Network Processing with Bayesian Graphical Models

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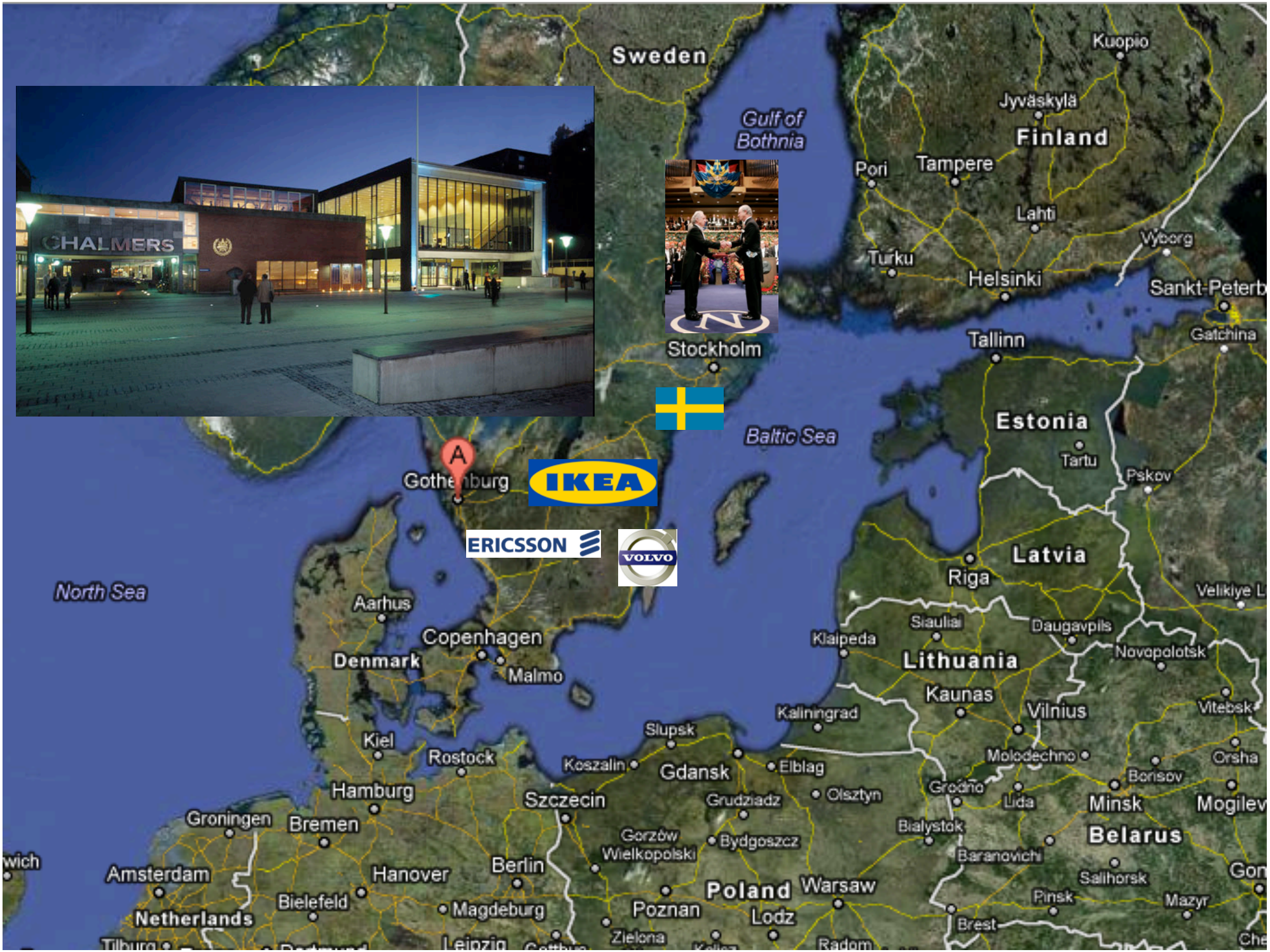
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Synchronization, positioning, channel estimation, modulation

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Data compression, hardware modeling

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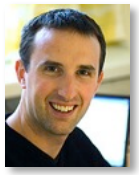
Tommy Svensson
Coded modulation, medium access, cooperative communications



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Fredrik Brännström
Error-control coding, modulation, iterative processing

Assistant Professor
Giuseppe Durisi
Network information theory, compressed sensing

Assistant Professor
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Coding, iterative processing, compressed sensing, cooperative communications

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Positioning, optics, Bayesian inference

Post Doc
Rocco di Taranto
Location-aware communications

Post Doc
Per Landin
Power amplifiers

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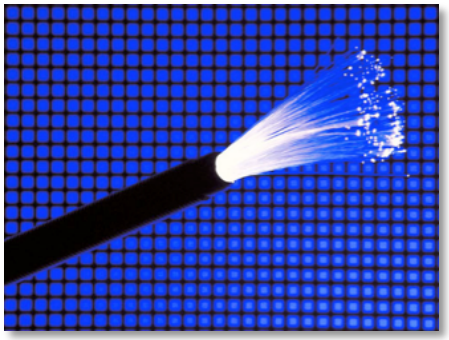
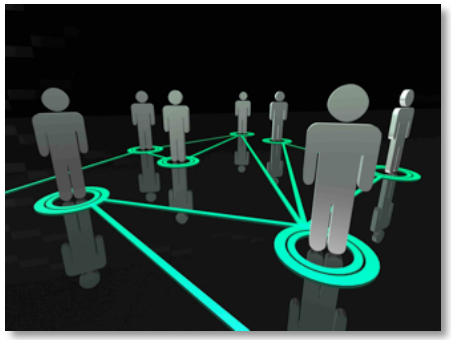
Our research

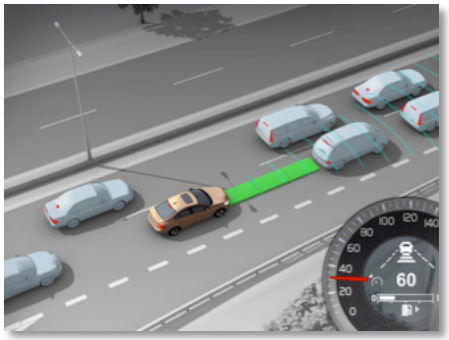


Fiber-optical communication

Wireless communication



Cooperative networks



Vehicular communication

Today's tutorial has applications in all these areas

Goals

At the end of this tutorial, you should be able to:

1. *Recognize and formulate* inference problems
2. *Solve* inference problems using factor graphs
3. *Design* algorithms for centralized and distributed processing
4. *Describe* the bigger picture

Do-it-yourself problems

- Some problems will not be solved in the tutorial
- Problems are shown by

- Most solutions are at the end of the tutorial

Outline

- Applications
- Background and terminology
- Bayesian detection
- **Tool 1: Bayesian graphical models**
  - Basics
  - Recipe for Bayesian inference
  - Practicalities
  - A worked example: a digital receiver
- **Tool 2: Belief consensus**
  - Basics
  - Convergence
- Applications revisited
- Variational interpretation
- Summary and conclusions

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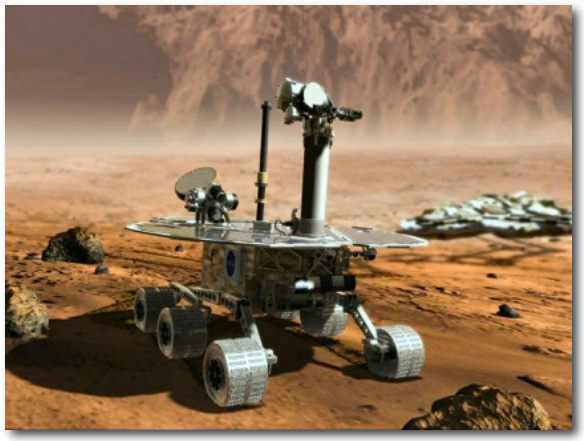
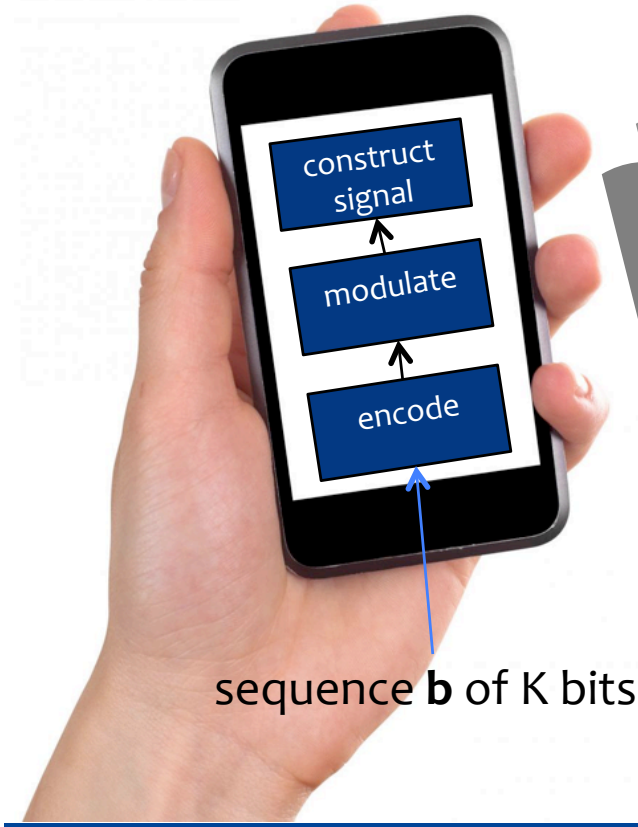
Applications



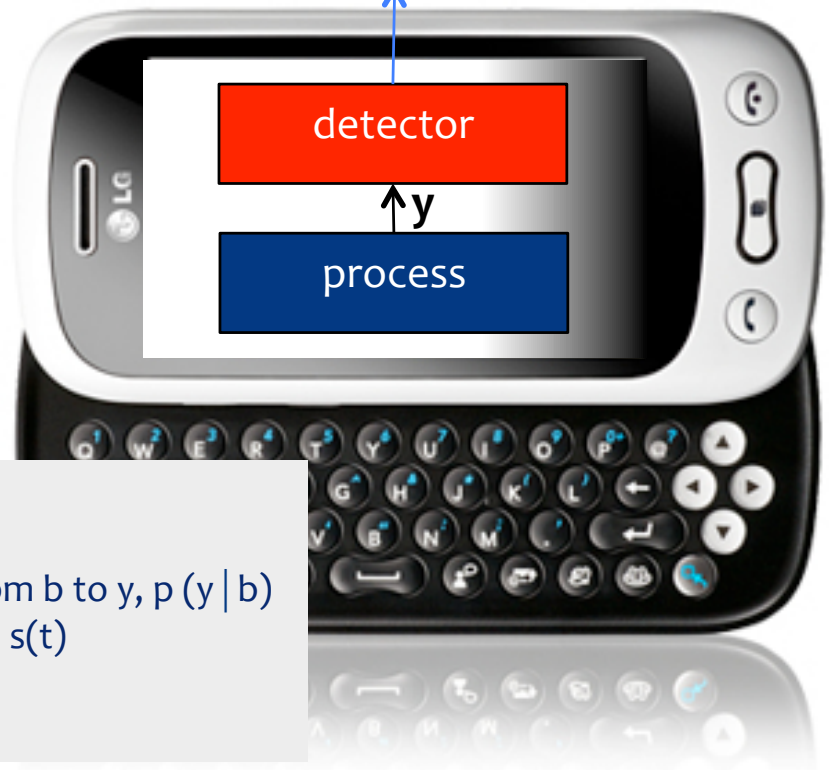
Application 1: receiver design

- Data detection problem: recover **b** from **y** (optimally)
- Many variations: codes, mapping, channels, antennas, users



sequence **b** of K bits

estimate of **b**



The receiver knows
A priori distribution p(b)
Probabilistic mapping from b to y, p (y | b)

- Conversion from b to s(t)
- Channel (linear)
- Noise statistics

Application 2: cooperative localization

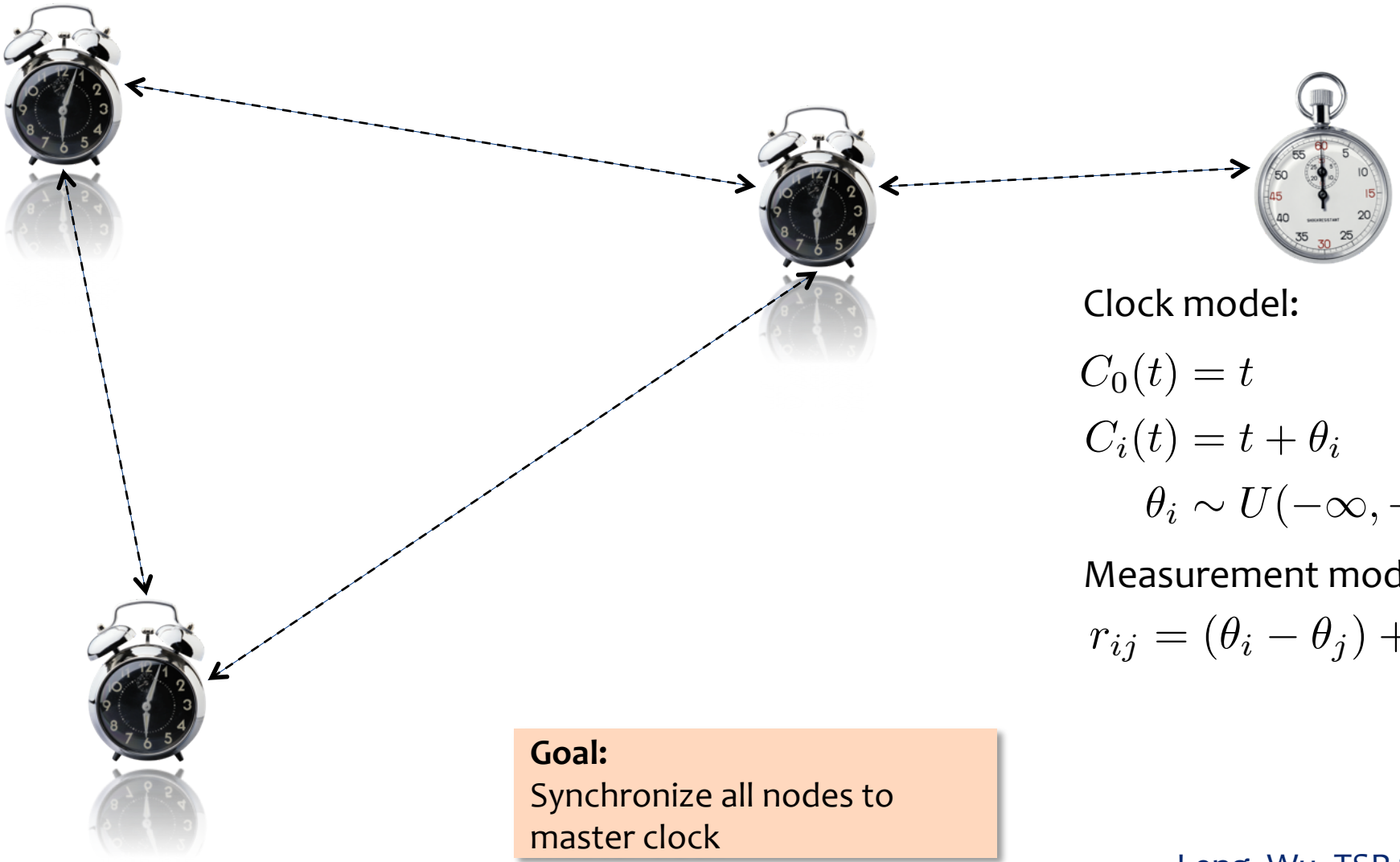
Scenario

- N mobile nodes (agents) with **unknown** positions $x_1(t),\ldots, x_N(t)$
- M reference nodes (anchors) with **known** positions $x_A(t)$, $x_B(t)$, $x_C(t)$, …
- **Measurements** $z_{ij}(t)$ between $x_i(t)$ and $x_j(t)$, where j is mobile or reference

Goals

- For every mobile node to determine its own position, given all measurements up to now

Application 3: network synchronization



Clock model:

$$C_0(t) = t$$

$$C_i(t) = t + \theta_i$$

$$\theta_i \sim U(-\infty, +\infty)$$

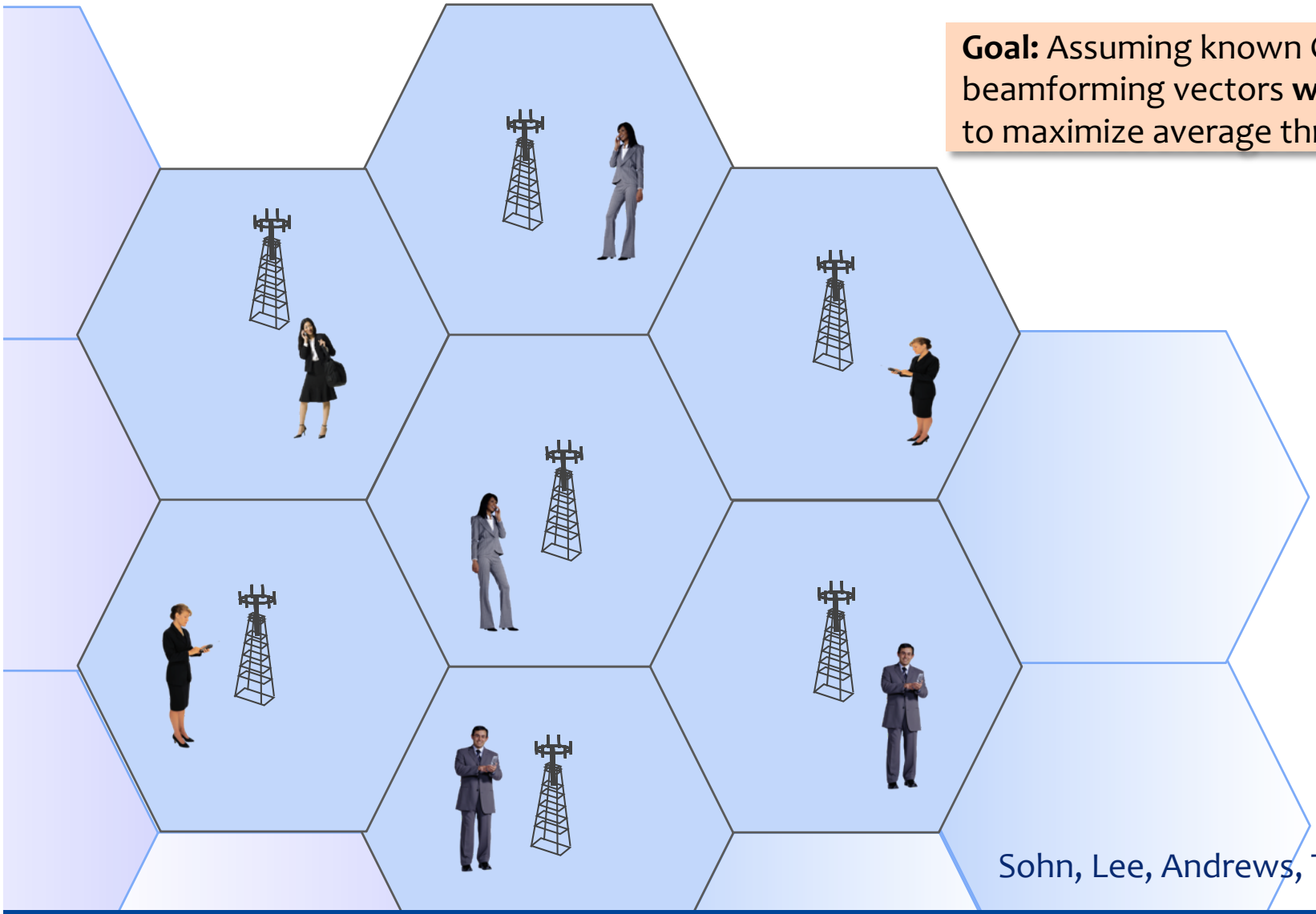
Measurement model:

$$r_{ij} = (\theta_i - \theta_j) + n_{ij}$$

Goal:
Synchronize all nodes to master clock

Leng, Wu, TSP 2011

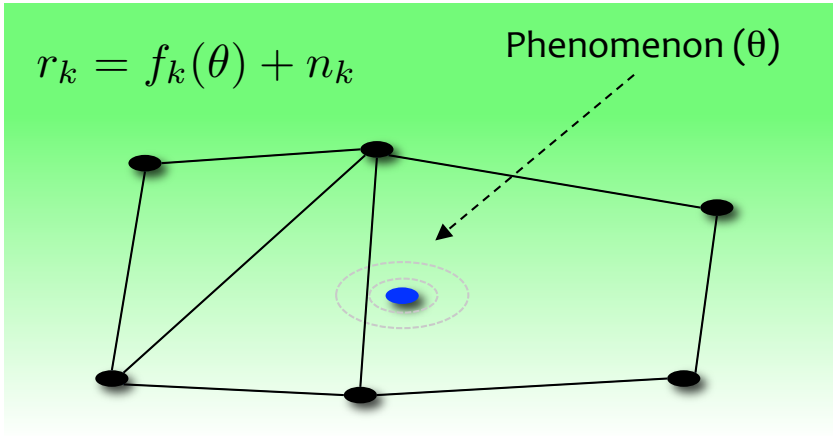
Application 4: distributed beamforming

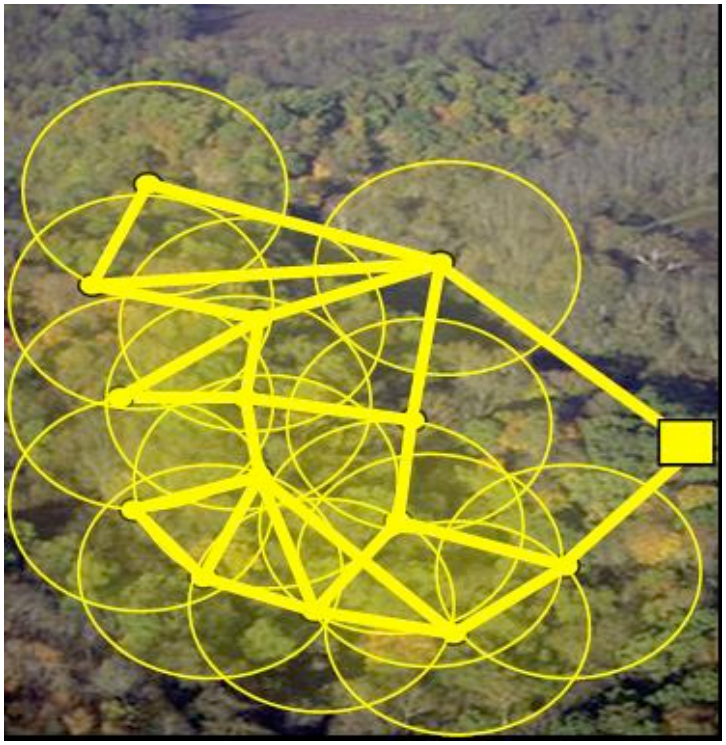


Goal: Assuming known CSI, design beamforming vectors $\mathbf{w}_j$ for every BS to maximize average throughput.

