main Theorem

- (OMP with Admissible Measurement matrix.) Fix $\delta \in (0,1)$, and choose $M = \Omega(K log(N/\delta))$. Suppose that $\mathbf{x}$ is an arbitrary $K$ −sparse vector in $\mathcal{R}^N$, and draw a random $M \times N$ admissible measurement matrix $\mathbf{A}$ independent from the vector. Given the measurement vector $\mathbf{y} = \mathbf{Ax}$. Then, OMP can reconstruct the support set with probability exceeding $1 - \delta$.

# Admissible Measurement Matrices

- An admissible measurement matrix for $K$ −sparse vectors in $\mathcal{R}^N$ is an $M \times N$ random matrix $\mathbf{A}$ with four properties.

(M0) Independence : The columns of $\mathbf{A}$ are stochastically independent.

(M1) Normalization : $\mathbb{E}\left[\|\mathbf{a}_i\|_2^2\right] = 1 \text{ for } j = 1, \cdots, N.$

(M2) Joint correlation : Let $\{\mathbf{u}^t\}$ be a sequence of vectors whose $l_2$ norms do not exceed one. Let $\mathbf{a}$ be a column of $\mathbf{A}$ that is independent from $\{\mathbf{u}^t\}$. Then,

$$\mathbb{P}\left\{\max_t \left|\left\langle \mathbf{a}, \mathbf{u}^t \right\rangle\right| \leq \varepsilon\right\} \geq 1 - 2K \exp\left(-c\varepsilon^2 M\right)$$

(M3) Smallest singular value: Given an $M \times K$ submatrix $\boldsymbol{Z}$ from $\mathbf{A}$, the largest singular value $\sigma_{min}(\mathbf{Z})$ satisfies $\mathbb{P}\left\{\sigma_{\min}(\mathbf{Z}) \geq 0.5\right\} \geq 1 - \exp\left(-cM\right)$

# The proof of the main Theorem-1

- First, let us define the greedy ratio at the $l^{\text{th}}$ iteration:

$$\rho\left(\mathbf{r}^{l}\right) := \frac{\max_{i \notin \mathcal{I}} \left|\left\langle \mathbf{r}^{l}, \mathbf{a}_{i} \right\rangle\right|}{\max_{i \in \mathcal{I}} \left|\left\langle \mathbf{r}^{l}, \mathbf{a}_{i} \right\rangle\right|}$$

- OMP correctly selects an index belonging to the support set if $\rho\left(\mathbf{r}^{l}\right) < 1.$

- OMP correctly reconstructs the support set when the event $E_{succ} := \max_{l \le K} \rho\left(\mathbf{r}^{l}\right) < 1$ occurs

- We aim to obtain the probability

$$\mathbb{P}\left\{E_{succ}\right\} := \mathbb{P}\left\{\max_{l \le K} \rho\left(\mathbf{r}^{l}\right) < 1\right\}$$

$$\ge \mathbb{P}\left\{\max_{l \le K} \rho\left(\mathbf{r}^{l}\right) < 1 \cap \sigma_{\min}\left(\mathbf{A}_{\mathcal{I}}\right) \ge 0.5\right\}$$

- Owing to (M3), we can solve LS within the $K^{\text{th}}$ iterations.

# The proof of the main Theorem-2

- Continuously, we aim to consider the probability

$$\mathbb{P}\left\{\max_{l\le K}\rho\left(\mathbf{r}^{l}\right)<1\middle|\sigma_{\min}\left(\mathbf{A}_{\mathcal{I}}\right)\ge 0.5\right\}$$