Sohn, Lee, Andrews, TCOM, 2011

Application 5: wireless sensor network





Goal:
Jointly compute estimate of θ

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  - Basics
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Background information

Bayesian inference

- Bayesian inference

  Unknown **x**, observation **y**, model θ

$$p(\mathbf{x},\mathbf{y}|\theta) = p(\mathbf{x}|\mathbf{y},\theta)p(\mathbf{y}|\theta)$$
$$= p(\mathbf{y}|\mathbf{x},\theta)p(\mathbf{x}|\theta)$$

$$p(x_k|\mathbf{y},\theta) = \sum_{\text{not } x_k} p(\mathbf{x}|\mathbf{y},\theta)$$

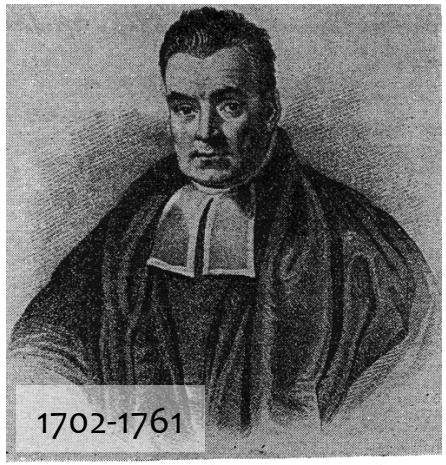
posterior of **x**

likelihood of θ

likelihood of **x** given θ

prior of **x**

marginal posterior of x_k



Rev. T. Bayes

Example: Nick is worried he is sick. He read in the newspaper 0.1% of the people is his city have contracted a deadly disease. He goes to the hospital to get tested. The doctor tells him the test is 95% reliable (i.e., it will give a correct result 95% of the time). The test is applied to Nick and turns out to be positive. With what probability does Nick have the disease?

The deadly disease

- We need to compute *p(Nick is sick|test is positive)*
- Applying Bayes' rule

$$p(S|P) = \frac{p(P|S) \times p(S)}{p(P)} \qquad p(H|P) = \frac{p(P|H) \times p(H)}{p(P)}$$

$$= \frac{0.95 \times 0.001}{p(P)} \qquad = \frac{0.05 \times 0.999}{p(P)}$$

$$= \frac{0.00095}{p(P)} \qquad = \frac{0.04995}{p(P)}$$

- So that

$$p(H|P) + p(S|P) = 1$$

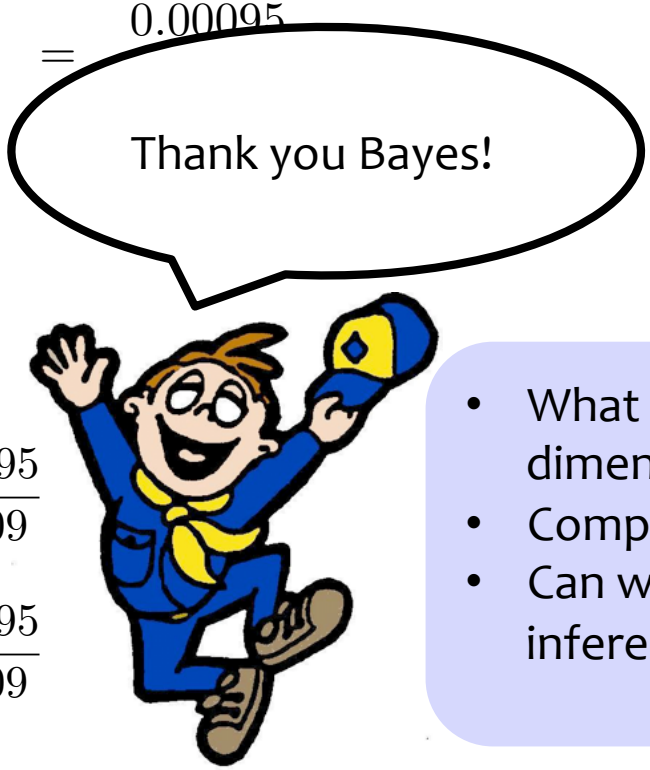
$$\frac{0.04995 + 0.00095}{p(P)} = 1$$

$$0.0509 = p(P)$$

$$\Rightarrow$$

$$P(S|P) = \frac{0.00095}{0.0509} = 0.02$$

$$P(H|P) = \frac{0.04995}{0.0509} = 0.98$$



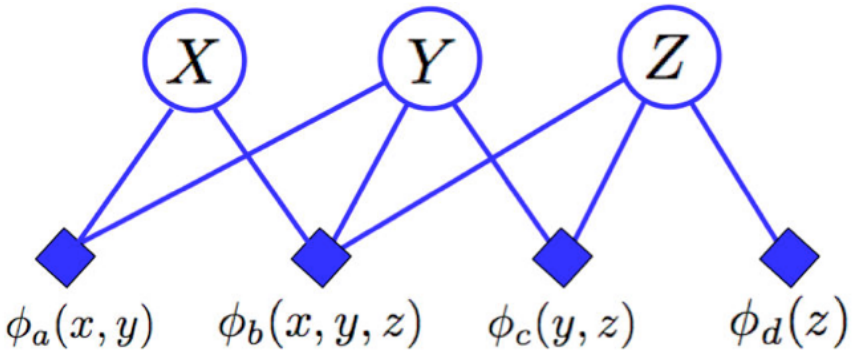
- What about high-dimensional problems?
- Complexity?
- Can we automate inference?

Graphical models

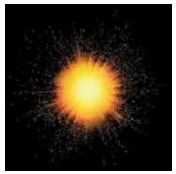
- Allow
  - Bayesian inference in rigorous, efficient, yet almost automated way
  - Modularity of problems and solutions
  - Graphical view of algorithms

- Applications
  - Every field that involves estimation/detecting something
  - Examples: communications, computer vision, bioinformatics

- New view of
  - Viterbi algorithm
  - BCJR algorithm
  - Turbo decoder
  - LDPC decoder
  - Kalman filter
  - Particle filter
  - ….



Graphical models: a biased history




Forney


Pearl


MacKay


Kschischang
Frey, Loeliger

1948 1963 1973 1981 1988 1993 1995 1996 2001 2005


Shannon


Gallager


Tanner


Berrou & Glavieux


Wiberg


Yedidia
Freeman
Weiss

Capacity
LDPC
trellis
Tanner graph
BP
turbo
LDPC
FG and SPA
decoding on graphs
GBP

Background material

- Papers
  - "Factor graphs and the sum-product algorithm," Kschischang, Frey, Loeliger, *IEEE Transaction in Information Theory*, 2001
  - "Constructing free energy approximations and generalized belief propagation algorithms," Yedidia, Freeman, Weiss, *IEEE Transactions on Information Theory*, 2005
  - Martin J. Wainwright and Michael I. Jordan (2008) "Graphical Models, Exponential Families, and Variational Inference", *Foundations and Trends in Machine Learning*: Vol. 1: No 1–2, pp 1-305.
- Books
  - *Probabilistic Graphical Models: Principles and Techniques*, Koller and Friedman, MIT press, 2009
  - *Graphical models for machine learning and digital communication*, Frey, MIT press, 1998
  - *Information theory, inference and learning algorithms*, MacKay, Cambridge University Press, 2003
  - *Wireless communication systems: advanced techniques for signal reception*, Wang, Poor, Prentice Hall, 2003
  - *Iterative Receiver Design*, Wymeersch, Cambridge University Press, 2007

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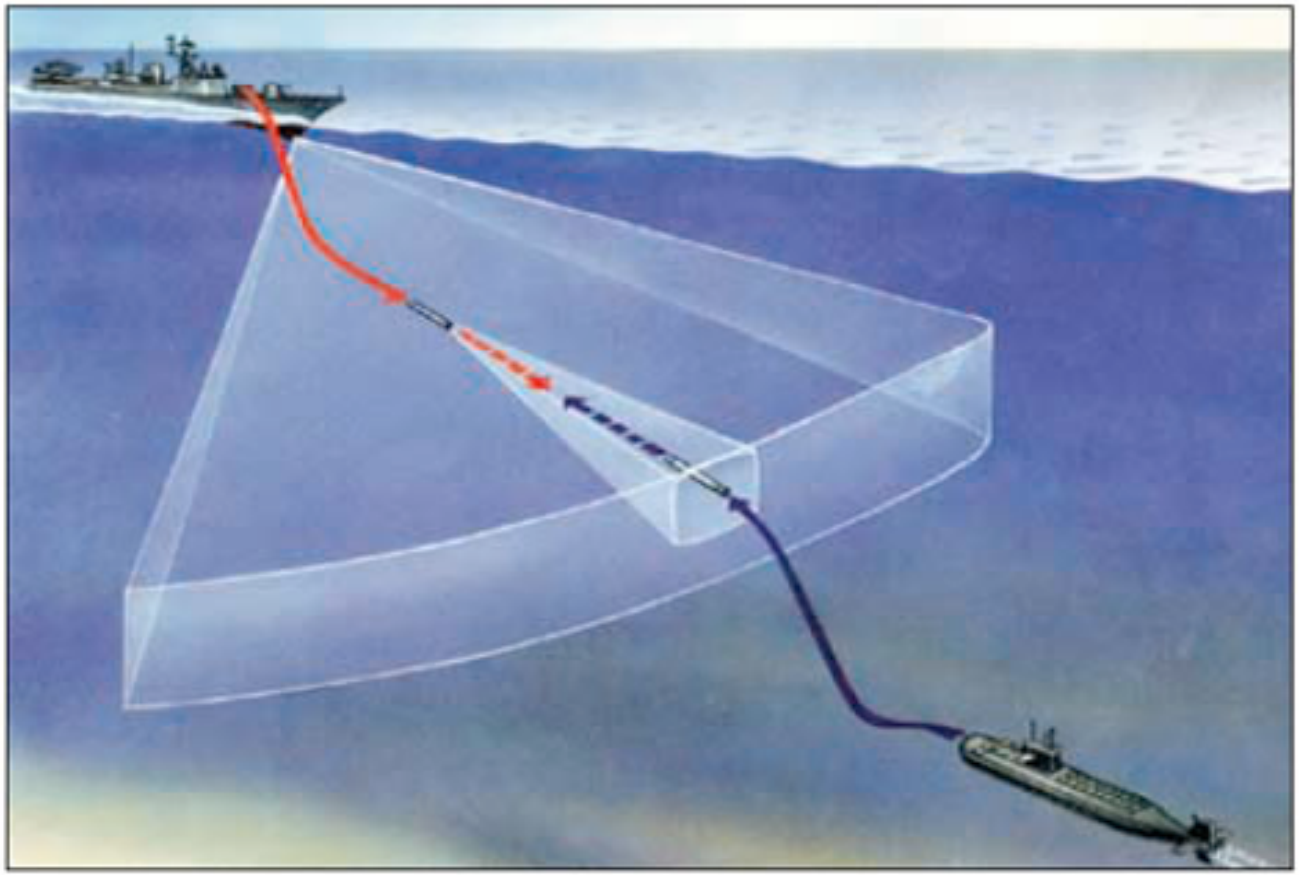
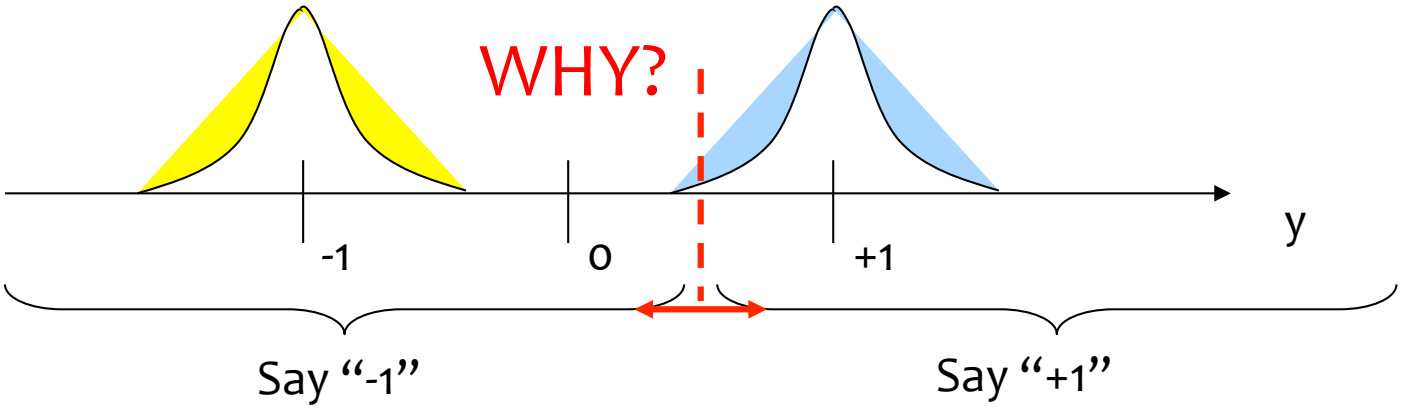
Detection Theory



Optimal detection

- Parameter to be estimated $\mathbf{b} \in \mathcal{B}$
  - known prior $p_B(\mathbf{b})$
- Observation $\mathbf{y} \in \mathcal{Y}$
  - known $p_{Y|B}(\mathbf{y}|\mathbf{b})$
- Estimator/detector $\mathbf{b}^*(\mathbf{y})$: $\mathcal{Y} \rightarrow \mathcal{B}$
  - For every possible **y**, associate an estimate
  - Designed to minimize a cost
- Example
  - $b \in \{-1,+1\}$, $p_B(+1)=0.1$, $p_B(-1)=0.9$
  - $y=b+n$, $n \sim N_n(0,1) \Rightarrow p_{Y|B}(y|b) = N_y(b,1)$

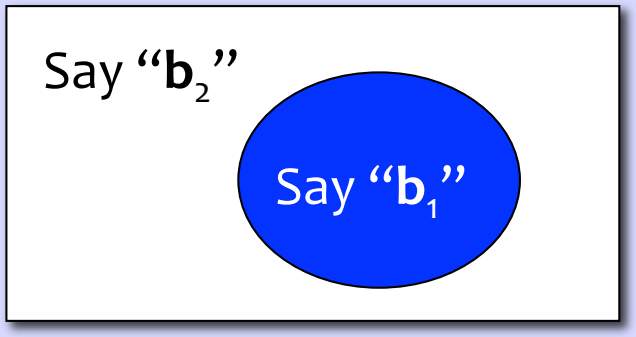


Optimal detection

- For every possible **y**, find best $\mathbf{b}^*(\mathbf{y})$
- What is "best"?
  - Correct: $\mathbf{b}^*(\mathbf{y})=\mathbf{b}$ has cost $C_1 \geq 0$
  - Wrong: $\mathbf{b}^*(\mathbf{y})=\mathbf{b}$ has cost $C_2 > C_1$
  - General: cost function $c(\boldsymbol{b},\boldsymbol{b}^*(\boldsymbol{y}))$: $\mathcal{B}\times\mathcal{B}\rightarrow R$
  - Minimize expected cost (why not max cost?)

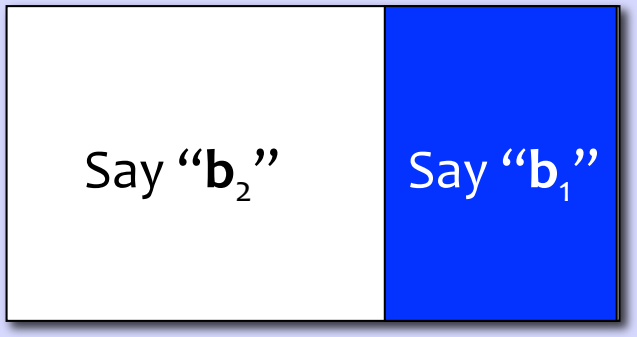
For a given cost function $c(\boldsymbol{b},\boldsymbol{b}^*(\boldsymbol{y}))$

$\mathcal{Y}$, set of possible **y**



⇒ Expected cost C

$\mathcal{Y}$, set of possible **y**



⇒ Expected cost C'

Optimal detection

- Expected cost for given detector **b*(y)**

$$\begin{aligned}\mathcal{C} &= \mathbb{E}_{\mathbf{B},\mathbf{Y}}\left\{c(\mathbf{B},\mathbf{b}^*(\mathbf{Y}))\right\} \\ &= \sum_{\mathbf{b},\mathbf{y}} p_{\mathbf{B},\mathbf{Y}}(\mathbf{b},\mathbf{y})c(\mathbf{b},\mathbf{b}^*(\mathbf{y})) \\ &= \sum_{\mathbf{y}} p_{\mathbf{Y}}(\mathbf{y}) \underbrace{\left(\sum_{\mathbf{b}} p_{\mathbf{B}|\mathbf{Y}}(\mathbf{b}|\mathbf{y})c(\mathbf{b},\mathbf{b}^*(\mathbf{y}))\right)}_{\mathcal{C}(\mathbf{y})}\end{aligned}$$

<Bayes' Rule: p(a,b)=p(a)p(b|a)>

Minimize this w.r.t. **b*(y)**

MAP detection

- Special case: cost 1 when incorrect decision, cost 0 otherwise

$$\begin{aligned} \mathcal{C}(\mathbf{y}) &= \sum_{\mathbf{b}} p_{\mathbf{B}|\mathbf{Y}}(\mathbf{b}|\mathbf{y}) c(\mathbf{b}, \mathbf{b}^*(\mathbf{y})) \\ &= \sum_{\mathbf{b} \neq \mathbf{b}^*(\mathbf{y})} p_{\mathbf{B}|\mathbf{Y}}(\mathbf{b}|\mathbf{y}) \\ &= 1 - p_{\mathbf{B}|\mathbf{Y}}(\mathbf{b}^*(\mathbf{y})|\mathbf{y}) \end{aligned}$$

- Leads to the maximum a posteriori (MAP) rule

$$\mathbf{b}^*(\mathbf{y}) = \arg\max_{\mathbf{b}} p_{\mathbf{B}|\mathbf{Y}}(\mathbf{b}|\mathbf{y})$$

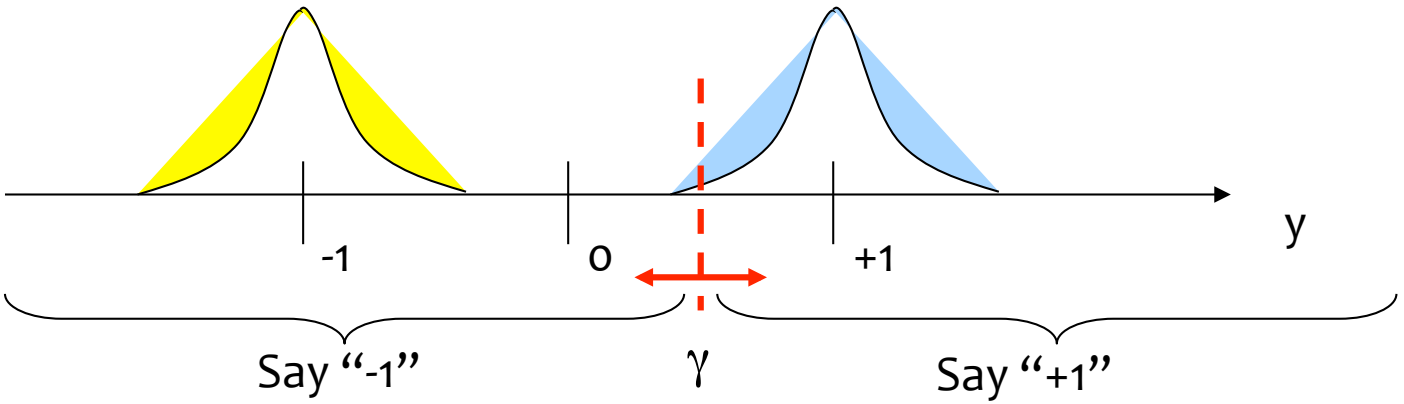
MAP detection - example

- $b \in \{-1,+1\}$, $p_B(1)=p_1$, $p_B(-1)=1-p_1$
- $y=b+n$, $n \sim N_n(0,1) \Rightarrow p_{Y|B}(y|b) = N_y(b,1)$

$$
\begin{aligned}
b^*(y) &= \arg\max_b p_{B|Y}(b|y) \\
&= \arg\max_b p_{Y|B}(y|b)p_B(b) \\
&= \arg\max_b \ln\left(p_{Y|B}(y|b)p_B(b)\right) \\
&= \arg\max_b -\frac{|y-b|^2}{2} + \ln p_B(b) \\
&= \arg\max_b yb + \ln p_B(b)
\end{aligned}
$$

Say "+1"
$\Leftrightarrow y+\ln(p_1) > -y+\ln(1-p_1)$
$\Leftrightarrow y > 0.5\ln((1-p_1)/p_1)$
$\Leftrightarrow y > \gamma$

Note:
$\gamma=0$ when $p_1=0.5$
$\gamma>0$ when $p_1<0.5$
$\gamma<0$ when $p_1>0.5$



MAP detector is optimal

- MAP detector is optimal in the following sense: *among all detectors, the MAP detector minimizes the probability of making an error*

- Two variations
  - Minimizing the error rate of the packet **b** (FER)

$$\mathbf{b}^*(\mathbf{y}) = \arg\max_{\mathbf{b}} p_{\mathbf{B}|\mathbf{Y}}(\mathbf{b}|\mathbf{y})$$

  - Minimizing the error rate of the individual bits (BER)

$$b_k^*(\mathbf{y}) = \arg\max_{b_k \in \{0,1\}} p_{B_k|\mathbf{Y}}(b_k|\mathbf{y}), \forall k$$

Why two MAP detectors?

Problem

- Maximizing‡ over all **b**: complexity exponential in length of **b**

$$\mathbf{b}^*(\mathbf{y}) = \arg\max_{\mathbf{b}} p_{\mathbf{B}|\mathbf{Y}}(\mathbf{b}|\mathbf{y})$$

Solution

- Minimize BER instead of FER

$$b_k^*(\mathbf{y}) = \arg\max_{b_k \in \{0,1\}} p_{B_k|\mathbf{Y}}(b_k|\mathbf{y}), \forall k$$

Problem

- It is hard to find $p(b_k \mid \mathbf{y})$

Solution

- How to find $p(b_k \mid \mathbf{y})$ is one of the topics of this tutorial
- $p(b_k \mid \mathbf{y})$ is marginal of $p(\mathbf{b} \mid \mathbf{y})$
- Factor graphs can help in computing marginals

‡Computing $p(\mathbf{b}|\mathbf{y})$ (up to constant) for given **y** and **b** is easy

How about continuous variables?

- Suppose $\mathbf{b} \in C^N$
- Terminology: "detection" → "estimation"
- Different cost functions lead to different estimators

- MAP estimator (for $\delta \to 0$)

$$c(\mathbf{b}, \mathbf{b}^*(\mathbf{y})) = \begin{cases} 0 & \|\mathbf{b} - \mathbf{b}^*(\mathbf{y})\| < \delta \\ 1 & \text{else} \end{cases}$$

$$\mathbf{b}^*(\mathbf{y}) = \arg\max_{\mathbf{b}} p_{\mathbf{B}|\mathbf{Y}}(\mathbf{b}|\mathbf{y})$$

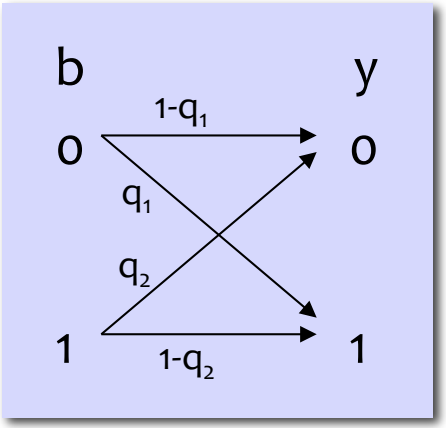
- MMSE estimator

$$c(\mathbf{b}, \mathbf{b}^*(\mathbf{y})) = \|\mathbf{b} - \mathbf{b}^*(\mathbf{y})\|^2$$

$$\mathbf{b}^*(\mathbf{y}) = \int \mathbf{b}\, p_{\mathbf{B}|\mathbf{Y}}(\mathbf{b}|\mathbf{y}) d\mathbf{b}$$

Two examples

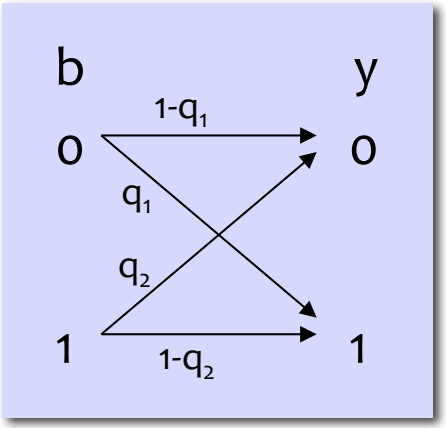
- Detection
  - Repetition code: **b**=[00], or **b**=[11], b a priori uniform
  - Binary asymmetrical channel
  - We receive **y**=[01]
  - Find MAP estimate of b



- Estimation
  - Coin with bias q, $p_Q(q) = U(1/2-\delta,1/2+\delta)$
  - Toss N times, observe **y** = sequence of heads and tails, with N_H heads
  - Find MAP estimate of q (prob of heads)

Example: detection

- Repetition code: **b**=[00], or **b**=[11], b a priori uniform
- Binary asymmetrical channel
- **y**=[01]
- Find MAP estimate of b



$$
\begin{aligned}
b^*([0,1]) &= \arg\max_b p_{B|\mathbf{Y}}(b|0,1) \\
&= \arg\max_b p_{\mathbf{Y}|B}(0,1|b) \\
&= \arg\max_b p_{\mathbf{Y}|B_1,B_2}(0,1|b,b) \\
&= \arg\max_b \underbrace{p_{Y_1|B_1}(0|b)p_{Y_2|B_2}(1|b)}_{\phi(b)}
\end{aligned}
$$

$\Rightarrow \phi(0)=(1-q_1)q_1$ and $\phi(1)=(1-q_2)q_2$

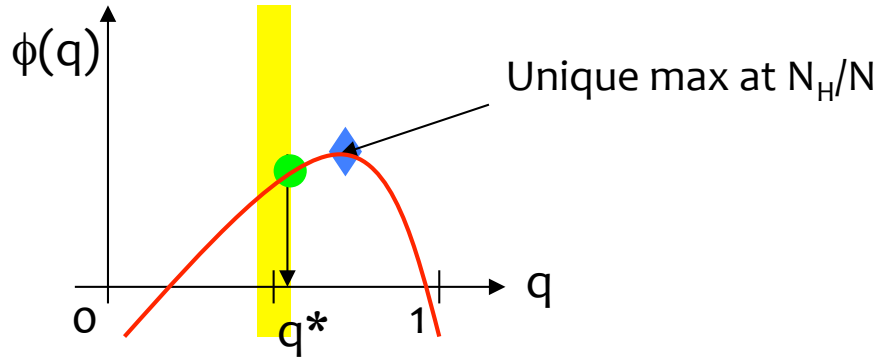
Example:

- Suppose $q_1=0.01$, $q_2=0.05$, then $\phi(0)=0.0099$ and $\phi(1)=0.0475$
- Decide: "1"

Example: estimation

- Coin with bias q, $p_Q(q) = U(1/2-\delta, 1/2+\delta)$
- Toss N times, observe **y** = sequence of heads and tails, with N_H heads
- Find MAP estimate of q (prob of heads)

$$
\begin{aligned}
q^*(\mathbf{y}) &= \arg\max_{q\in[0,1]} p_{Q|\mathbf{Y}}(q|\mathbf{y}) \\
&= \arg\max_{q\in[0,1]} p_{\mathbf{Y}|Q}(\mathbf{y}|q)p_Q(q) \\
&= \arg\max_{q\in[1/2-\delta,1/2+\delta]} q^{N_\mathrm{H}}(1-q)^{(N-N_\mathrm{H})} \\
&= \arg\max_{q\in[1/2-\delta,1/2+\delta]} \underbrace{N_\mathrm{H}\ln(q) + (N-N_\mathrm{H})\ln(1-q)}_{\phi(q)}
\end{aligned}
$$



Outline

- Applications
- Background and terminology
- Bayesian detection
- Tool 1: Bayesian graphical models
  - Basics
  - Recipe for Bayesian inference
  - Practicalities
  - A worked example: a digital receiver
- Tool 2: Belief consensus
  - Basics
  - Convergence
- Applications revisited
- Variational interpretation
- Summary and conclusions

Factor graphs and message passing

Motivation

- Three important problems
  - MAP configuration
    $$\hat{\mathbf{x}} = \arg\max_{\mathbf{x}} p(\mathbf{x}|\mathbf{y})$$
  - Marginal posteriors
    $$p(x_i|\mathbf{y}) = \sum_{\sim x_i} p(\mathbf{x}|\mathbf{y})$$
  - Probability of y
    $$p(\mathbf{y})$$

- Complexity exponential in dimension of x
- Complexity can be reduced by exploiting conditional independence
- Bayesian graphical models is a tool to do this **systematically**

Example

- Consider the following distribution over binary variables

$$p(x_1, x_2, x_3, x_4, \mathbf{y}) = \prod_{i=1}^{4} p(x_i) \times \prod_{i=2}^{4} p(\mathbf{y}_i | x_1, x_i)$$

- Solve the three problems

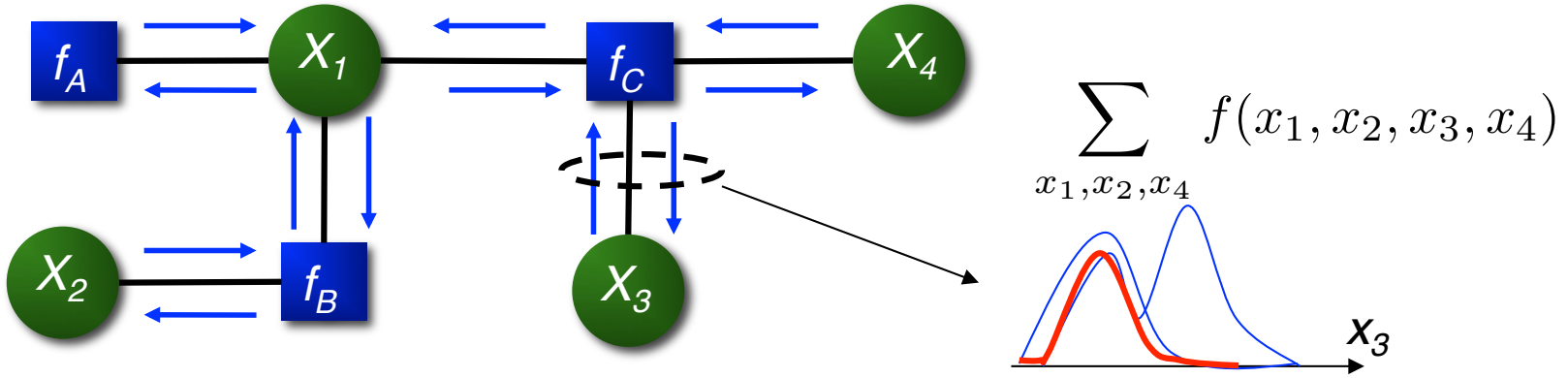
$$\hat{\mathbf{x}} = \arg\max_{\mathbf{x}} p(\mathbf{x}, \mathbf{y})$$

$$\hat{x}_i = \arg\max_{x_i} \underbrace{\max_{\sim x_i} p(\mathbf{x}, \mathbf{y})}_{g(x_i)}$$

$$p(x_i, \mathbf{y}) = \sum_{\sim x_i} p(\mathbf{x}, \mathbf{y})$$

Factor graphs: high level

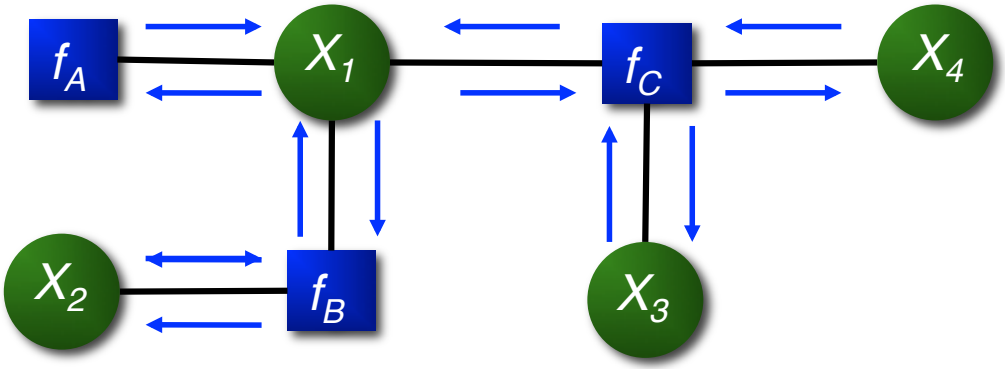
- Factorization $f(x_1, x_2, x_3, x_4) = f_A(x_1) f_B(x_1, x_2) f_C(x_1, x_3, x_4) \geq 0$

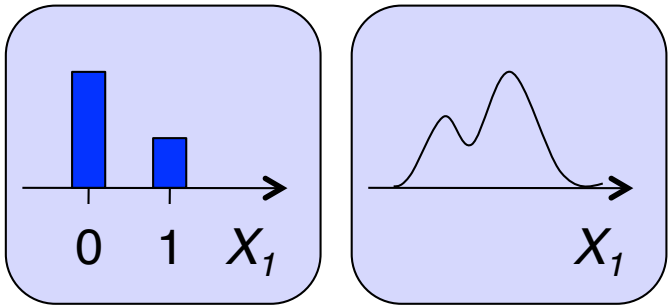


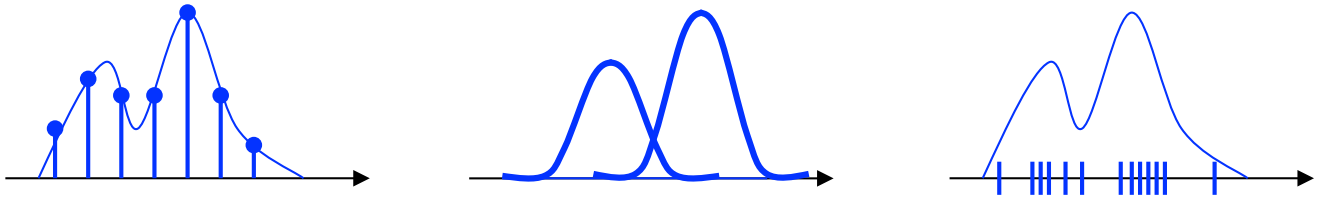
- The sum-product algorithm
  - Variable vertex to factor vertex
    $$\mu_{X_1 \to f_C}(x_1) = \mu_{f_A \to X_1}(x_1) \times \mu_{f_B \to X_1}(x_1)$$
  - Factor vertex to variable vertex
    $$\mu_{f_C \to X_1}(x_1) = \sum_{x_3, x_4} f_C(x_1, x_3, x_4) \times \mu_{X_3 \to f_C}(x_3) \times \mu_{X_4 \to f_C}(x_4)$$
- Marginals
    $$g_{X_3}(x_3) = \mu_{f_C \to X_3}(x_3) \times \mu_{X_3 \to f_C}(x_3)$$
    $$= \sum_{x_1, x_2, x_4} f(x_1, x_2, x_3, x_4)$$

Factor graphs: high level

- Messages are functions



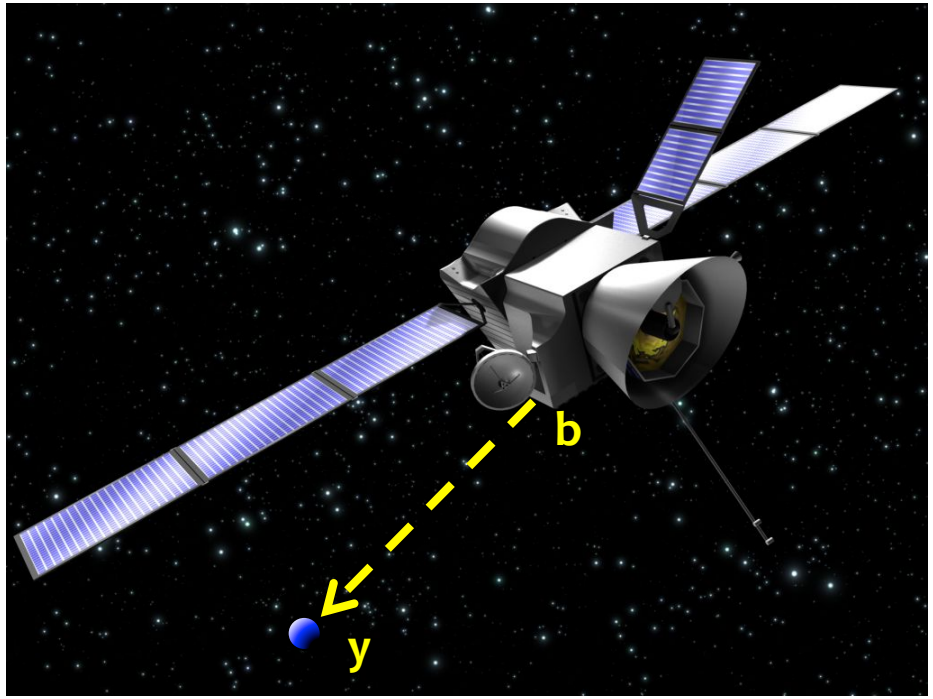


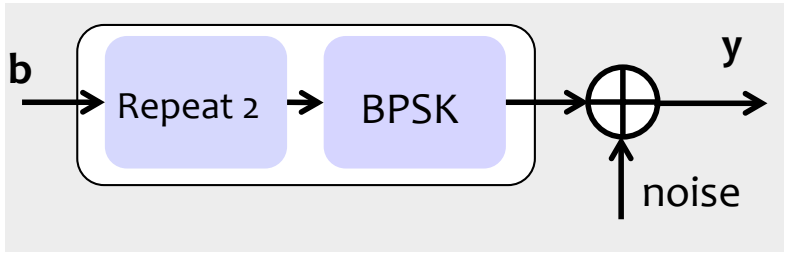


- Messages are often normalized and transformed (e.g., to LLR)
- When factor graph has cycles, SPA may fail
- Replacing "sum" with "max" yields max-marginals (MPA)

$$\sum_{x_1,x_2,x_4} f(x_1,x_2,x_3,x_4) \longrightarrow \max_{x_1,x_2,x_4} f(x_1,x_2,x_3,x_4)$$

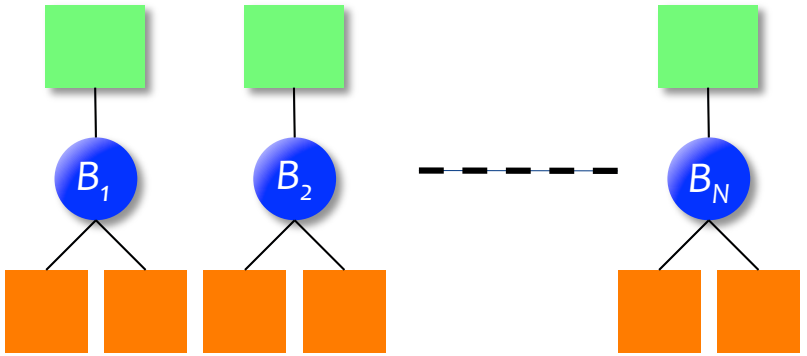
Factor graphs: high level





$$p(\mathbf{b}|\mathbf{y}) \propto p(\mathbf{b})p(\mathbf{y}|\mathbf{b})$$

$$= \prod_{k=1}^{N} p(b_k)p(y_{2k-1}|b_k)p(y_{2k}|b_k)$$



Optimal sequence detection:

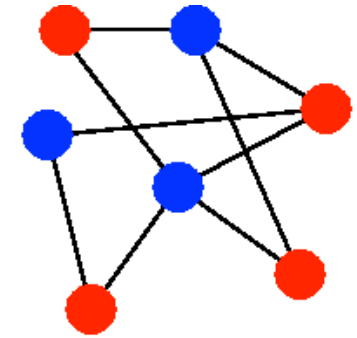
$$\hat{\mathbf{b}} = \arg\max_{\mathbf{b}\in\mathbb{B}^N} p(\mathbf{b}|\mathbf{y})$$

Optimal bit-by-bit detection:

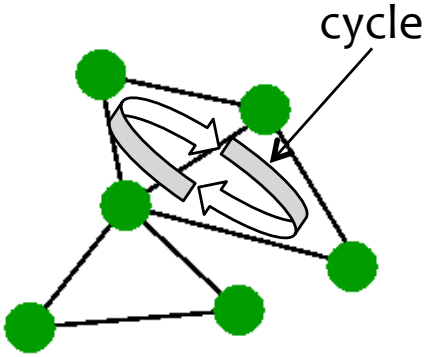
$$\hat{b}_k = \arg\max_{b_k\in\mathbb{B}} p(b_k|\mathbf{y})$$

What is a factor graph?

- Graphs
  - Graph G=(V,E)
  - Vertices, edges
  - Bipartite graph



A) A Bipartite Graph



B) A non–Bipartite Graph

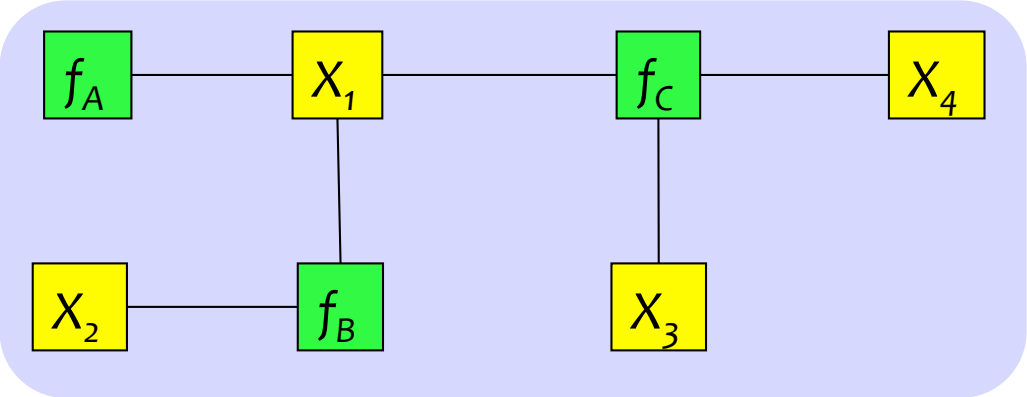
- Factor graph

A non-unique graph that represents the factorization of a real-valued function f(**x**)

Factor graph: construction

- Factorization $f(x_1, x_2, x_3, x_4) = f_A(x_1) f_B(x_1, x_2) f_C(x_1, x_3, x_4)$

- Factor graph

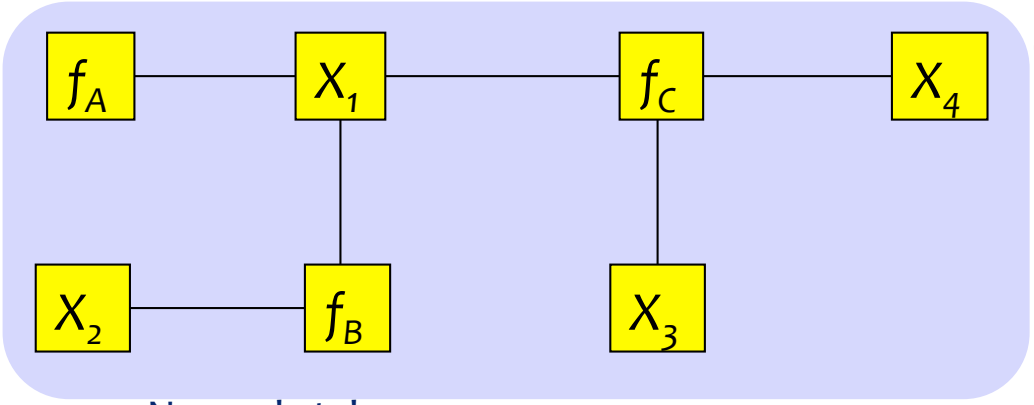


Construction

1. Vertex for every variable x_i (labeled with X_i)
2. Vertex for every factor f_k (labeled with f_k)
3. Edge between vertex x_i and vertex f_k when x_i appears in f_k