- For this end, we consider the greedy ratio at the $l^{\text{th}}$ iteration. Then, we have

$$\rho\left(\mathbf{r}^{l}\right)=\frac{\max_{i\notin\mathcal{I}}\left|\left\langle\mathbf{r}^{l},\mathbf{a}_i\right\rangle\right|}{\max_{i\in\mathcal{I}}\left|\left\langle\mathbf{r}^{l},\mathbf{a}_i\right\rangle\right|}=\frac{\max_{i\notin\mathcal{I}}\left|\left\langle\mathbf{r}^{l},\mathbf{a}_i\right\rangle\right|}{\left\|\mathbf{A}_{\mathcal{I}}^{T}\mathbf{r}^{l}\right\|_{\infty}}\le\frac{\sqrt{K}\max_{i\notin\mathcal{I}}\left|\left\langle\mathbf{r}^{l},\mathbf{a}_i\right\rangle\right|}{\left\|\mathbf{A}_{\mathcal{I}}^{T}\mathbf{r}^{l}\right\|_{2}}$$

- Now, we simplify the upper bound of the greedy ratio. First, let us define $\mathbf{r}^{l}:=\mathbf{u}^{l}\left\|\mathbf{A}_{\mathcal{I}}^{T}\mathbf{r}^{l}\right\|_{2}/0.5.$ Then, the upper bound becomes

$$\frac{\sqrt{K}\max_{i\notin\mathcal{I}}\left|\left\langle\mathbf{r}^{l},\mathbf{a}_i\right\rangle\right|}{\left\|\mathbf{A}_{\mathcal{I}}^{T}\mathbf{r}^{l}\right\|_{2}}=\frac{\sqrt{K}\max_{i\notin\mathcal{I}}\left|\left\langle\mathbf{u}^{l}\left\|\mathbf{A}_{\mathcal{I}}^{T}\mathbf{r}^{l}\right\|_{2}/0.5,\mathbf{a}_i\right\rangle\right|}{\left\|\mathbf{A}_{\mathcal{I}}^{T}\mathbf{r}^{l}\right\|_{2}}$$

$$=2\sqrt{K}\max_{i\notin\mathcal{I}}\left|\left\langle\mathbf{u}^{l},\mathbf{a}_i\right\rangle\right|.$$

# The proof of the main Theorem-3

- Owing to M3, we have $\left\|\mathbf{A}_{\mathcal{I}}^{T}\mathbf{r}^{l}\right\|_{2} / \left\|\mathbf{r}^{l}\right\|_{2} \geq \sigma_{\min}\left(\mathbf{A}_{\mathcal{I}}\right) \geq 0.5.$

- Then, we can show that the $l^2$ norm of the vector $\mathbf{u}^l$ is always less than one.

$$\mathbf{u}^{l} = 0.5\mathbf{r}^{l} / \left\|\mathbf{A}_{\mathcal{I}}^{T}\mathbf{r}^{l}\right\|_{2} \leq \mathbf{r}^{l} / \left\|\mathbf{r}^{l}\right\|_{2}$$

- Now, we have

[redacted]

$$= \mathbb{P}\left\{\max_{i \notin \mathcal{I}} \max_{l \leq K} \left|\left\langle \mathbf{u}^{l}, \mathbf{a}_{i} \right\rangle\right| < \frac{1}{2\sqrt{K}} \middle| \sigma_{\min}\left(\mathbf{A}_{\mathcal{I}}\right) \geq 0.5\right\}$$

$$\geq \prod_{i \notin \mathcal{I}} \mathbb{P}\left\{\max_{l \leq K} \left|\left\langle \mathbf{u}^{l}, \mathbf{a}_{i} \right\rangle\right| < \frac{1}{2\sqrt{K}} \middle| \sigma_{\min}\left(\mathbf{A}_{\mathcal{I}}\right) \geq 0.5\right\}$$

$$\geq \left[1 - 2K \exp\left(-cM / (4K)\right)\right]^{N-K}$$

# The proof of the main Theorem-4

- In addition, we have $\mathbb{P}\left\{\sigma_{\min}(\mathbf{Z}) \geq 0.5\right\} \geq 1-\exp(-cM)$.
- Thus, we finally obtain

$$\mathbb{P}\left\{E_{succ}\right\} \geq \left[1-2K\exp\left(-cM/(4K)\right)\right]^{N-K}\left[1-\exp(-cM)\right].$$