Conventional vs normal FG

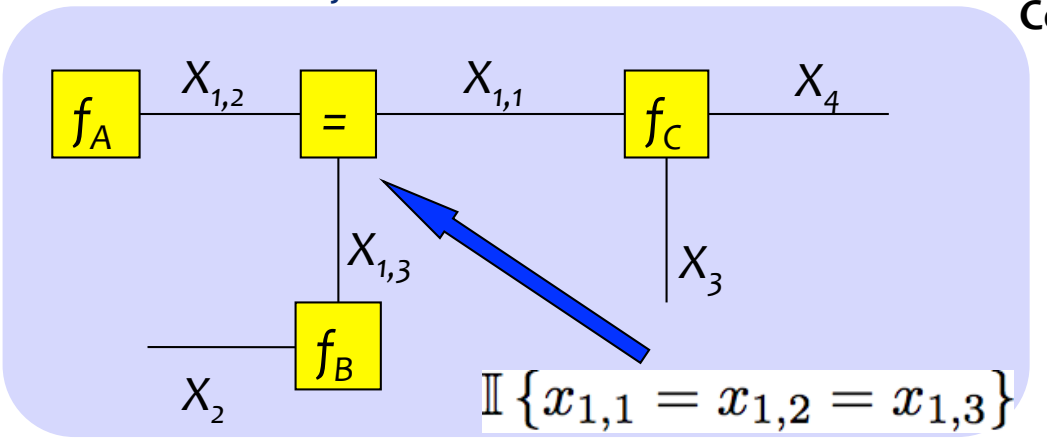
- Factorization $f(x_1, x_2, x_3, x_4) = f_A(x_1) f_B(x_1, x_2) f_C(x_1, x_3, x_4)$
- Conventional style



Construction

1. Vertex for every variable x_i (labeled with X_i)
2. Vertex for every factor f_k (labeled with f_k)
3. Edge between vertex x_i and vertex f_k when x_i appears in f_k

- Normal style



Construction

1. Vertex for every factor f_k (labeled with f_k)
2. An edge for every variable x_i connecting the factors in which it appears
3. x_i appears in 1 factor f_k: half-edge connected to vertex f_k, label X_i
4. x_i appears in 2 factors f_k,f_l: edge between f_k and f_l, label X_i
5. x_i appears in M>2 factors: a vertex (label "="), M edges (labeled $X_{i,1}$,…,$X_{i,M}$), connected to the M factors

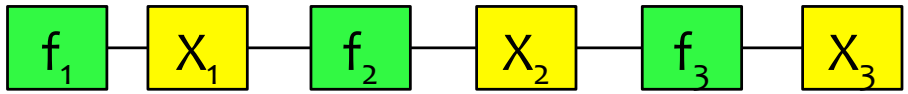
Indicator function: 1 when argument is true, 0 otherwise

Example

1. Draw the FG of the following distribution

$$p(x_1, x_2, x_3, x_4, \mathbf{y}) = \prod_{i=1}^{4} p(x_i) \times \prod_{i=2}^{4} p(\mathbf{y}_i | x_1, x_i)$$

2. Write down *a* function corresponding to the following FG

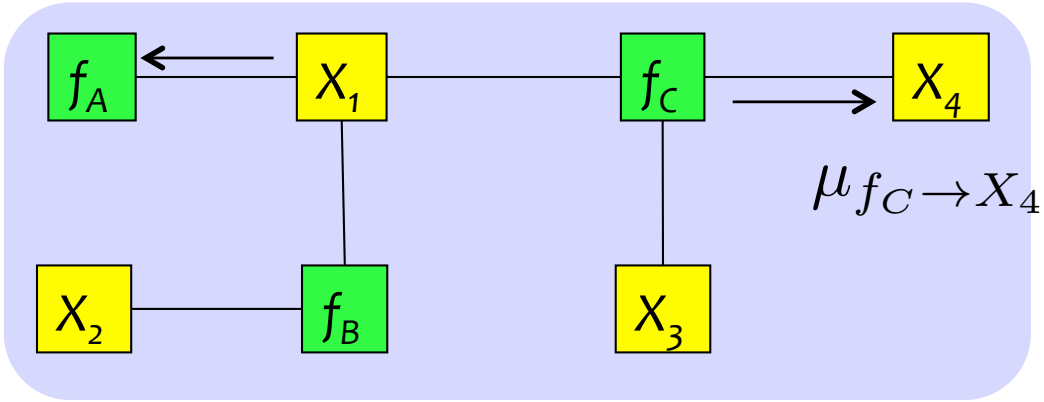


3. Can you come up with a reasonable distribution?

Message passing on factor graphs

- Messages are functions

$\mu_{X_1 \to f_A}(x_1)$



- If X_4 is binary and X_1 ternary

$$\mu_{X_1 \to f_A}(\cdot) = [\mu_{X_1 \to f_A}(-1) \quad \mu_{X_1 \to f_A}(0) \quad \mu_{X_1 \to f_A}(+1)]$$

$$\mu_{f_C \to X_4}(\cdot) = [\mu_{f_C \to X_4}(0) \quad \mu_{f_C \to X_4}(1)]$$

Sum-product and max-product algorithms

- There are many ways to compute messages
  - MAP configuration: max-product algorithm (MPA)
  - Marginal distributions: sum-product algorithm (SPA)
  - Normalization constant: sum-product algorithm

- Sum-marginal of a function f(**x**)

$$g_{X_k}^{\text{sum}}(x_k) = \sum_{\sim\{x_k\}} f(\mathbf{x})$$

- Max-marginal of a function f(**x**)

$$g_{X_k}^{\max}(x_k) = \max_{\sim\{x_k\}} f(\mathbf{x})$$

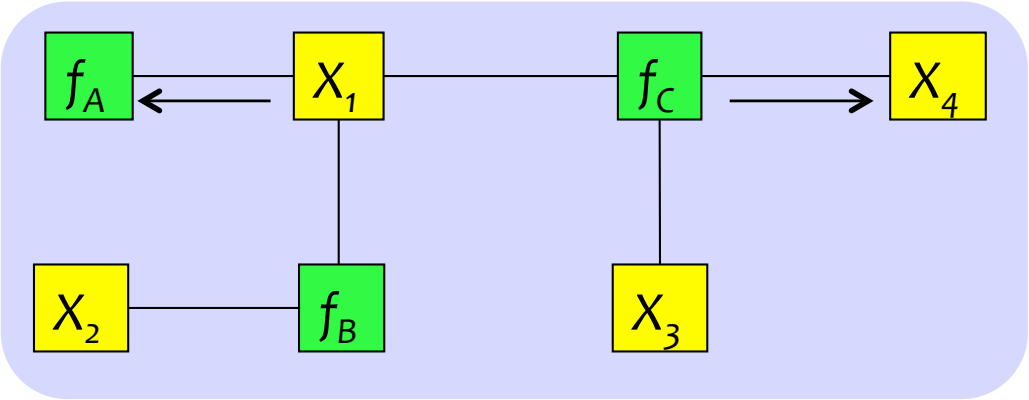
When $f(\mathbf{x}) = p(\mathbf{x}, \mathbf{y})$
Then

$$g_{X_4}^{\text{sum}}(x_4) = p(x_4, \mathbf{y})$$

and

$$g_{X_k}^{\max}(x_k) = \max_{\sim\{x_k\}} p(\mathbf{x}, \mathbf{y})$$

The sum-product algorithm (=belief propagation)



1. Message from variable to factor vertex

$$\mu_{X_1 \to f_A}(x_1) = \mu_{f_B \to X_1}(x_1) \times \mu_{f_C \to X_1}(x_1)$$

2. Message from factor to variable vertex

$$\mu_{f_C \to X_4}(x_4) = \sum_{x_1, x_3} f_C(x_1, x_3, x_4) \mu_{X_1 \to f_C}(x_1) \mu_{X_3 \to f_C}(x_3)$$

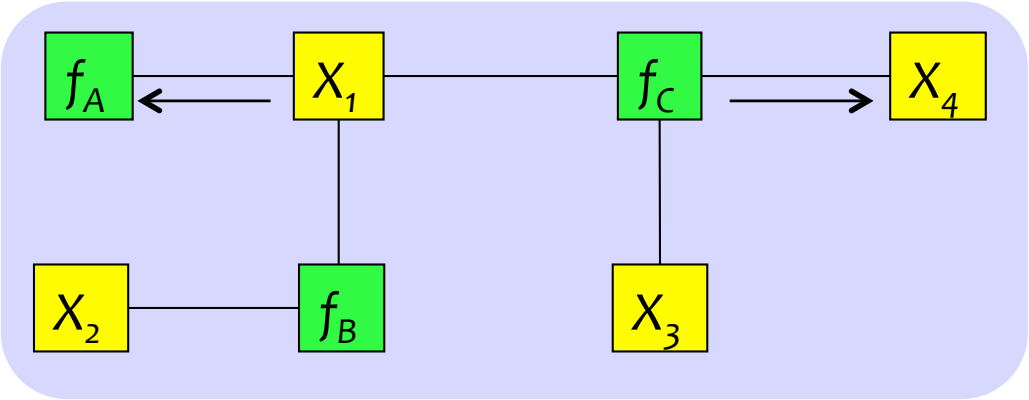
3. Sum-marginal

$$g_{X_1}^{\text{sum}}(x_1) = \mu_{f_A \to X_1}(x_1) \times \mu_{X_1 \to f_A}(x_1)$$

$$g_{X_4}^{\text{sum}}(x_4) = \mu_{f_C \to X_4}(x_4) \times \mu_{X_4 \to f_C}(x_4)$$

Initialization: Messages are initialized from leaves. Outgoing message only computed when incoming messages are available. Alternative: always transmit messages.

The max-product algorithm



1. Message from variable to factor vertex

$$\mu_{X_1 \to f_A}(x_1) = \mu_{f_B \to X_1}(x_1) \times \mu_{f_C \to X_1}(x_1)$$

2. Message from factor to variable vertex

$$\mu_{f_C \to X_4}(x_4) = \max_{x_1, x_3} f_C(x_1, x_3, x_4) \mu_{X_1 \to f_C}(x_1) \mu_{X_3 \to f_C}(x_3)$$

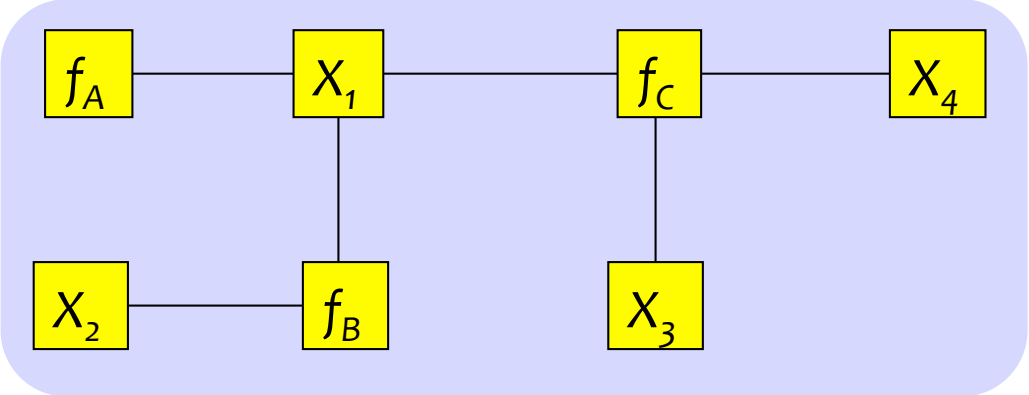
3. Max-marginal

$$g_{X_1}^{\max}(x_1) = \mu_{f_A \to X_1}(x_1) \times \mu_{X_1 \to f_A}(x_1)$$
$$g_{X_4}^{\max}(x_4) = \mu_{f_C \to X_4}(x_4) \times \mu_{X_4 \to f_C}(x_4)$$

Initialization: Messages are initialized from leaves. Outgoing message only computed when incoming messages are available. Alternative: always transmit messages.

The sum-product algorithm

- $f(x_1,x_2,x_3,x_4)= f_A(x_1)f_B(x_1,x_2,x_3)f_C(x_3,x_4)$
  - Assume all variables are defined over the set $\{0,1\}$
  - Functions can be represented as a vector of size 2: $[\mu(0),\mu(1)]$



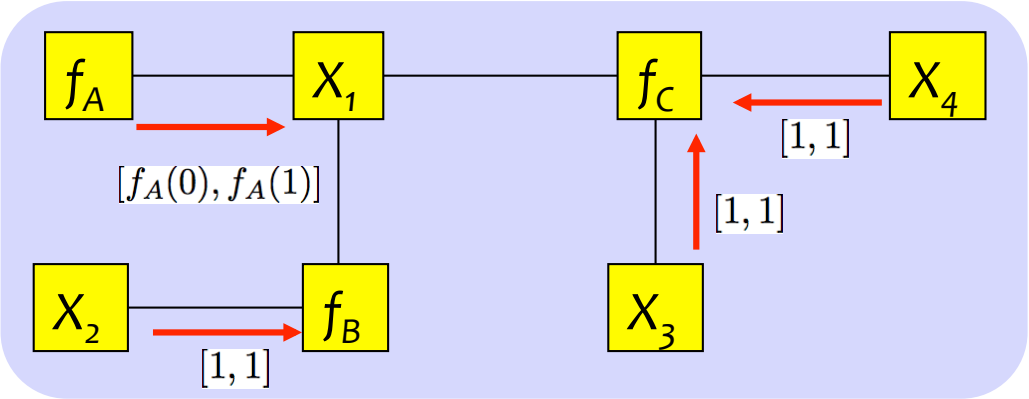
3 phases

1. Initialization
2. Message computation
3. Termination

$$\mu_{f_k \to X_i}(x_i) = \sum_{\sim\{x_i\}} \left(f_k(x_1, \ldots, x_D) \prod_{j \neq i} \mu_{X_j \to f_k}(x_j) \right)$$

Phase 1: initialization

- Messages start from the edge of the graph
  - Variable vertices of degree 1 send the message "1" over the domain: **m**=[1,1]
  - Factor vertices of degree 1 (say $f_k(x_i)$) send message: **m**=[$f_k(0)$,$f_k(1)$]



3 phases

1. Initialization
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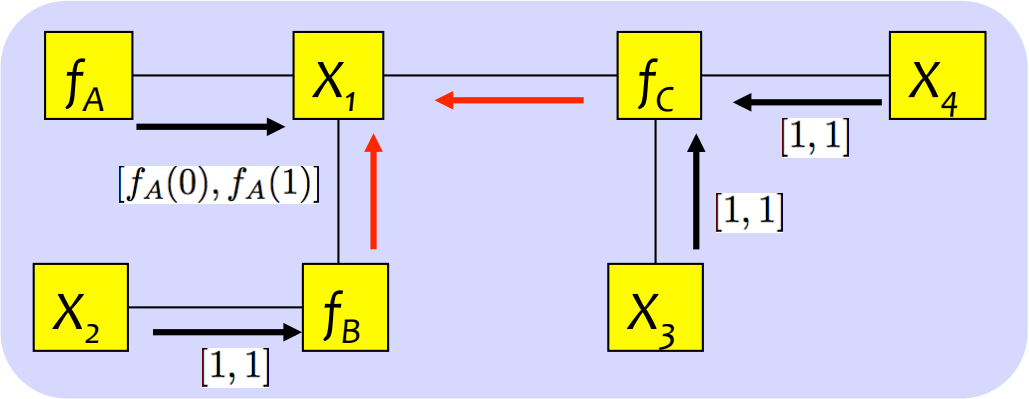
$$\mu_{f_k \to X_i}(x_i) = \sum_{\sim\{x_i\}} \left(f_k(x_1, \ldots, x_D) \prod_{j \neq i} \mu_{X_j \to f_k}(x_j) \right)$$

Phase 2: message computation

- Compute outgoing message when all other incoming messages are available



$$\mu_{f_B \to X_1}(x_1) = \sum_{x_2 \in \{0,1\}} f_B(x_1, x_2)\mu_{X_2 \to f_B}(x_2)$$

$$\mu_{f_C \to X_1}(x_1) = \sum_{x_3, x_4 \in \{0,1\}} f_C(x_1, x_3, x_4)\mu_{X_3 \to f_C}(x_3)\mu_{X_4 \to f_C}(x_4)$$

$$= \sum_{x_3, x_4 \in \{0,1\}} f_C(x_1, x_3, x_4)$$

---

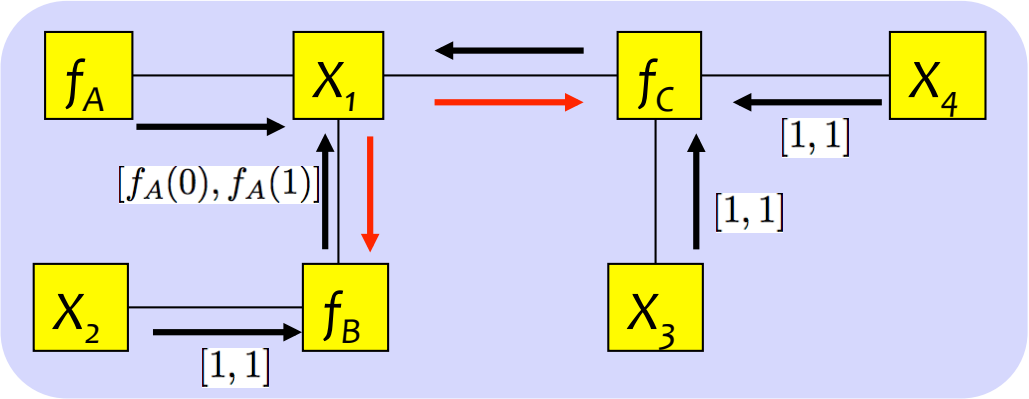
3 phases

1. Initialization
2. Message computation
3. Termination

$$\mu_{f_k \to X_i}(x_i) = \sum_{\sim\{x_i\}} \left(f_k(x_1, \ldots, x_D) \prod_{j \neq i} \mu_{X_j \to f_k}(x_j) \right)$$

Phase 2: message computation

- Compute outgoing message when all other incoming messages are available



$$\mu_{X_1 \to f_C}(x_1) = \mu_{f_A \to X_1}(x_1)\mu_{f_B \to X_1}(x_1)$$

$$\mu_{X_1 \to f_B}(x_1) = \mu_{f_A \to X_1}(x_1)\mu_{f_C \to X_1}(x_1)$$

---

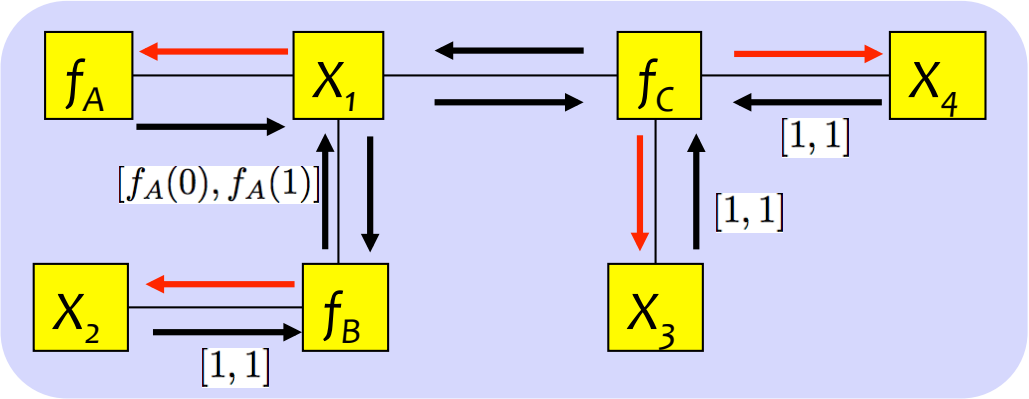
3 phases

1. Initialization
2. Message computation
3. Termination

$$\mu_{f_k \to X_i}(x_i) = \sum_{\sim\{x_i\}} \left(f_k(x_1, \ldots, x_D) \prod_{j \neq i} \mu_{X_j \to f_k}(x_j) \right)$$

Phase 2: message computation

- Compute outgoing message when all other incoming messages are available



$$\mu_{f_C \to X_3}(x_3) = \sum_{x_1, x_4 \in \{0,1\}} f_C(x_1, x_3, x_4) \mu_{X_1 \to f_C}(x_1) \mu_{X_4 \to f_C}(x_4)$$

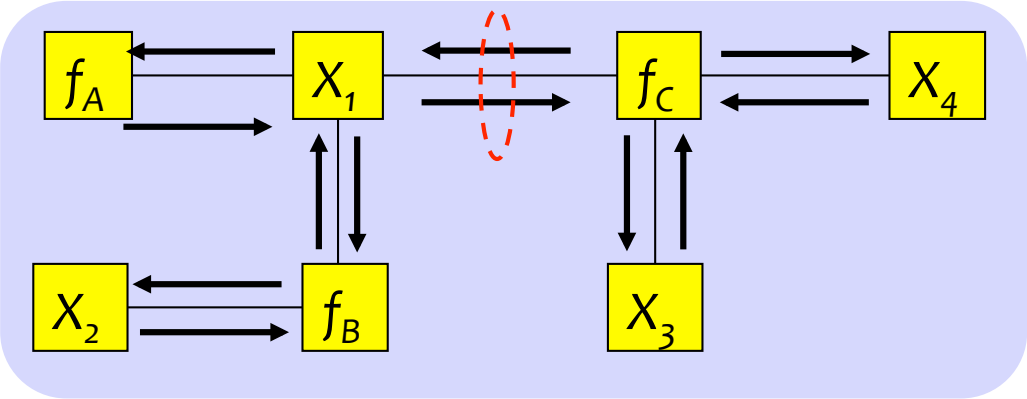
3 phases

1. Initialization
2. Message computation
3. Termination

$$\mu_{f_k \to X_i}(x_i) = \sum_{\sim\{x_i\}} \left(f_k(x_1, \ldots, x_D) \prod_{j \neq i} \mu_{X_j \to f_k}(x_j) \right)$$

Phase 3: termination

- Compute outgoing message when all other incoming messages are available



$$\mu_{X_1 \to f_C}(x_1)\mu_{f_C \to X_1}(x_1) = \sum_{x_2,x_3,x_4} f(x_1,x_2,x_3,x_4)$$
$$= \mu_{X_1 \to f_A}(x_1)\mu_{f_A \to X_1}(x_1)$$

---

3 phases

1. Initialization
2. Message computation
3. Termination

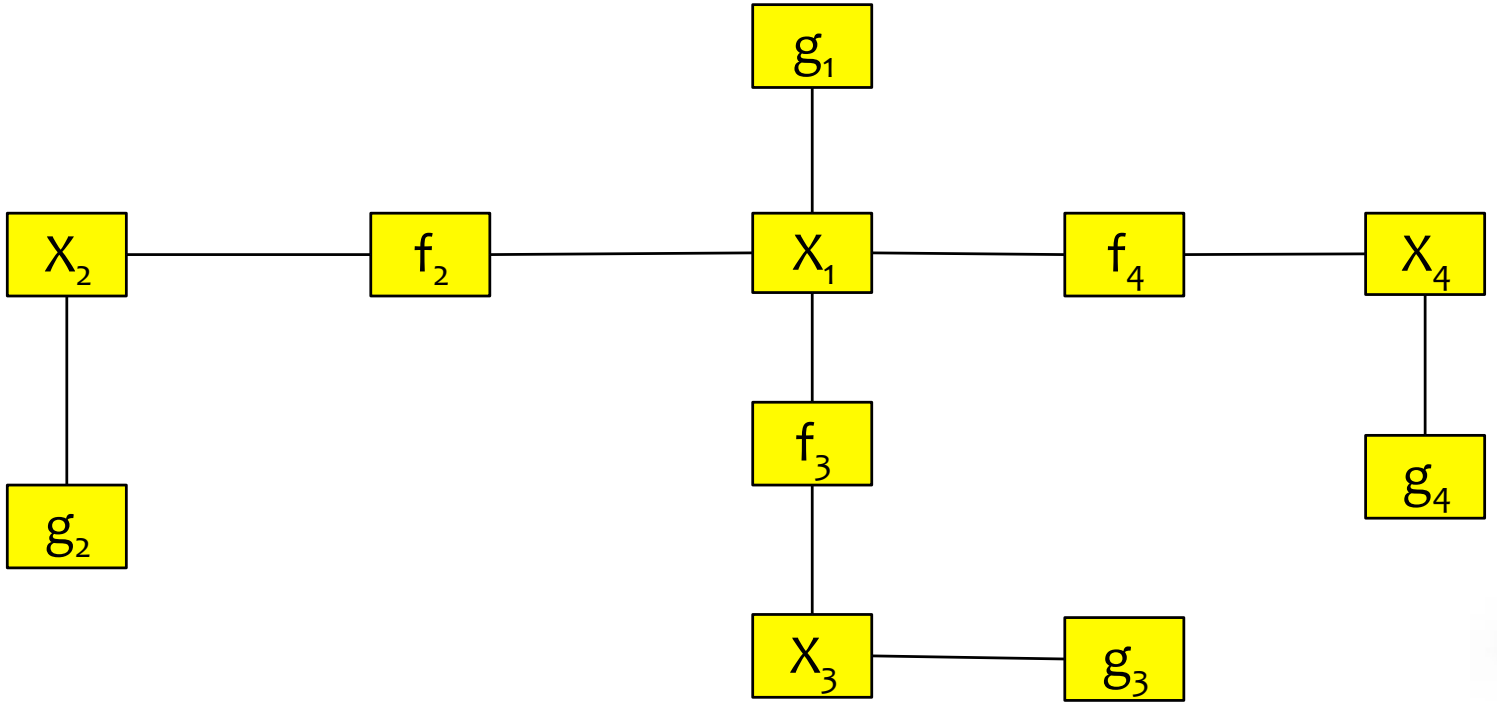
$$g_{X_i}(x_i) = \mu_{f_k \to X_i}(x_i) \times \mu_{X_i \to f_k}(x_i)$$

Example

- Consider $p(x_1, x_2, x_3, x_4, \mathbf{y}) = \prod_{i=1}^{4} p(x_i) \times \prod_{i=2}^{4} p(\mathbf{y}_i | x_1, x_i)$

  $g_i(x_i)$ $f_i(x_1, x_i)$

- Write down the sum-product rules
- Verify the solution



Sum-product and max-product algorithm

- Sum-product algorithm
  - Aim: compute (sum-) marginals $g_{X_i}(x_i) = \sum_{\sim\{x_i\}} f(\mathbf{x})$
  - Method: distributive law for real numbers
    $a(b+c) = ab + ac$

- Max-product algorithm
  - Aim: compute max-marginals $g_{X_i}(x_i) = \max_{\sim\{x_i\}} f(\mathbf{x})$
  - Method: distributive law for non-negative real numbers
    $a\max(b,c) = \max(ab, ac); a, b, c \geq 0$

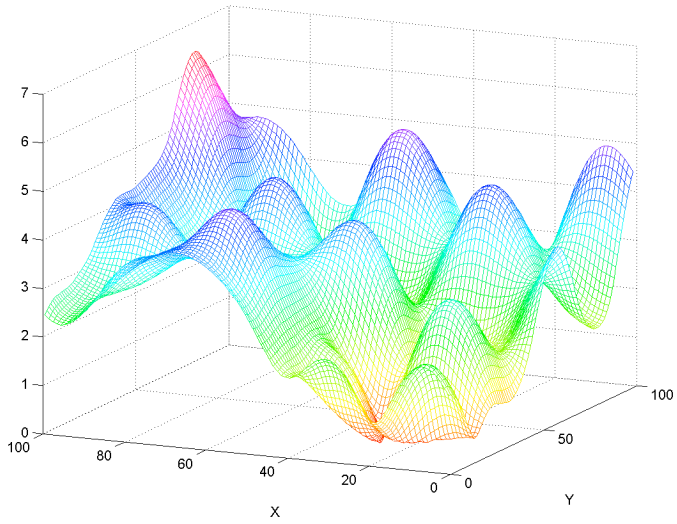
- Log-domain versions (see later)
  - Max*-sum
  - Max-sum

- Generalization: any suitable (F,⊕,⊗)

Sum-product and max-product algorithm

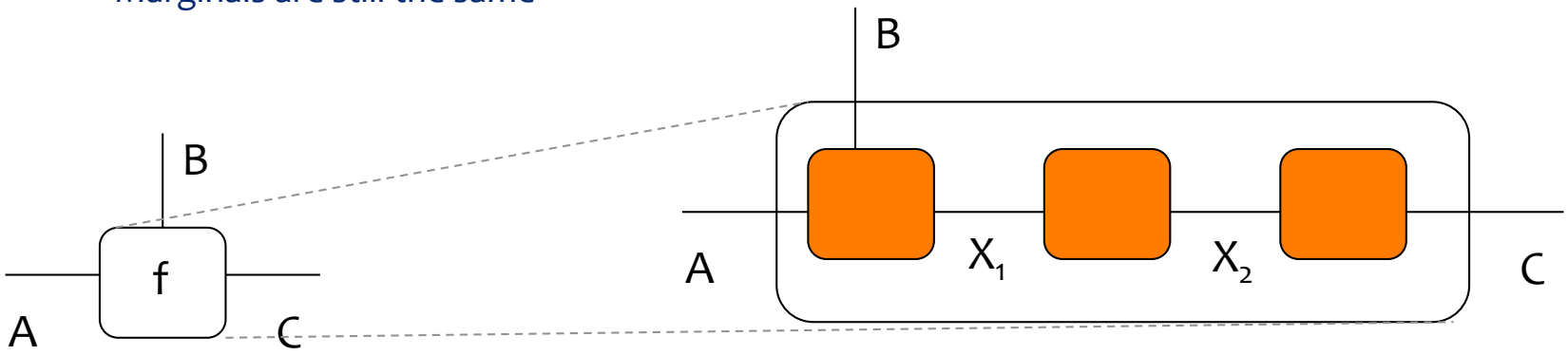
- Message passing rules
  - Replace all sums in SPA by max
  - Outcome is max-marginals

- Usage
  - assume f(**x**) has global maximum: f(**x***)
  - Concatenation of modes of max-marginals = **x***



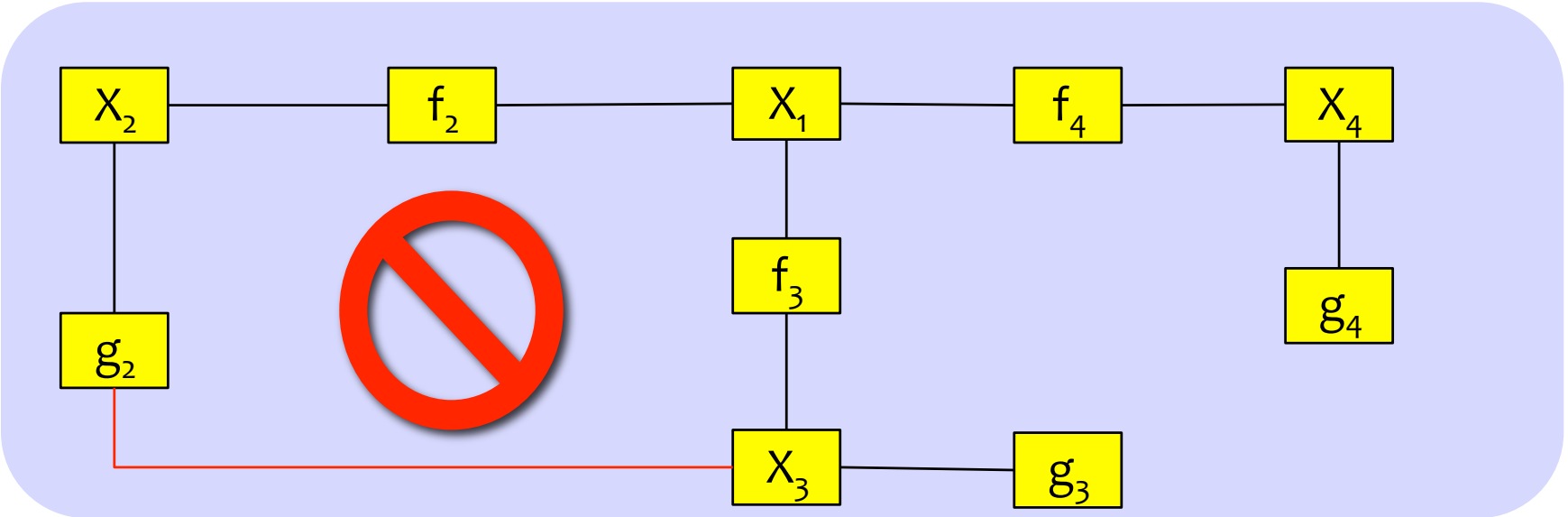
Aside: *opening* of vertices

- Given a vertex f(A,B,C) in a larger factor graph of a joint distribution p(A,B,C,… |**Y**=**y**)
- We can replace f(A,B,C) with a factorization g(A,B,C,X_1,X_2) where
  - X_1, X_2 do not appear elsewhere
  - Summing out X_1 and X_2 yields again f(A,B,C)
  - Internal factor graph is a tree
- Outcome of [S/M]PA is not affected:
  - Messages over A, B and C are still the same
  - Marginals are still the same



Factor graphs with cycles

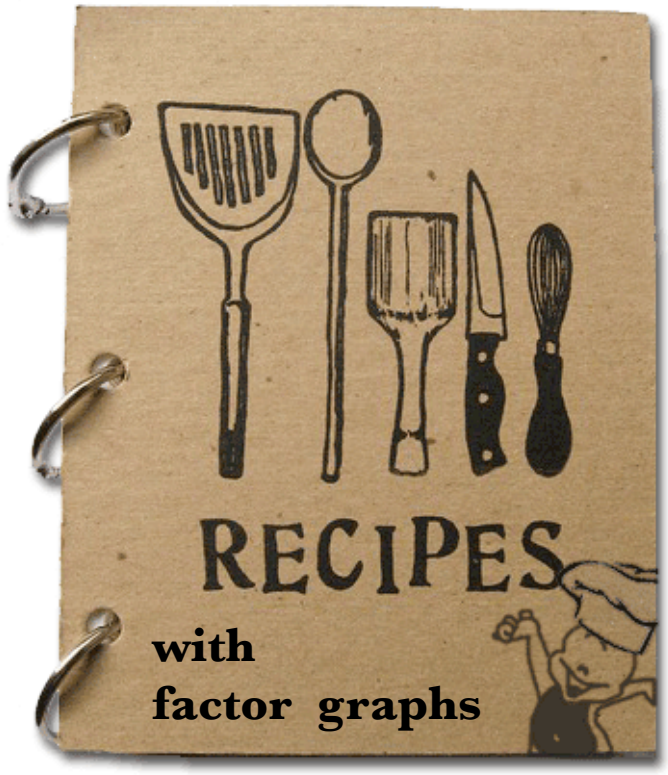
- Problem: When the factor graph has cycles: [S/M]PA gets stuck
- Solution: add artificial messages (e.g., set to "1")
- New Problem 1: XPA keeps running forever
- Solution: stop XPA after some time, and compute marginals
- New Problem 2: XPA becomes very unstable; messages tend to very small or very large values; marginals are generally completely incorrect
- Conclusion: don't use XPA for factor graphs with cycles!



Outline

- Applications
- Background and terminology
- Bayesian detection
- Tool 1: Bayesian graphical models
  - Basics
  - Recipe for Bayesian inference
  - Practicalities
  - A worked example: a digital receiver
- Tool 2: Belief consensus
  - Basics
  - Convergence
- Applications revisited
- Variational interpretation
- Summary and conclusions

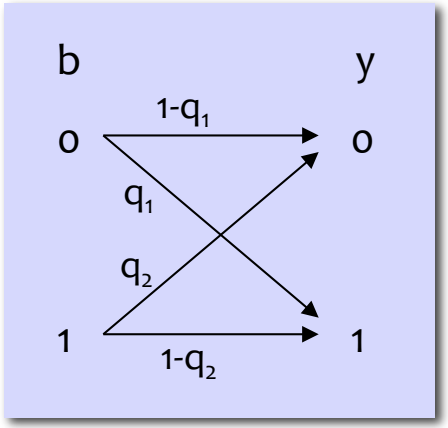
Inference using factor graphs



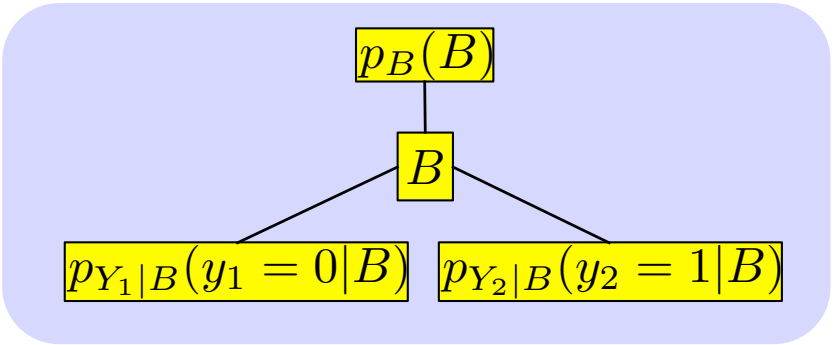
Inference using factor graphs

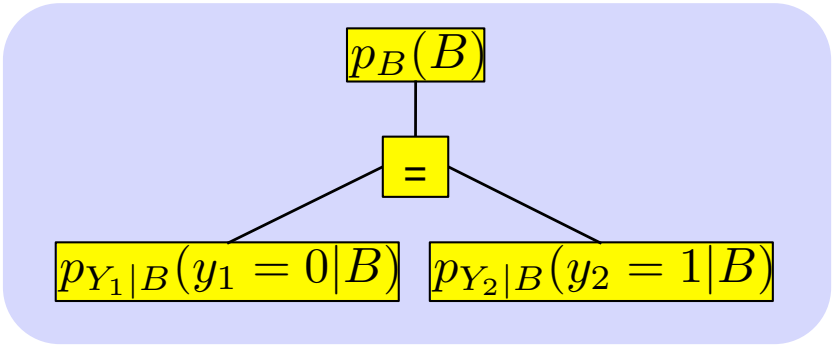
- Statistical inference
  - Parameter of interest **x**, a priori $p_X(\mathbf{x})$
  - Observation **y**, probabilistic mapping $p_{Y|X}(\mathbf{y}|\mathbf{x})$
  - Model *M*, encapsulating any additional assumptions
  - **x**,**y** can be discrete, continuous, hybrid
- 5 problems of interest
  1. Likelihood of the model $p(\mathbf{Y}=\mathbf{y}|\mathcal{M})$ √
  2. Marginal a posteriori distributions (MAPD) $p(X_k|\mathbf{Y}=\mathbf{y},\mathcal{M})$ √
  3. Certain characteristics of the MAPD (e.g., mean, mode) √
  4. A posteriori distribution (APD) $p(\mathbf{X}|\mathbf{Y}=\mathbf{y},\mathcal{M})$ X
  5. Certain characteristics of the APD (e.g., mean, mode) √

Running example



- Repetition code: **b**=[00], or **b**=[11], b a priori uniform
- Binary asymmetrical channel
- **y**=[01]
- Find
  - MAP estimate of b (mode of MAPD $p_{X|Y}(\mathbf{x}|\mathbf{y})$)
  - likelihood of receiving **y** ($p_Y(\mathbf{y})$, M=[q_1,q_2])
- Factorization $p_{B,\mathbf{Y}}(b,\mathbf{y}) = p_{\mathbf{Y}|B}(\mathbf{y}|b)p_B(b)$
  $= p_{Y_1|B}(y_1|b)p_{Y_2|B}(y_2|b)p_B(b)$
- Factor graphs





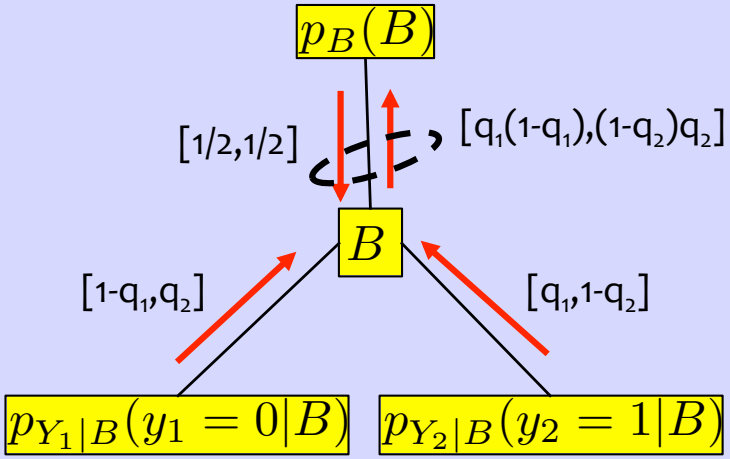
Problem 1 - likelihood of the model

Recipe

1. Factorize the joint distribution $p_{X,Y}(\mathbf{x},\mathbf{y}) = p_{Y|X}(\mathbf{y}|\mathbf{x})p_X(\mathbf{x})$ (add variables if necessary)
2. Create a factor graph of this factorization ($\mathbf{y}$ is given, it is a parameter, not a variable!)
3. Perform the SPA, this yields marginals $p_{Xk,Y}(x_k,\mathbf{y})$
4. ‡Take any k, sum over x_k, this gives $p_Y(\mathbf{y})$

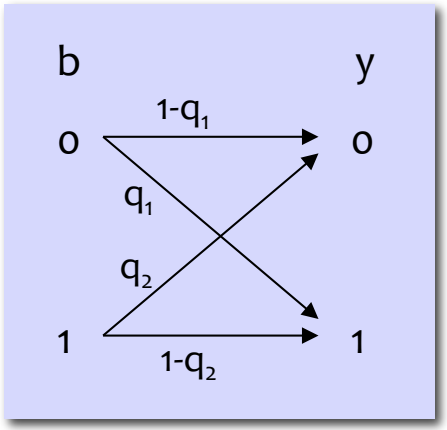
$p_{B,Y}(b=0, \mathbf{y}=[0,1]) = 0.5\, q_1(1-q_1)$
$p_{B,Y}(b=1, \mathbf{y}=[0,1]) = 0.5\, q_2(1-q_2)$

$p_Y(\mathbf{y}=[0,1]) = 0.5(q_1(1-q_1)+(1-q_2)q_2)$



Recall:

- $p(a|b) \propto p(a,b)$
- $p(a) = \Sigma_b p(a,b)$



‡ $\Sigma_a p_{A,B}(a,b) = p_B(b)$

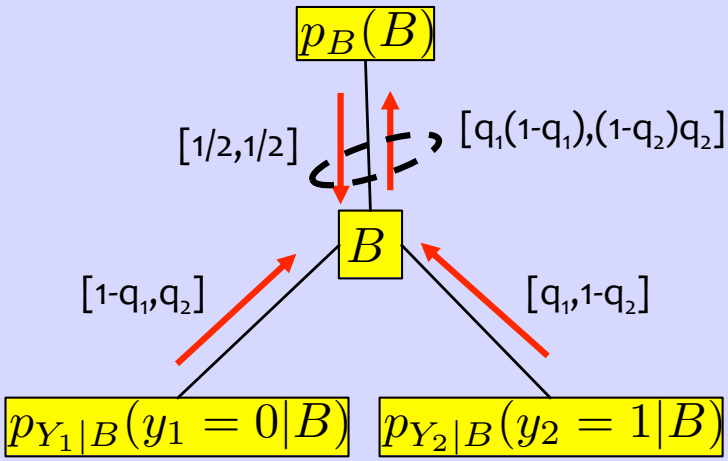
Problems 2,3 - marginal a posteriori distributions

Recipe

1. Factorize the joint distribution $p_{X,Y}(\mathbf{x},\mathbf{y}) = p_{Y|X}(\mathbf{y}|\mathbf{x})p_X(\mathbf{x})$ (add variables if necessary)
2. Create a factor graph of this factorization ($\mathbf{y}$ is given, it is a parameter, not a variable!)
3. Perform the SPA, this yields marginals $p_{X_k,Y}(x_k,\mathbf{y})$
4. For the MAPD of X_k, normalizing $p_{X_k,Y}(x_k,\mathbf{y})$ gives $p_{X_k|Y}(x_k|\mathbf{y})$

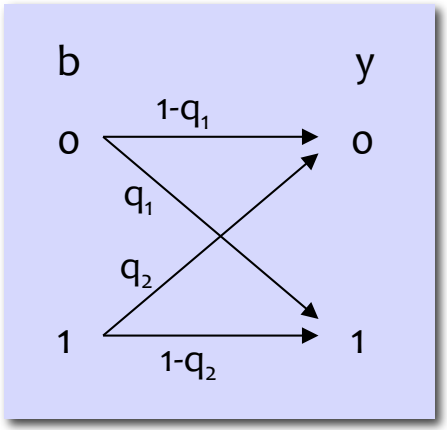
$p_{B,Y}(b=0, y=[0,1]) = 0.5\, q_1(1-q_1)$
$p_{B,Y}(b=1, y=[0,1]) = 0.5\, q_2(1-q_2)$