- To simplify the lower bound, we apply the inequality $(1-x)^n \geq 1-kn$ for $n \geq 1$ and $x \leq 1$. Then, for $K(N-K) \leq N^2/4$, we have

$$\mathbb{P}\left\{E_{succ}\right\} \geq 1-2K(N-K)\exp\left(-cM/(4K)\right)-\exp(-cM).$$

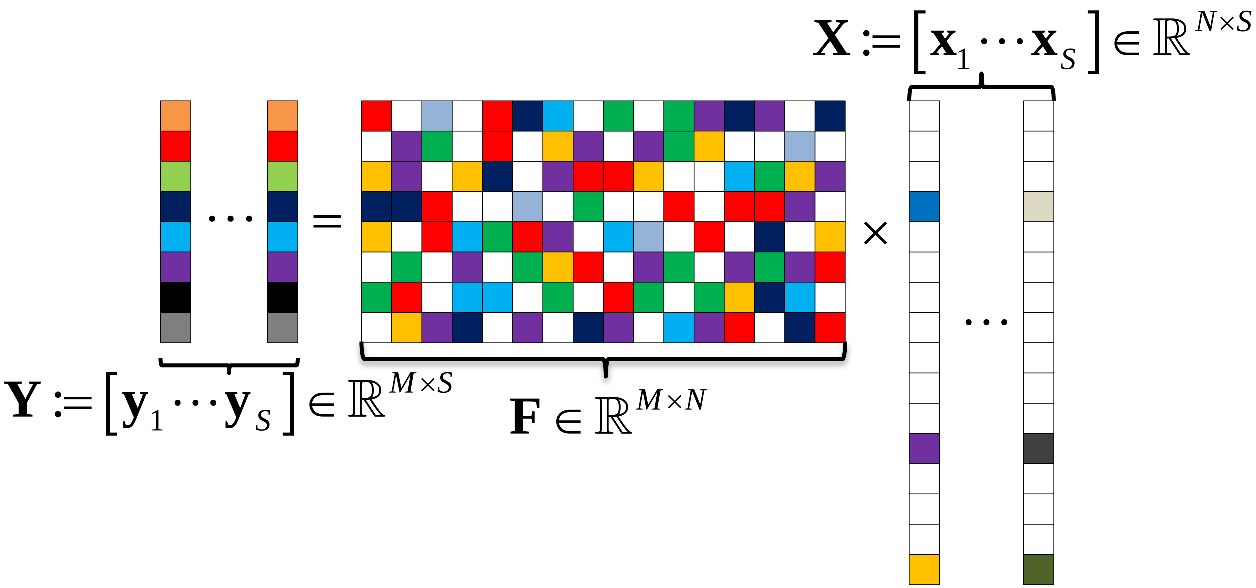
- By again simplifying the above lower bound, we have

$$\mathbb{P}\left\{E_{succ}\right\} \geq 1-N^2\exp(-cM/K).$$

- Finally, we can see that the choice $M=\Omega(Klog(N/\delta))$ is sufficient to reduce the failure probability below $\delta$.