$p_{B|Y}(0|[0,1]) = q_1(1-q_1)/(q_1(1-q_1)+(1-q_2)q_2)$
$p_{B|Y}(1|[0,1]) = q_2(1-q_2)/(q_1(1-q_1)+(1-q_2)q_2)$



Recall:

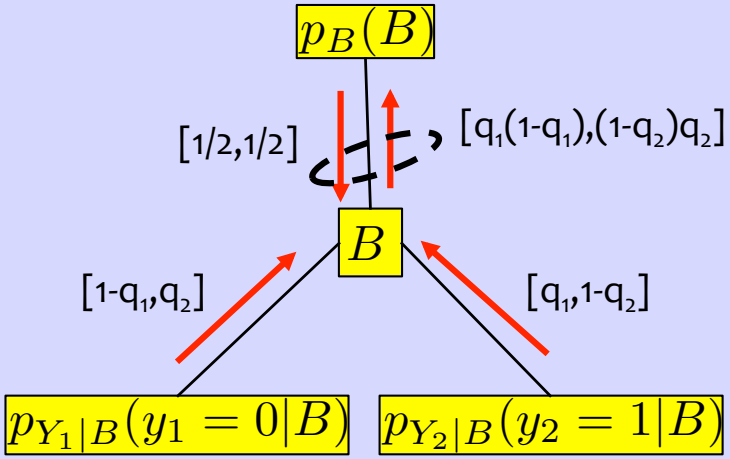
- $p(a|b) \propto p(a,b)$
- $p(a) = \Sigma_b p(a,b)$

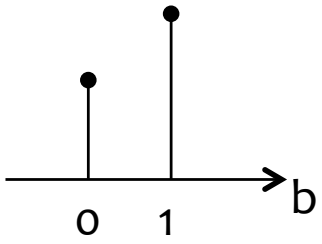


Problem 5 – mode of the a posteriori distribution

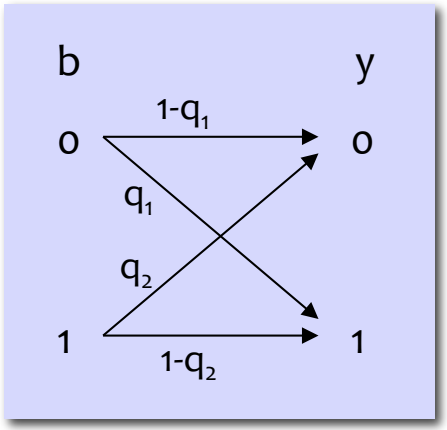
Recipe

1. Factorize the joint distribution $p_{X,Y}(\mathbf{x},\mathbf{y}) = p_{Y|X}(\mathbf{y}|\mathbf{x})p_X(\mathbf{x})$ (add variables if necessary)
2. Create a factor graph of this factorization (**y** is given, it is a parameter, not a variable!)
3. Perform the MPA, this yields max-marginals $q_k(x_k)$
4. The the mode of the max-marginals and concatenate

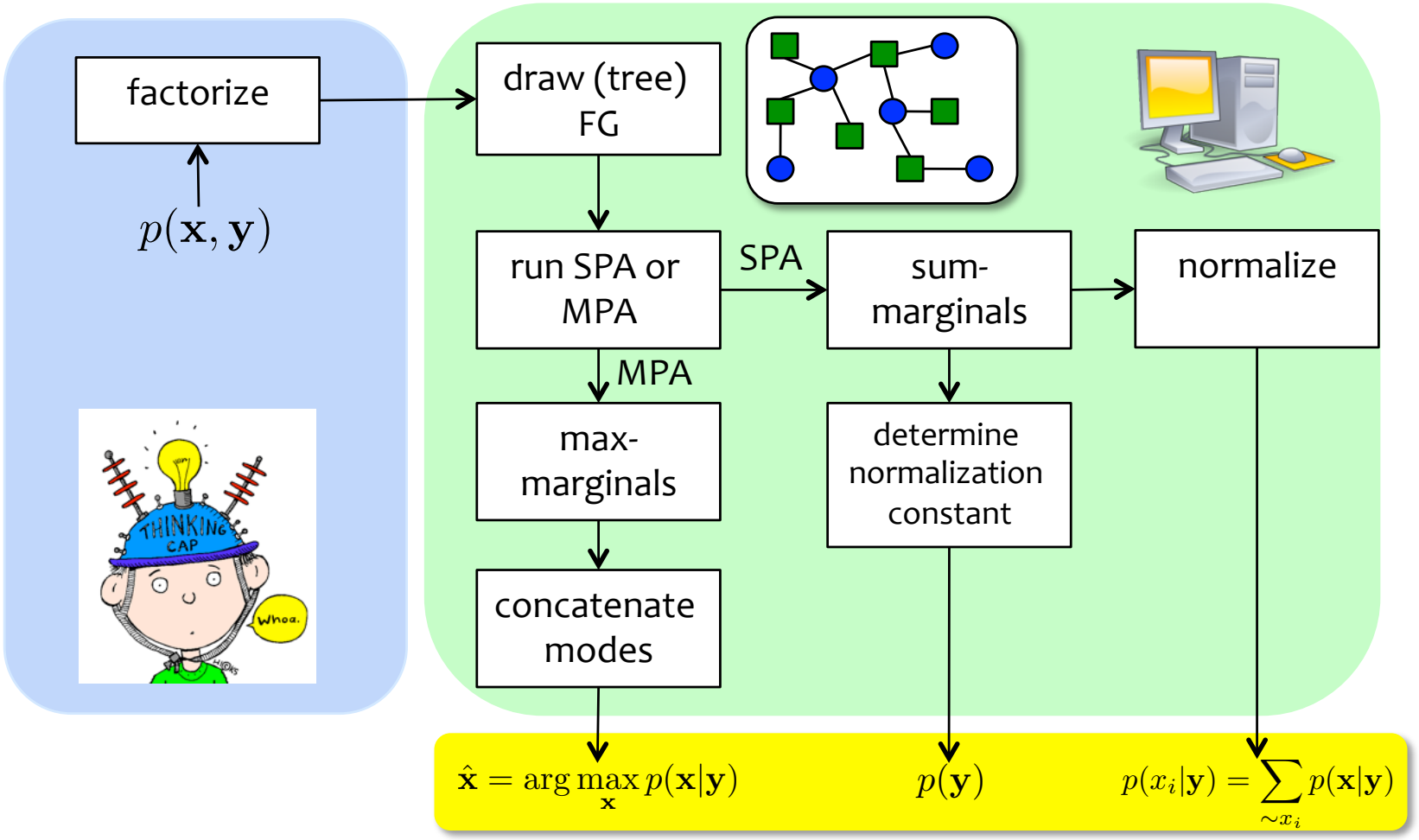




q(b=0) = 0.5 $q_1(1-q_1)$
q(b=1) = 0.5 $q_2(1-q_2)$



Inference recipe



Factor Graphs and Inference: Examples

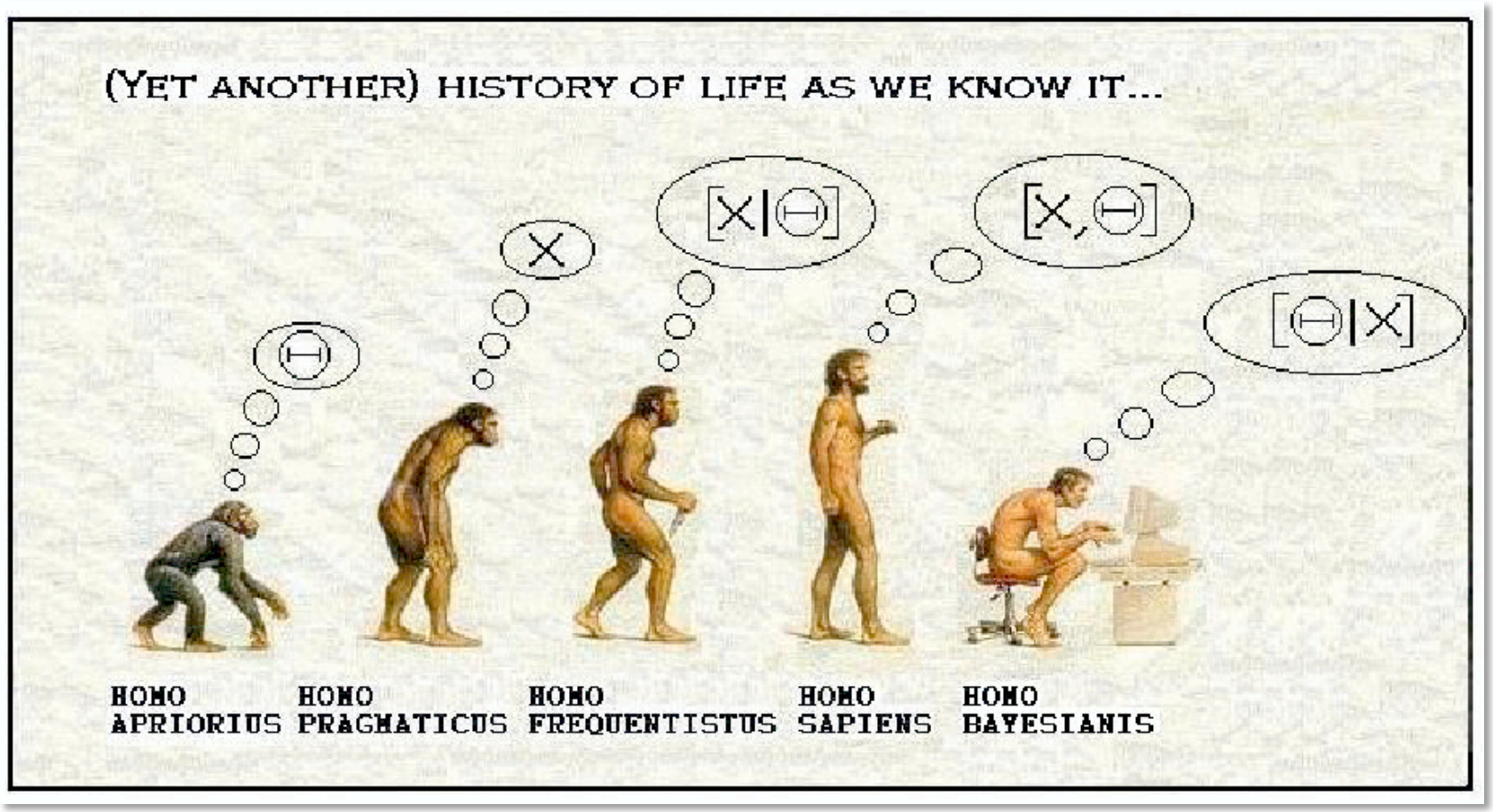


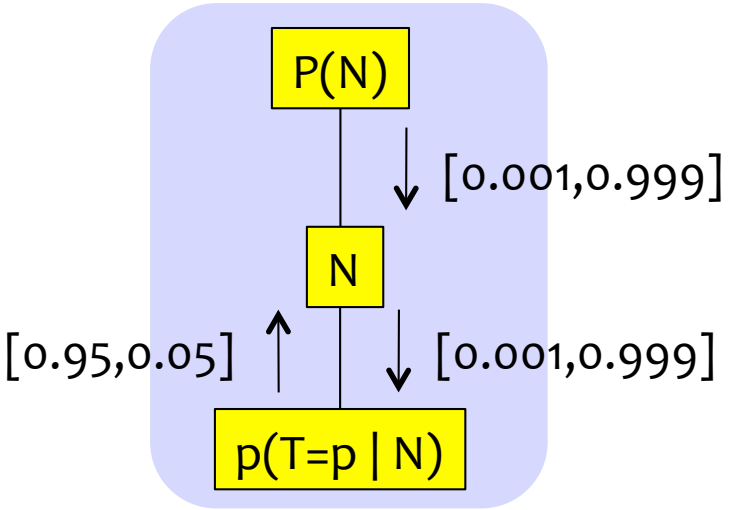
Image: Mike West, Duke University

The deadly disease - revisited

- Problem

  Nick is worried he is sick. He read in the newspaper 0.1% of the people is his city have contracted a deadly disease. He goes to the hospital to get tested. The doctor tells him the test is 95% reliable (i.e., it will give a correct result 95% of the time). The test is applied to Nick and turns out to be positive. With what probability does Nick have the disease?

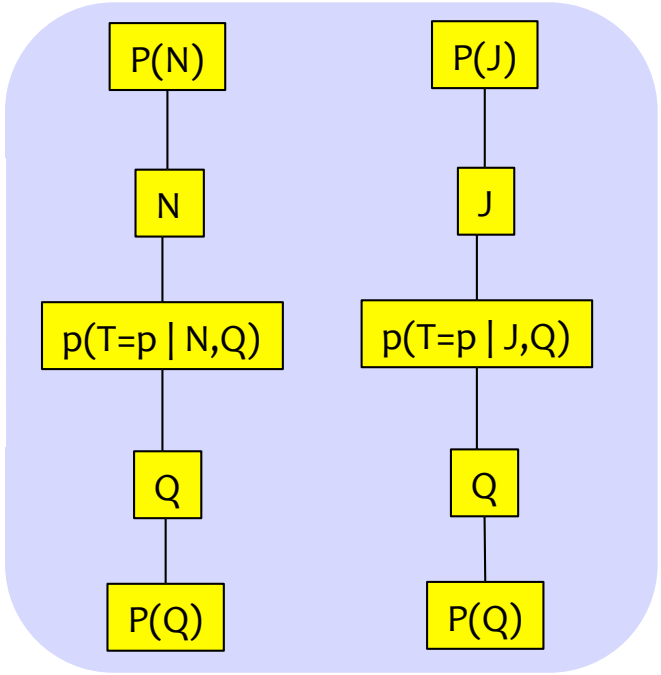
- Factor graph
  - Observation: "test is positive" = "T=p"
  - Unknown: "Nick's health" = "N" ∈ {h,s}
  - Determine p(N|T=p) = p(N,T=p)/P(T=p)
  - Draw factor graph of p(N, T=p), perform SPA

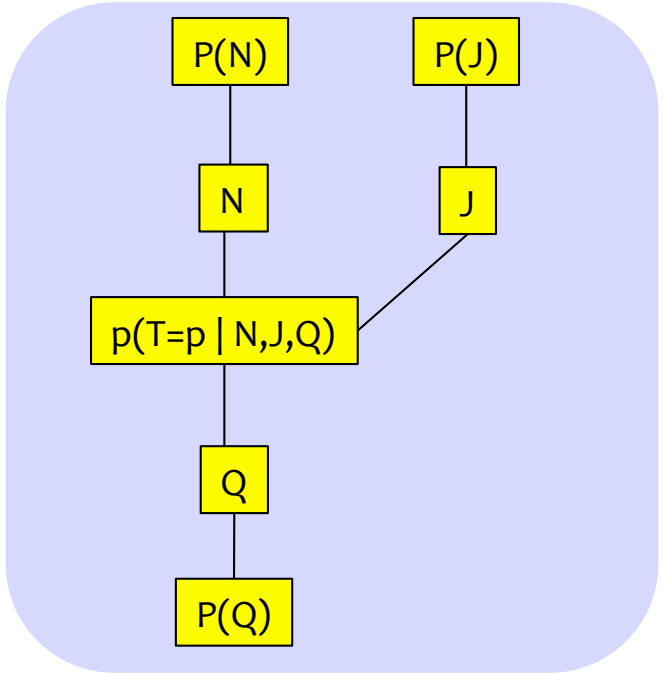


$$
\begin{aligned}
p(N=s,T=p) &= 0.95 \times 0.001 \\
p(N=h,T=p) &= 0.05 \times 0.999 \\
p(N=h|T=p) &= \frac{p(N=h,T=p)}{p(T=p)} \\
&= \frac{p(N=h,T=p)}{p(N=h,T=p)+p(N=s,T=p)} \\
&= 0.98
\end{aligned}
$$

The deadly disease -- variations

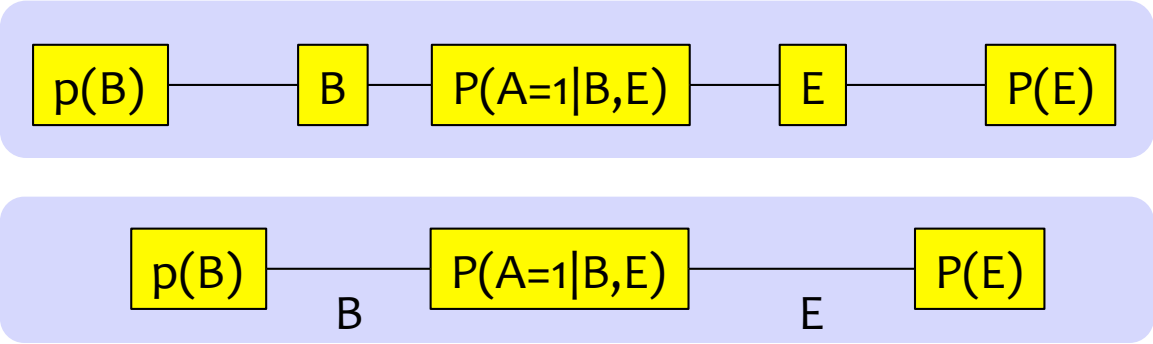
- Variation 1
  - Quality of test; Q ∈ {Good,Bad}
- Variation 2
  - Nick and Jane both get tested, result is positive
    - Independently *OR*
    - From the same test (e.g., same bad/good batch)



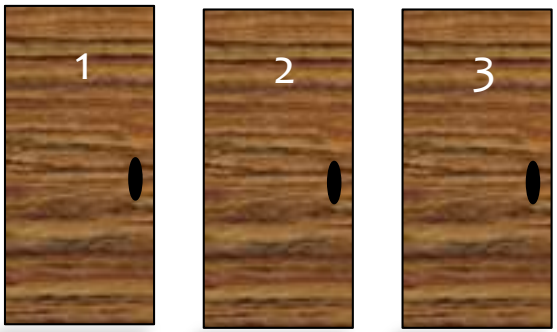


Example: The burglar alarm problem

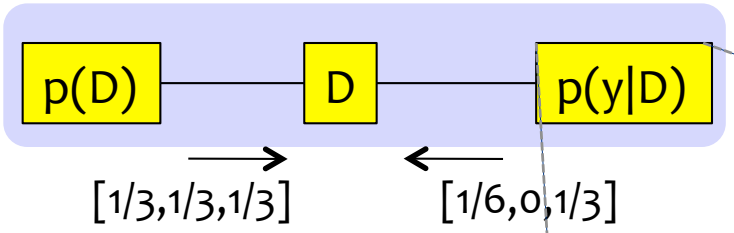
- Setup
  - A neighbor calls you at work, saying your burglar alarm is ringing
  - You know this could be due to either a burglar or an earthquake
  - Burglar and earthquake are a priori equally likely (probability 1/100)
- The following can trigger alarm, with the associate probabilities:
  - neither burglar nor earthquake = 1/1000
  - Earthquake, but no burglar = 1/10
  - Burglar, but no earthquake = 1/7
  - Both burglar and earthquake = 9/10
- Questions
  - What is the likelihood of the alarm ringing?
  - What is the APD of a burglary happening
  - What is the APD of an earthquake happening?



Example: *The Monty Hall Problem*



- Problem
  - Behind one door is a prize. Pick a door (say, door 1).
  - Host reveals one other door (say, door 2), showing a chicken.
  - Should you switch to door 3?
- Factor graph solution
  - Unknown: door holding the prize: $D \in \{1,2,3\}$
  - Observation: Choice C =1, Fake F = 2: y=[C=1,F=2]



$p(D=1|C=1,F=2)=C * 1/6$
$p(D=2|C=1,F=2)=C * 0$
$p(D=3|C=1,F=2)=C * 1/3$

$p(C=1,F=2|D=1)=p(F=2|D=1,C=1)p(C=1|D=1)=1/2*1/3$
$p(C=1,F=2|D=2)=p(F=2|D=2,C=1)p(C=1|D=2)=0*1/3$
$p(C=1,F=2|D=3)=p(F=2|D=3,C=1)p(C=1|D=3)=1*1/3$

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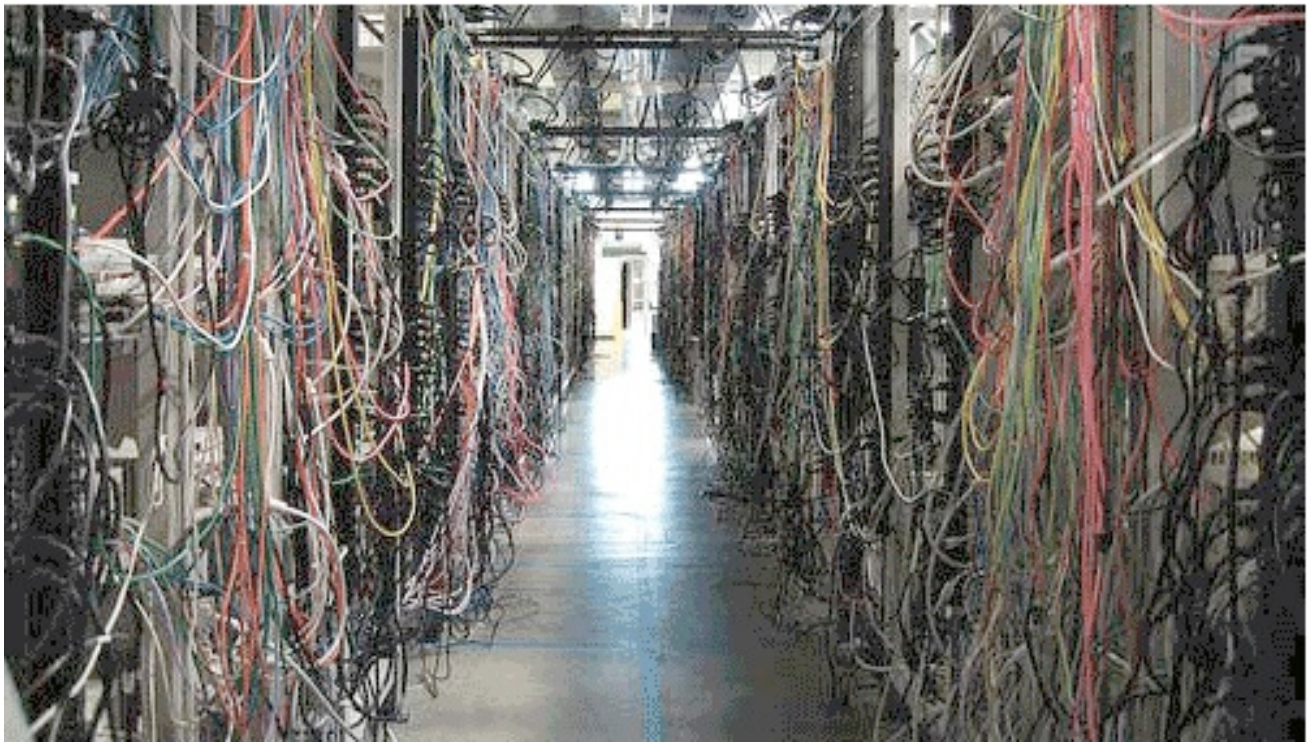
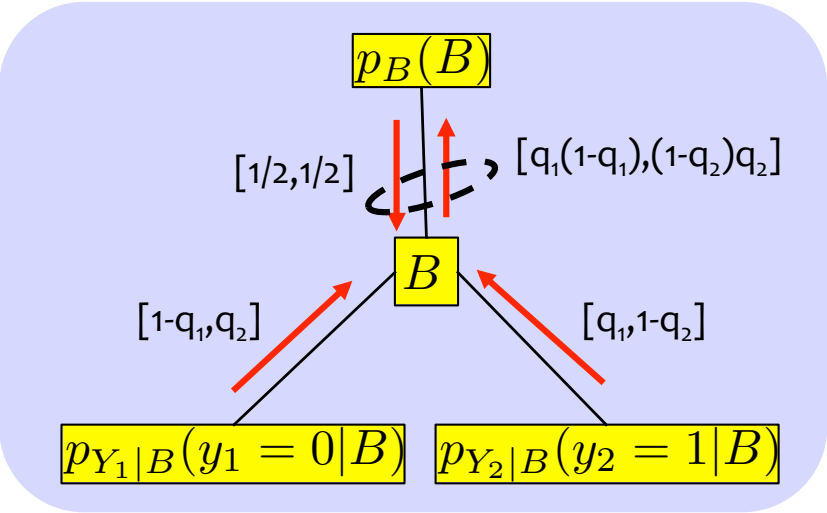
Practicalities



Message representation

Messages and their representations

- In inference problems
  - Initialization: messages are likelihoods or pmfs
  - SPA: multiplication and addition of messages
- Numerical stability issue
  - Messages get smaller and smaller in magnitude
- Several approaches to solve this
  - Normalization
  - Log-domain processing



Very small in magnitude

$p_{B|Y}(0|[0,1]) = q_1(1-q_1)/(q_1(1-q_1)+(1-q_2)q_2)$
$p_{B|Y}(1|[0,1]) = q_2(1-q_2)/(q_1(1-q_1)+(1-q_2)q_2)$

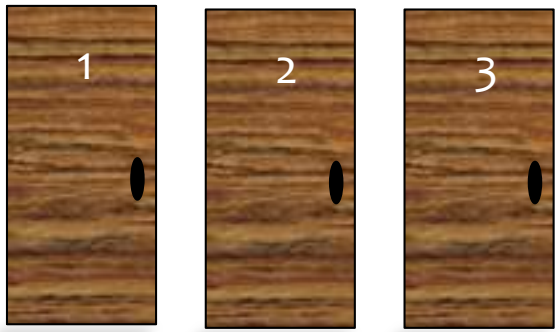
Messages representations

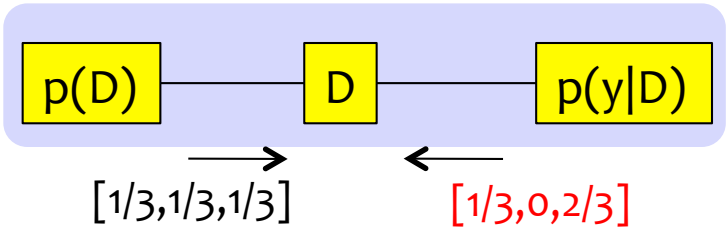
- Normalization
  - M = [a; b] becomes [a/(a+b); b/(a+b)] with normalization constant (NC) 1/(a+b)
  - Does not affect outcome of SPA (as long as we keep track of NCs)
  - Forgetting NCs does not affect marginal posteriors
  - Forgetting NCs affects likelihood of model

- Log-domain
  - M = [a; b] becomes [log(a) log(b)]
  - Efficient computation rules (“max-star”)

- Normalization + log-domain
  - M=[a; b] becomes [log(a/b)]
  - Log-likelihood ratio (LLR)
  - Popular for binary variables

Example: normalization





- Problem
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  - Host reveals one other door (say, door 2), showing a chicken.
  - Should you switch to door 3?
- Factor graph solution
  - Unknown: door holding the prize: $D \in \{1,2,3\}$
  - Observation: Choice C =1, Fake F = 2: y=[C=1,F=2]

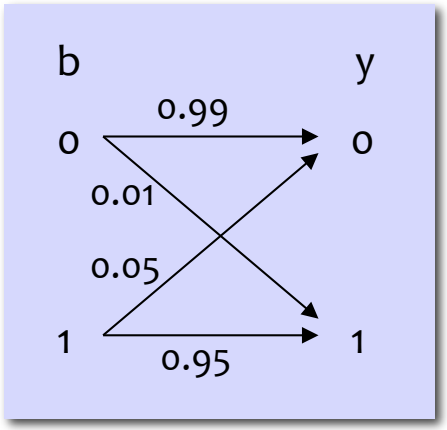
$p(D=1|C=1,F=2)=C * 1/6$
$p(D=2|C=1,F=2)=C * 0$
$p(D=3|C=1,F=2)=C * 1/3$

SWITCH!!!

$p(D=1|C=1,F=2)=C * 1/3$
$p(D=2|C=1,F=2)=C * 0$
$p(D=3|C=1,F=2)=C * 2/3$

SWITCH!!!

Example: normalization

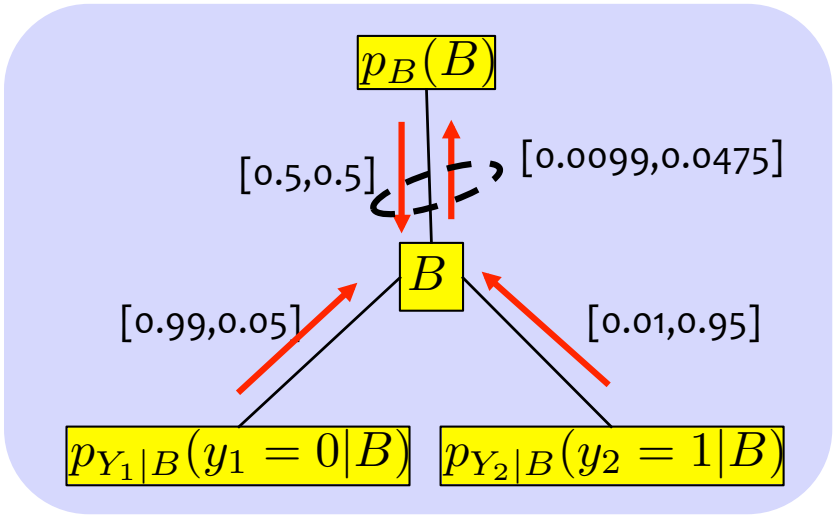


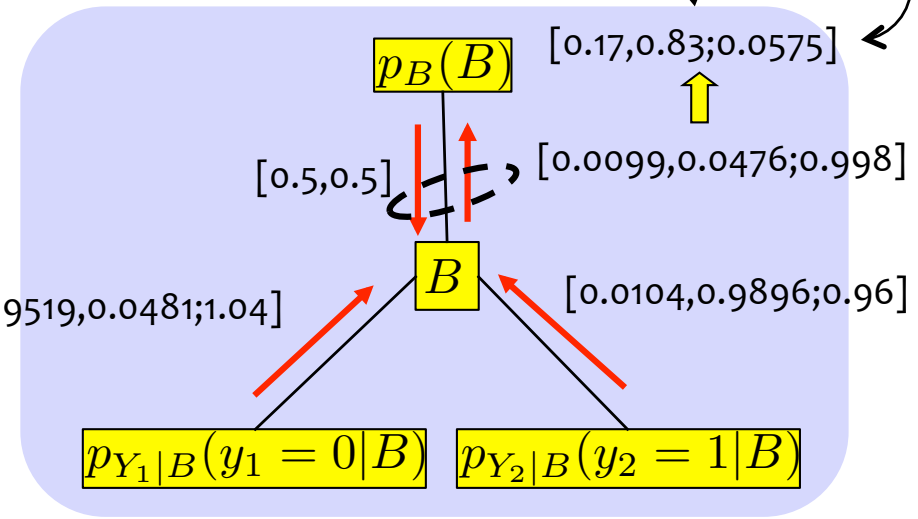
- Dividing all elements of the message with a constant, such that sum=1
  - Messages can be interpreted as distributions
  - Example: **m**=[0.001,0.02] → normalized = [0.0476,0.9524]
  - Normalization constant 0.021: **m'** = [0.0476,0.9524;0.021]

Without normalization:

$p_{B|Y}(0|[0,1])$= 0.0099 /(0.0099+0.0475)=0.17
$p_{B|Y}(1|[0,1])$= 0.0475 /(0.0099+0.0475)=0.83

$p_Y([0,1])$= 0.0099+0.0475=0.0575





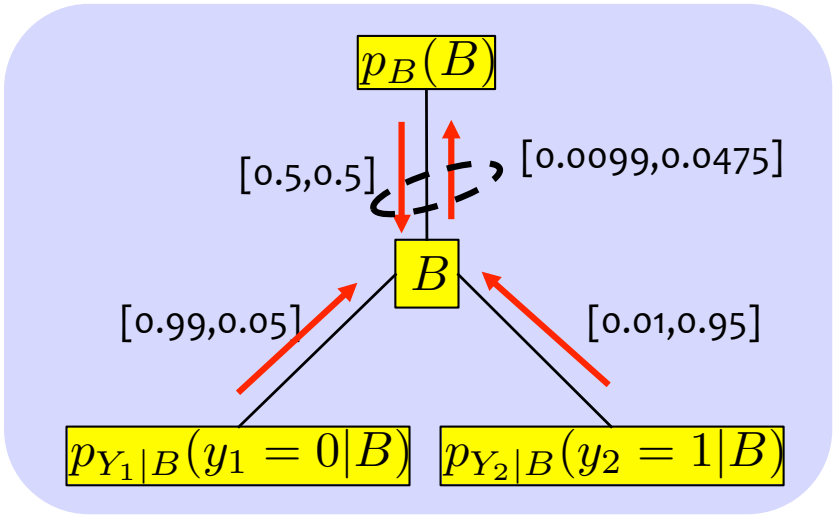
Example: normalization

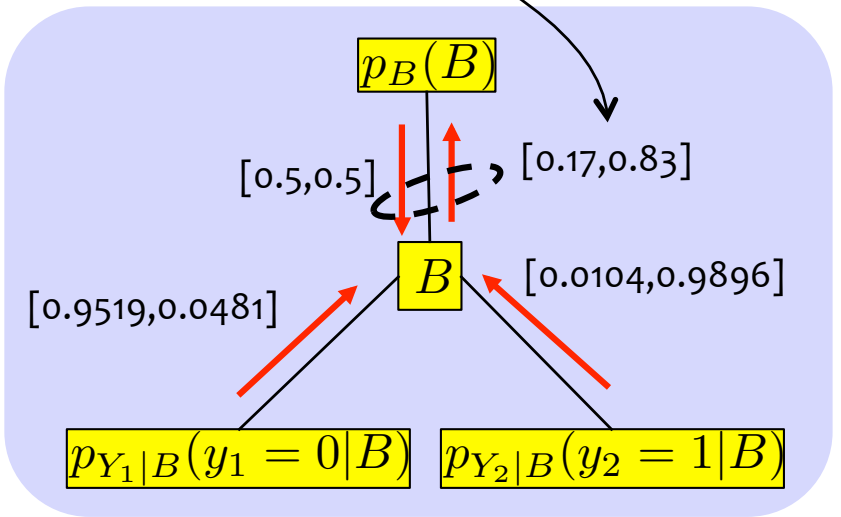
- Forgetting normalization constant is OK

Without normalization:

$p_{B|Y}(0|[0,1])$= 0.0099 /(0.0099+0.0475)=0.17
$p_{B|Y}(1|[0,1])$= 0.0475 /(0.0099+0.0475)=0.83

$p_Y([0,1])$= 0.0099+0.0475=0.0575 ???





Example: log-domain

- Instead of **m**, store log(**m**)
- Increased dynamic range
- Product become sum

Probability domain:

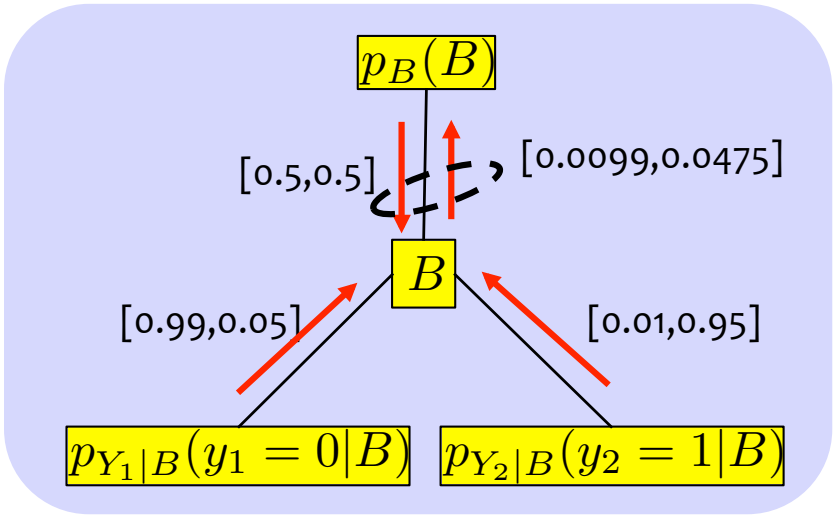
$p_{B|Y}(0|[0,1])$= 0.0099 /(0.0099+0.0475)=0.17
$p_{B|Y}(1|[0,1])$= 0.0475 /(0.0099+0.0475)=0.83

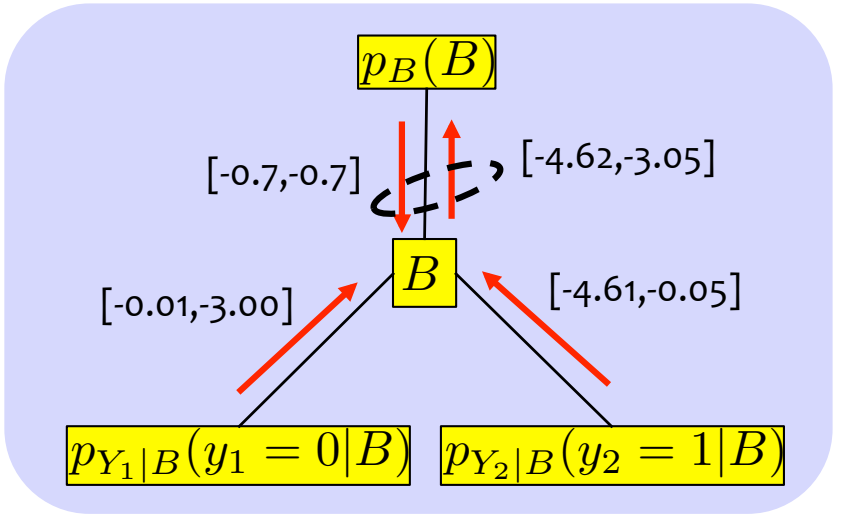
$p_Y([0,1])$= 0.0099+0.0475=0.0575

Log-domain:

$p_{B|Y}(0|[0,1])$= exp(-4.62)/(exp(-4.62)+exp(-3.05))=0.17
$p_{B|Y}(1|[0,1])$= exp(-3.05) /(exp(-4.622)+exp(-3.05))=0.83

$p_Y([0,1])$= exp(-4.62)+exp(-3.05)=0.0575





Log-domain messages

- Go back and forth between messages and log-domain

$$\mu_{f_k \to X_i}(x_i) = \sum_{\sim\{x_i\}} \left(f(x_1, \ldots, x_D) \prod_{j \neq i} \mu_{X_j \to f_k}(x_j) \right)$$

$$L_{f_k \to X_i}(x_i) = \log \left\{ \sum_{\sim\{x_i\}} \left(f(x_1, \ldots, x_D) \prod_{j \neq i} e^{L_{X_j \to f_k}(x_j)} \right) \right\}$$

- Can be avoided by using the Jacobian logarithm (aka max-star)

$$L_{f_k \to X_i}(x_i) = \mathbb{M}_{\sim\{x_i\}} \left(\log f(x_1, \ldots, x_D) + \sum_{j \neq i} L_{X_j \to f_k}(x_j) \right)$$

  - Defined recursively

$$\mathbb{M}(L_1, \ldots, L_L) = \mathbb{M}(L_1, \mathbb{M}(L_2, \ldots, L_L))$$
$$\mathbb{M}(L_1, L_2) = \max(L_1, L_2) + \log\left(1 + e^{-|L_1 - L_2|}\right)$$

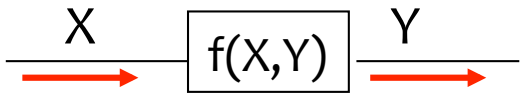
  - With core operation: 1 max + 1 table look-up
  - Very efficient

- Example for variables x_1, x_2 defined over $\{-1,0,1\}$

$$\mathbb{M}_{\sim\{x_1\}}(f(x_1, x_2)) = \mathbb{M}(f(x_1, -1), f(x_1, 0), f(x_1, +1))$$

Continuous variables

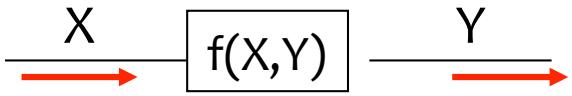
- Interpret messages a probability mass functions
- 3 approaches
  - Make them discrete, and use known techniques
  - Parametric representation (e.g., Gaussian mixture); message = vector of parameters
  - Non-parametric representation; message = vector of (weighted) samples
- We focus on non-parametric representation for simple factor:



We assume we have a representation of the incoming message on X: $\{w_k, x_k\}$, k=1..N

Non-parametric representation: method 1

- Given N samples $\{w_k,x_k\}$ of $p_X(x)$



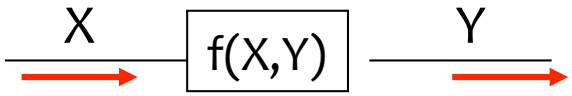
$$
\begin{aligned}
p_Y(y) &= \gamma \int f(x,y) p_X(x) dx \\
&= \int p_{Y|X}(y|x) p_X(x) dx \\
&\approx \int p_{Y|X}(y|x) \sum_{k=1}^{N} w_k \delta(x - x_k) dx \\
&= \sum_{k=1}^{N} w_k p_{Y|X}(y|x_k)
\end{aligned}
$$

- For every x_k, draw a sample y_k from some nice pdf, weight v_k

$$
\begin{aligned}
y_k &\sim q_{Y|X}(y|x_k) \\
v_k &\propto w_k \frac{p_{Y|X}(y|x_k)}{q_{Y|X}(y|x_k)} \quad \} \text{ unknown} \\
&\propto w_k \frac{f(x_k,y)}{q_{Y|X}(y|x_k)} \quad \} \text{ known}
\end{aligned}
$$

Non-parametric representation: method 2

- Given N samples $\{w_k,x_k\}$ of $p_X(x)$



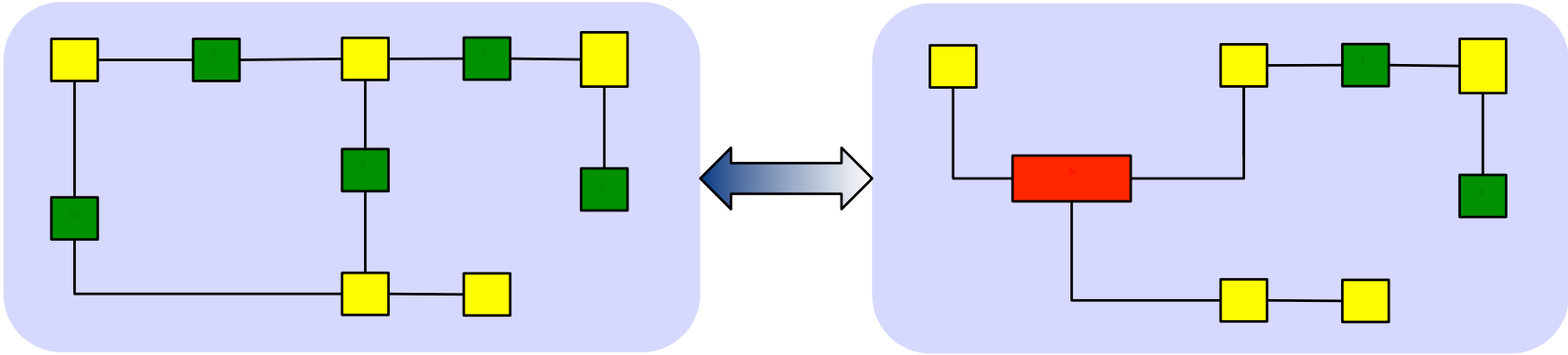
$$
\begin{aligned}
p_Y(y) &= \gamma \int f(x,y) p_X(x) dx \\
&= \int p_{Y|X}(y|x) p_X(x) dx \\
&\approx \int p_{Y|X}(y|x) \sum_{k=1}^{N} w_k \delta(x - x_k) dx \\
&= \sum_{k=1}^{N} w_k p_{Y|X}(y|x_k)
\end{aligned}
$$

- Draw x_k from $p_X(x)$, draw a sample y_k from some nice pdf, weight v_k

$$
\begin{aligned}
x_k &\sim p_X(x) \\
y_k &\sim q_{Y|X}(y|x_k) \\
v_k &\propto \frac{p_{Y|X}(y|x_k)}{q_{Y|X}(y|x_k)} \quad \text{unknown} \\
&\propto \frac{f(x_k,y)}{q_{Y|X}(y|x_k)} \quad \text{known}
\end{aligned}
$$