# New researches problems

1. Can we establish a sufficient condition for Simultaneously Orthogonal Matching Pursuit?



# Simultaneously Orthogonal Matching Pursuit

**Initialization**

$i = 0, \mathbf{x}_s^0 = \mathbf{0}_N$

$\mathbf{r}_s^0 = \mathbf{y}_s, \mathcal{Q} = \{ \ \}$

$i = i + 1$

**Main Iteration**

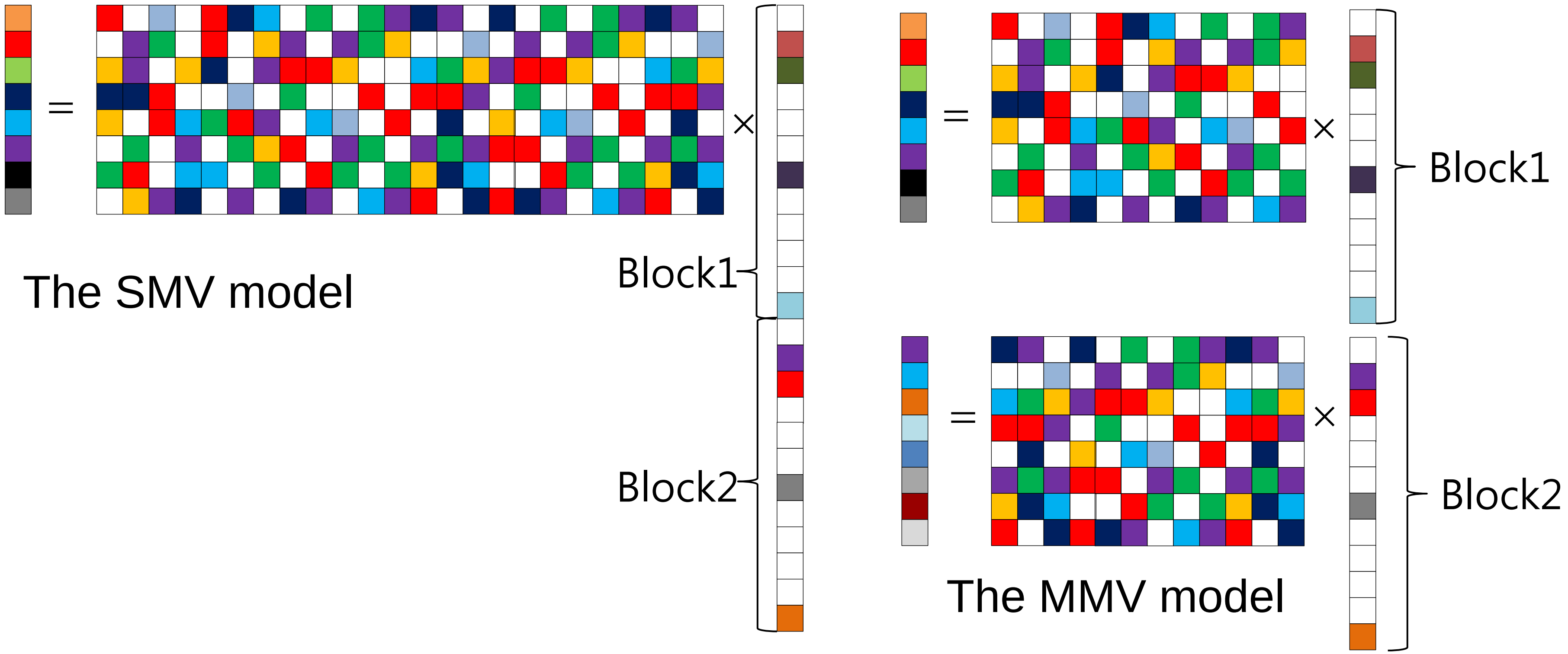
1. **Choose** $t = \arg\max_{j=1,\cdots,N} \sum_s \left| \left\langle \mathbf{r}_s^{i-1}, \mathbf{a}_j \right\rangle \right|$
2. **Update** $\mathcal{Q} = \mathcal{Q} \bigcup \{t\}$
3. **Least Square** $\mathbf{x}_s^i = \min_{\mathbf{x}} \left\| \mathbf{y}_s - \mathbf{A}\mathbf{x} \right\|_2^2$ s.t. $\mathrm{Supp}(\mathbf{x}) = \mathcal{Q}$
4. **Update** $\mathbf{r}_s^i = \mathbf{y}_s - \mathbf{A}\mathbf{x}_s^i$

$\sum_s \left\| \mathbf{r}_s \right\|_2^2 \le \varepsilon^2$

No

Yes

# New researches problems




2. Let $M_1$ be the number of measurements in the SMV model when OMP is exploited. Let $M_2$ be the total number of measurements in the MMV model when SOMP is exploited. What is the relation between $M_1$ and $M_2$?