Graphs with cycles

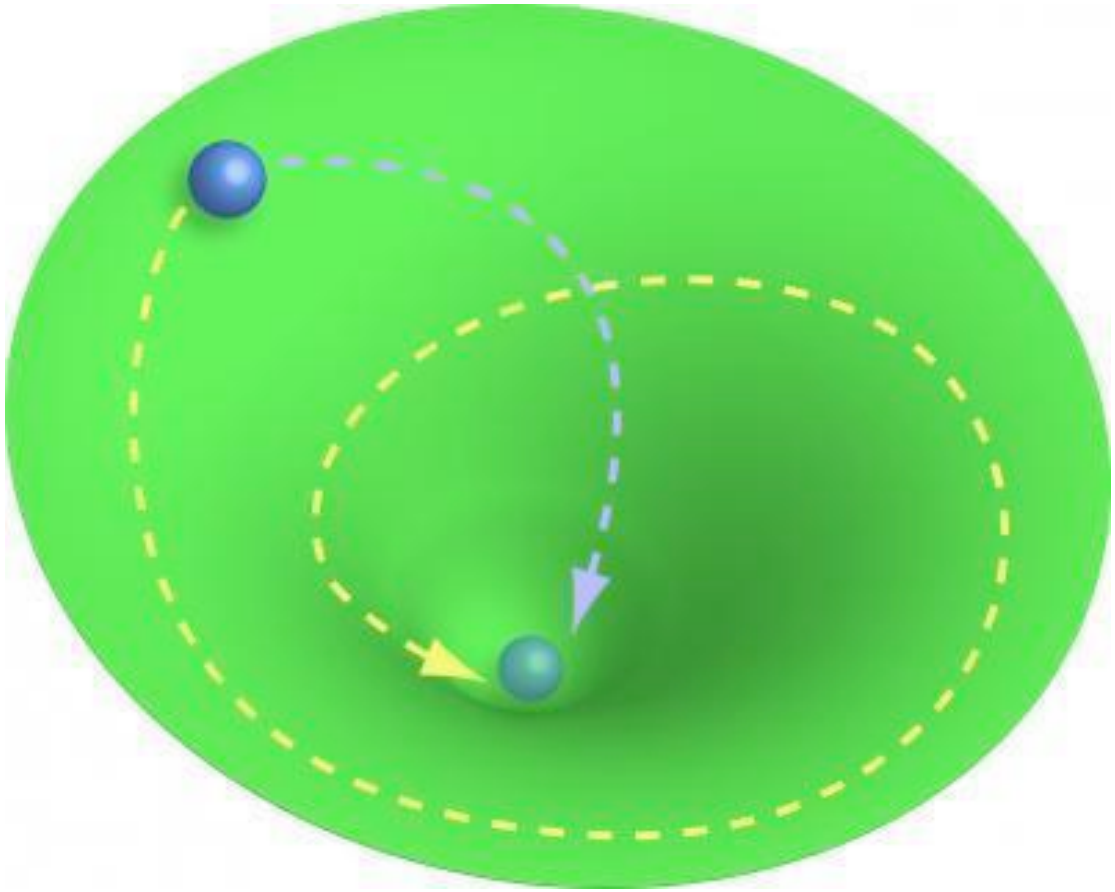
- Traditional approach
  - Convert to tree using junction tree methods



- Modern approach
  - when we normalize messages, SPA can still give good results
  - MAPD are not exact, but approximations (except means for Gaussian models)
  - Approximate MAPD are called "beliefs"
  - Possible to compute an approximation of the likelihood (using a Bethe free energy approximation)

- Many of the practical applications of SPA involve factor graphs with cycles
  - Turbo codes, LDPC codes, BICM-ID, MIMO detection, Multi-user detection

Convergence behavior



Turbo decoding example

- Typical behavior

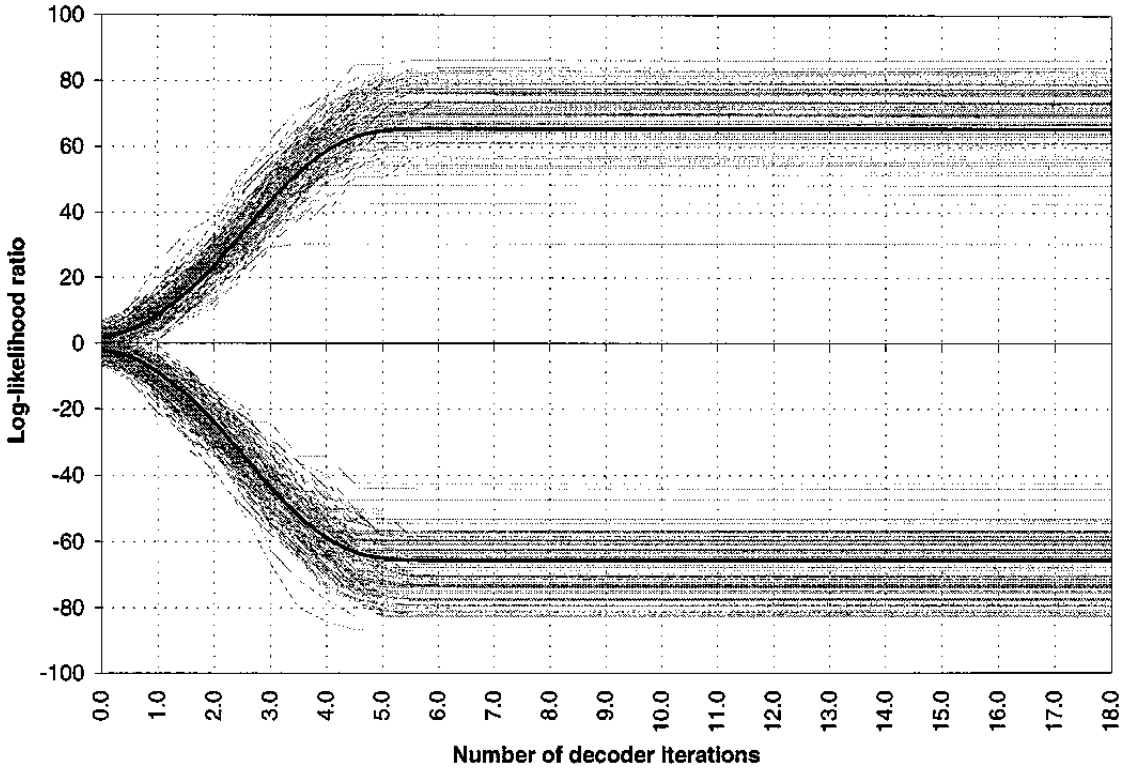


Fig. 1. Bit convergence of a typical frame in which all errors are corrected. (SNR = 1.77 dB, random interleaver).

Source: Reid, A.C.; Gulliver, T.A.; Taylor, D.P.; , "Convergence and errors in turbo-decoding," *Communications, IEEE Transactions on* , vol.49, no.12, pp.2045-2051, Dec 2001

Turbo decoding example

- Sometimes

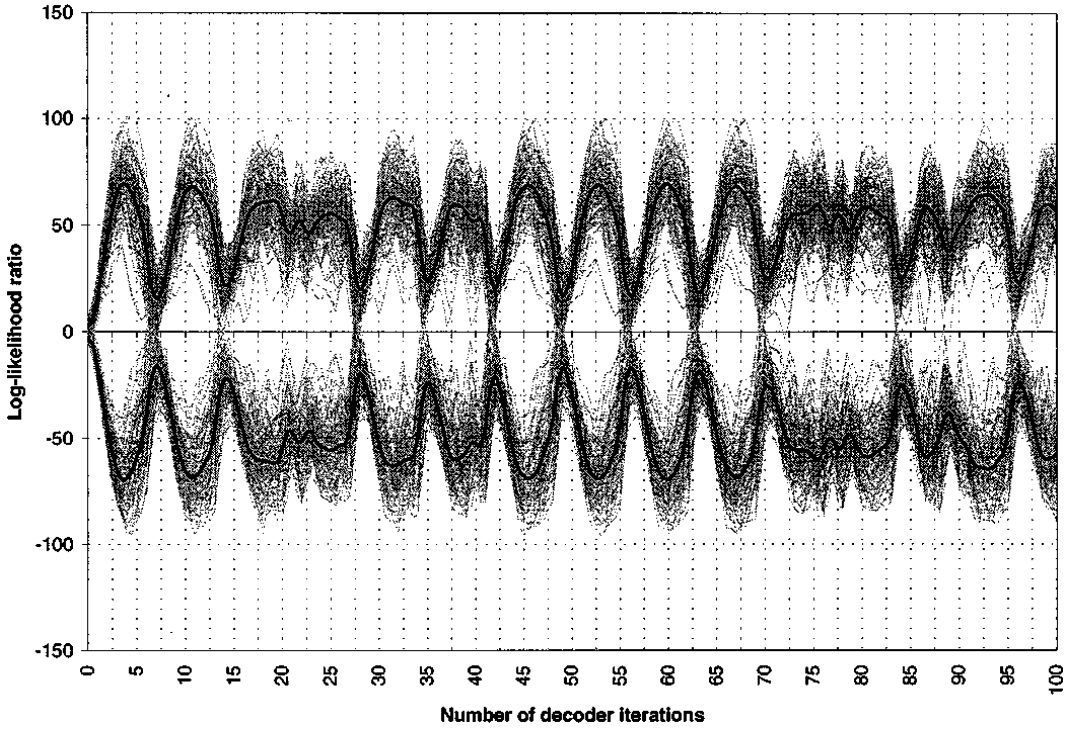


Fig. 11. Bit convergence of a frame which shows large oscillations to 100 iterations. (SNR= 3.27 dB, constrained interleaver).

Source: Reid, A.C.; Gulliver, T.A.; Taylor, D.P.; , "Convergence and errors in turbo-decoding," *Communications, IEEE Transactions on* , vol.49, no.12, pp.2045-2051, Dec 2001

Convergence results for SPA

- General models
  - Convergence is not guaranteed
  - Even when converged, no guarantee on quality of the marginals

- Gaussian models $p(\mathbf{x}) \propto \exp\left(-\mathbf{x}^T\mathbf{W}\mathbf{x} + \mathbf{b}^T\mathbf{x}\right)$

$$= \prod_{i,j} \exp(-w_{ij}x_i x_j) \times \prod_i \exp(b_i x_i)$$

1. Convergence is guaranteed when W is diagonal dominant (Weiss, Freeman, 2001)

$$|w_{ii}| > \sum_{j \neq i} |w_{ij}|, \forall i$$

2. Convergence is guaranteed when W=I-R, for which ρ(R)<1 (Malioutov, Johnsson, Willsky, 2006)
3. When convergence is achieved, the means of the marginals are correct
4. When convergence is achieved, the variances of the marginals are typically too small

- Non-Gaussian models (Mooij, Kappen, 2007)
  - Sufficient, not necessary conditions for convergence
  - No guarantee on quality of the solutions

Convergence results for SPA

- Rules of thumb
  - Avoid short cycles
  - Avoid strong interactions among variables

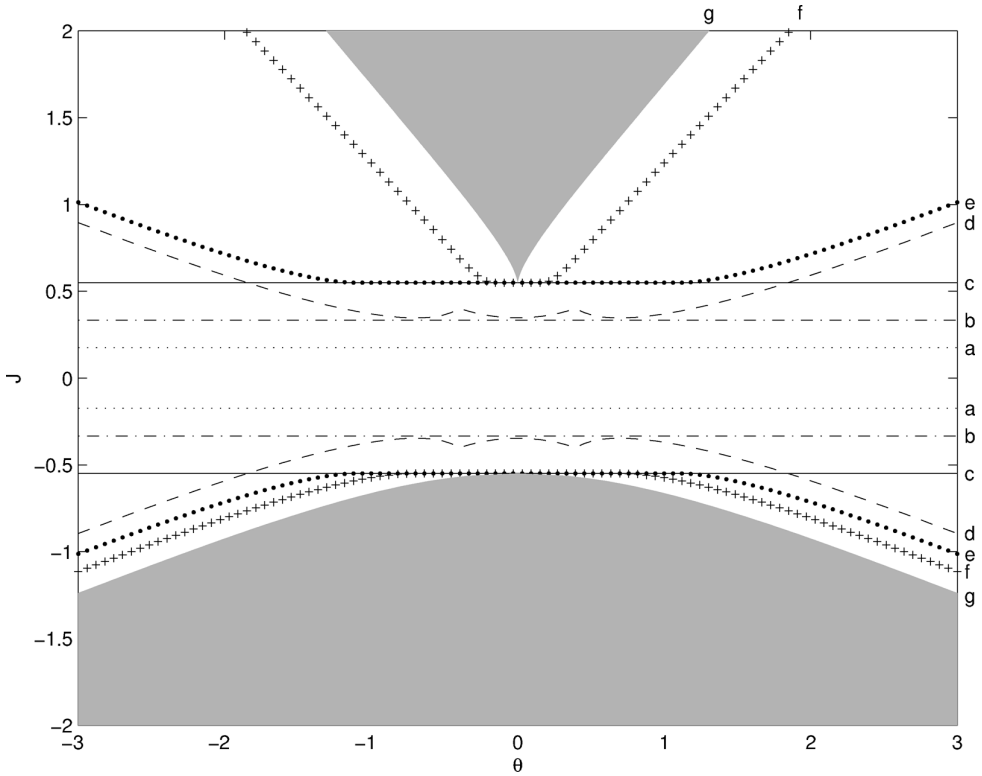
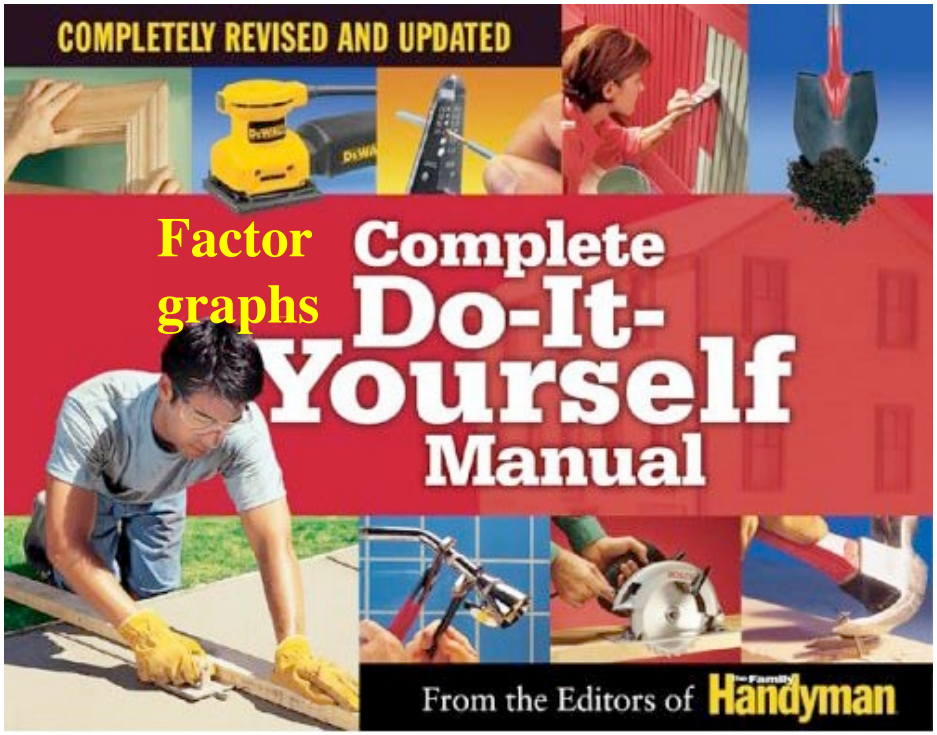


Fig. 4. Comparison of various BP convergence bounds for the fully connected $N = 4$ binary Ising model with uniform coupling J and uniform local field θ. a: Heskes' condition, b: Simon's condition, c: spectral radius condition, d: Dobrushin's condition, e: improved spectral radius condition for $m = 1$, f: improved spectral radius condition for $m = 5$, g: uniqueness of Gibbs' measure condition. See the main text (Section VI-A) for more explanation.

Outline

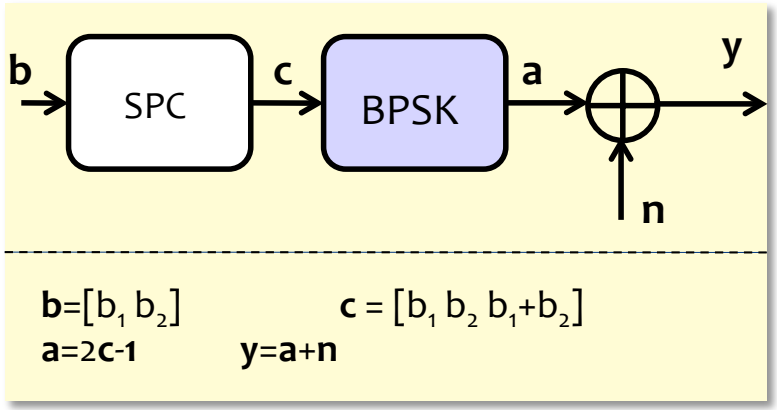
- Applications
- Background and terminology
- Bayesian detection
- Tool 1: Bayesian graphical models
  - Basics
  - Recipe for Bayesian inference
  - Practicalities
  - A worked example: a digital receiver
- Tool 2: Belief consensus
  - Basics
  - Convergence
- Applications revisited
- Variational interpretation
- Summary and conclusions

Example



Build your own factor graph receiver

- Transmitter



- Develop optimal bit-by-bit detector

$$\hat{b}_k = \arg\max_{b_k \in \mathbb{B}} p(b_k|\mathbf{y})$$

- Implement in MATLAB (less than 20 lines)

Now: MATLAB implementation

- In LLR domain

$$\lambda = \log \frac{\mu(1)}{\mu(0)}$$

$$= \log \mu(1) - \log \mu(0)$$

- Recall that

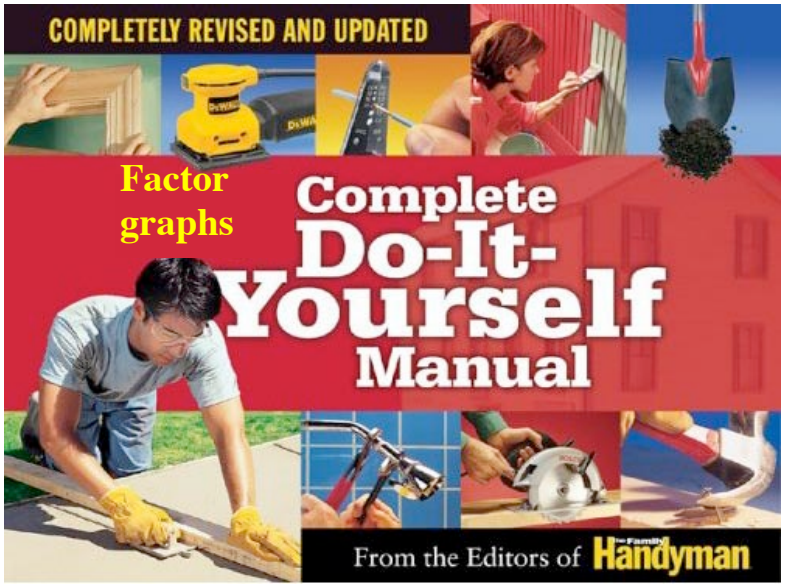
$$\mu_a \times \mu_b \rightarrow \log \mu_a + \log \mu_b$$

$$\mu_a + \mu_b \rightarrow \max^*(\log \mu_a, \log \mu_b)$$

- Other conversion rules

$$\log \mu_a(1) = +\lambda_a/2$$

$$\log \mu_a(0) = -\lambda_a/2$$



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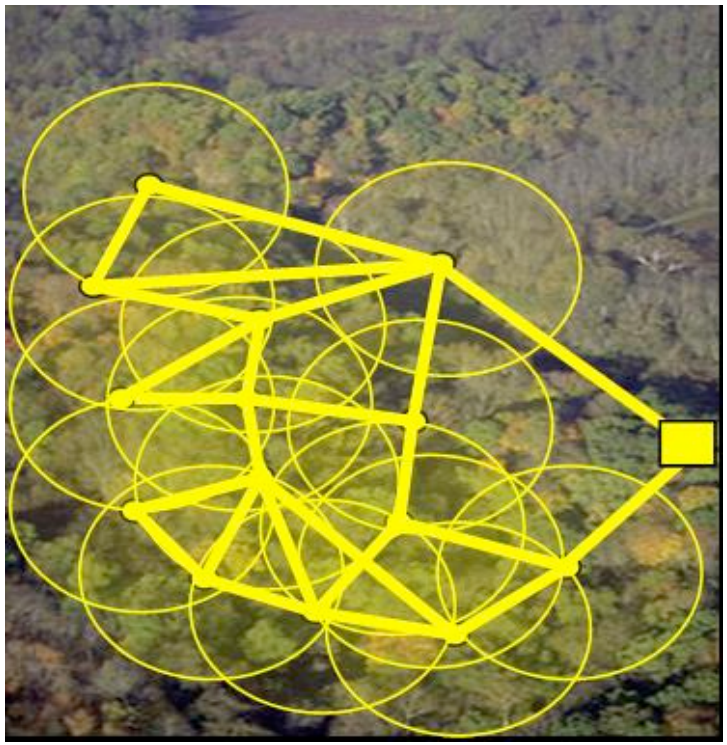
Motivation

- Many individual nodes/observers, form network
- How to perform distributed inference, optimization, or control without centralized processing?
- Connection to flocking behavior



Problem formulation

Model

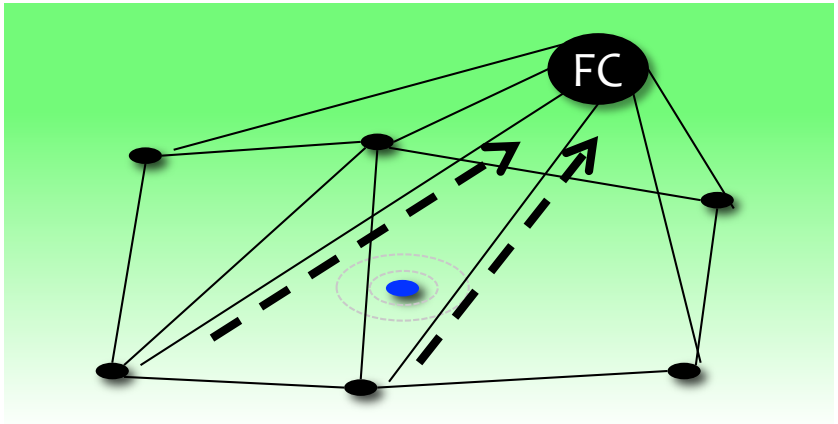
- N sensors, form a network, $\mathcal{N}_i$ = neighbors of node i
- Independent observation at sensor i: $y_i \in R^M$
- Prior p(x), $x \in \{0,1\}$, known to **all** sensors
- Likelihood $p(y_i|x)$ known **only** to sensor i
- No packet collisions

Goal

- Compute $p(x|\mathbf{y})$ at every sensor, without centralized processing

Centralized approach

- Version 1: broadcast to fusion center



- Version 2: routing to fusion center